اهداء

الى أعزائي الطلبة بقسم الرياضيات كلية الرياضيات والاعلام الآلي جامعة الشهيد مصطفى ابن بولعيد –باتنه- والى كل طلبة ل.م.د رياضيات أهدي هذه النسخة من دروس في الرياضيات سميتها: **-عناصر التحليل الرياضي-** حرصت أثناء تحضيرها على جمع معظم العناصر الأساسية في الرياضيات التحليلية والأساسية ليسانس-ماستر رياضيات وعملت على ادرج معظم المعلومات الضرورية داخل النسخة وخاصة ما يتعلق منها ببراهين النظريات الأساسية في كل فصل من فصول هذه النسخة. الوحدات المستهدفة حصريا: الطبولوجيا- الفضاءات المترية-المؤثرات الخطية. أنوي مستقلا ان شاء الله استكمالها بعناصر جديدة تتضمن وحدات التجويد بقيود وبدون قيود.

قراءاتكم لما تيسرمنها زاد لنا دعواتكم لنا بالتوفيق والسداد والقبول وشكرا

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Summary

1-Real Numbers

1.1-Introduction.

Numbers are a central element in mathematics. Among the different types of numbers, the set $\mathbb{N} = \{0,1,2,\dots\}$ of natural numbers, the set $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N})$ $[m \in (-\mathbb{N}) \Leftrightarrow \exists\, n \in \mathbb{N}; m = -n$ of relative numbers and the set $\mathcal{Q}$ of rational numbers, that is: $r \in \mathcal{Q}$, if there are $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$, p and q are prime to each other $(p \wedge q = 1)$, such that $r = \frac{p}{q}$. Starting from $\mathbb{Q}$, whose well-known properties are assumed, namely, $(\mathbb{Q}, +, ., \leq)$ is a totally ordered set, the total order relation $\leq$, defined on $\mathbb{Q}$ is compatible with the addition + and the multiplication ×, and that $\mathbb{Q}$ is Archimidean i.e., $\forall r \in \mathbb{Q}^*$ there exists $n \in \mathbb{N}^*$, such that $r < n$. The need to introduce a larger set than $\mathbb{Q}$, is motivated by the fact that $\sqrt{2} \notin \mathbb{Q}$. Indeed, if there exists $p \epsilon \mathbb{Z}$ and $q \epsilon \mathbb{N}^*$ with $p \wedge q = 1$, such that $p^2 = 2\,q^2$, then 2 divides p^2, as the square of an odd number is odd, also 2 divides p, so there exists $p' \epsilon \mathbb{Z}$, such that $p = 2p'$, hence $2p'^2 = q^2$and therefore 2 divides q, contradiction. Also, the two numbers e and π are not rational. In general, if p is a prime number, then $\sqrt{p}$ is not an rational number,...etc. Such numbers are called irrational numbers. The union of rational numbers and irrational numbers constitutes the set $\mathbb{R}$ of real numbers. The object of the following section, is to define the set of real numbers by a series of axioms, and to give a second motivation for the introduction of this set.

1.2-Axiomatic definition of real numbers.

Since, the set of real numbers, was introduced to complete the set $\mathbb{Q}$ of rational numbers, then we say that x is a real number if either $(x \in \mathbb{Q})$, or ($x \notin \mathbb{Q}$, x is said to be an irrational number). The intuition of their existence is ancient (since Pythagoras and his proof of the irrationality of $\sqrt{2}$). Their rigorous construction, dating from the $19iem$ century by Cantor and Dedekine. Note that we can define a real number from its decimal development, i.e. a real x can be seen as a relative integer constituting its integer part, separated by a comma, followed by an infinity of digits constituting its decimal part for example: $\pi = 3.1415926536\dots$. This definition called arithmetic representation of a real number poses a certain number of problems. Also, a real number can be defined as a limit of the so-called Cauchy sequences in $\mathbb{Q}$ (the density of $\mathbb{Q}$ in $\mathbb{R}$). One of the simplest definitions of $\mathbb{R}$ is the following axiomatic definition.

Definition 1.1. The set $\mathbb{R}$ of real numbers, provided with two internal laws: the addition noted +, the multiplication noted ×. and the ordering or a comparison relation noted ≤ (lower or equal), satisfies the following axioms.

1-$(\mathbb{R}^*, +, \times)$ is a commutative field.

The addition is such that $(\mathbb{R}, +)$ is an Abelian group.

$a_1)$ $\forall x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$. The addition is associative.

$a_2)$ $\forall x, y \in \mathbb{R}$, $x + y = y + x$. The addition is commutative.

$a_3)$ $\forall x \in \mathbb{R}$, $x + 0 = 0$. 0 is a neutral element for addition.

$a_4)$ $\forall x \in \mathbb{R}$, $x + (-x) = 0$. Each element x admits a symmetric for the addition noted $-x \in \mathbb{R}$.

