Correction contrôle continu

ANA. Num. Mat

Master I : M.A.

Ex1 (9P)

1- (2P) λ v.p de $A \iff \exists \bar{x} \neq 0 \,/\, A\bar{x} = \lambda \bar{x} \implies \exists \bar{x} \neq 0 \; B_\varepsilon \bar{x} = \frac{A}{\rho(A)+\varepsilon}\bar{x} = \frac{\lambda}{\rho(A)+\varepsilon}\bar{x}$.

$\iff \frac{\lambda}{\rho(A)+\varepsilon}$ v.p de B_ε

2- (2P) $\rho(B_\varepsilon) = \frac{\rho(A)}{\rho(A)+\varepsilon} < 1 \iff \lim\limits_{k\to\infty} \|B_\varepsilon^k\| = 0$ (Cours).

3) (2P) Soit λ v.p de $A \,/\, \rho(A) = |\lambda|$

$\rho(A)\|x\| = |\lambda|\,\|x\| = \|Ax\| \leq \|A\|.\|x\| \implies \rho(A) \leq \|A\|$

et $\rho(A^k) \leq \|A^k\|$.

or $\rho(A^k) = (\rho(A))^k$ donc $\rho(A) \leq \|A^k\|^{1/k}$ (*)

4) (3P) $\lim\limits_{k\to\infty} \|B_\varepsilon^k\| = 0 \iff \forall \varepsilon' > 0 \; \exists n_0 \,/\, \forall n \geq n_0 \; \|B_\varepsilon^k\| < \varepsilon'$

Pour $\varepsilon' = 1$ $\exists n_0 \,/\, \forall n \geq n_0 \quad \|B_\varepsilon^k\| \leq 1$

$\iff \exists n_0 \,/\, \forall n \geq n_0 \quad \frac{\|A^k\|}{(\rho(A)+\varepsilon)^k} \leq 1$

$\iff \exists n_0 \,/\, \forall n \geq n_0 \quad \|A^k\|^{1/k} \leq \rho(A)+\varepsilon$ (**)

(*) et (**) $\implies \forall \varepsilon > 0 \; \exists n_0 \,/\, \forall n \geq n_0 \quad \rho(A)-\varepsilon \leq \rho(A) \leq \|A^k\|^{1/k} \leq \rho(A)+\varepsilon$.

$\iff \lim\limits_{k\to\infty} \|A^k\|^{1/k} = \rho(A)$.

Ex2. (8P)

(3P) 1) On Trouve $M = \frac{I}{r}$, $N = \frac{I}{r} - A$, $B = I - rA$.

2) (2P) λ_i VP de $A \iff \exists x_i \neq 0 \,/\, Ax_i = \lambda_i x_i \iff \exists x_i \neq 0 \,/\, (I - rA)x_i = (1 - r\lambda_i)x_i$

$\iff (1 - r\lambda_i)$ v.p de $(I - rA)$.

3) C.N.S sur r pour que la méthode converge $\iff \rho(I - rA) < 1$

(3P) $\iff -1 < 1 - r\lambda < 1 \quad \forall \lambda$ v.p de A

$\iff 0 < r < \frac{2}{\lambda} \quad \forall \lambda$ v.p de A. (*)

or $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \implies \frac{2}{\lambda_n} \leq \cdots \leq \frac{2}{\lambda_1}$

donc (*) $\iff 0 < r < \frac{2}{\lambda_n}$.

X3) si $0 < r < \frac{1}{\max\limits_{1\leq i\leq n} a_{ii}}$ on a $\|B\|_\infty = \|(I - rA)\|_\infty < 1$