The multiplication is such that $(\mathbb{R}^*, \times)$, is an Abelian group, $(\mathbb{R}^* = \mathbb{R} \setminus \{0\})$.

$a_5)$ $\forall x, y, z \in \mathbb{R}$, $(x \times y) \times z = x \times (y \times z)$. Multiplication is associative.

$a_6)$ $\forall x, y \in \mathbb{R}$, $x \times y = y \times x$. Multiplication is commutative.

$a_7)$ $\forall x \in \mathbb{R}$, $x \times 1 = x$. 1 is a neutral element for multiplication.

$a_8)$ $\forall x \in \mathbb{R}^*$, $x \times x^{-1} = 1$. Each element x in $\mathbb{R}^*$ admits the reverse for the multiplication, noted x^{-1} or $\frac{1}{x} \in \mathbb{R}^*$.

Multiplication is distributive with respect to addition.
a_9) $\forall x, y, z \in \mathbb{R}, x \times (y+z) = (x \times y) + (x \times z)$.
2-$(\mathbb{R}, \leq)$is completely ordered.
a_{10}) $\forall x \in \mathbb{R}, x \leq x$. The ordering is reflexive.
a_{11}) $\forall x, y, z \in \mathbb{R}$, if $(x \leq y$ and $y \leq z)$ then $x \leq z$. The ordering is transitive.
a_{12}) $\forall x, y \in \mathbb{R}$, if $(x \leq y$ and $y \leq x)$ then $(x = y)$. The ordering is antisymmetric.
a_{13}) $\forall x, y \in \mathbb{R}, x \leq y$ or $y \leq x$. The comparison relation is a total ordering.
For every $x, y \in \mathbb{R}$, we write $x \leq y$ (x is less than or equal to y) or equivalently $y \geq x$ (y is upper than or equal to x), and the ordering $(x \leq y;\ x \neq y)$ is written $(x < y)$ (x is less than y), or (y is upper than x).
A real number x is said to be positive if $0 < x$, the set of positive real numbers is denoted by $\mathbb{R}_+^*$, x is said to be negative if $x < 0$, the set of negative real numbers is denoted by $\mathbb{R}_-^*$. In the sequel, for every $x, y \in \mathbb{R}$, we write $x - y$ instead of $x + (-y)$ and xy instead of $x \times y$.
3-Compatibility of the ordering $\leq$ with addition and multiplication.
a_{14}) $\forall x, y, x', y' \in \mathbb{R}$, satisfying $(x \leq y$ and $x' \leq y')$, we have $(x + x' \leq y + y')$. The ordering $\leq$ is compatible with addition.
a_{15}) $\forall x, y, x', y' \in \mathbb{R}^*$, satisfying $(x \leq y$ and $x' \leq y')$, we have $(xx' \leq yy')$. The ordering $\leq$ is compatible with multiplication.
As a consequence: for every x, y in $\mathbb{R}$, (if $x \leq y$ then $-y \leq -x$) and for every x, y in $\mathbb{R}^*$, (if $x \leq y$ then $y^{-1} \leq x^{-1}$).

1.3-Intervals, absolute value, bounded parts
Definition 1.2. A non-empty part E in $\mathbb{R}$ is an interval if, $\forall x, y \in E$ satisfying $x < y$, there exists $z \in E$ such that $x < y < z$.
If a, b and x_0 are three real numbers such that: $a < x_0 < b$. The unbounded open intervals of $\mathbb{R}$ are: $]-\infty, a[,\ ]b, +\infty[,\ \mathbb{R} =]-\infty, +\infty[$, and the open bounded interval of $\mathbb{R}$ is $]a, b[$. The unbounded closed intervals of $\mathbb{R}$ are: $]-\infty, a],\ [b, +\infty[,\ \mathbb{R} =]-\infty, +\infty[$ and the closed bounded interval of $\mathbb{R}$ is $[a, b]$. Neither open nor closed bounded intervals of $\mathbb{R}$ are $]a, b]$, $[a, b[$. In the case where $a = b$, $[a, a] = \{a\}$ and $]a, a[= \emptyset$. The numbers a and b are called the limits of the interval and $b - a$ is its length. The total order relation makes it possible to define the absolute value function in $\mathbb{R}$.
Definition 1.3. The absolute value in $\mathbb{R}$, is a function noted $|.|$, defined from $\mathbb{R}$ to $\mathbb{R}_+$ by:
$$\forall x \in \mathbb{R}, |x| = \begin{cases} x, \text{if } 0 \leq x; \\ -x, \text{if } x < 0, \end{cases}$$
As a direct consequence we have:
$\forall x \in \mathbb{R}, (x \leq |x|)$ and if $\alpha \in \mathbb{R}_+$ (fixed), $(|x| \leq \alpha \Leftrightarrow -\alpha \leq x \leq \alpha)$.
Proposition 1.1. The following are true, for every $x, y \in \mathbb{R}$:
1) $(\ |x| = 0 \Leftrightarrow x = 0)$.
2) $|xy| = |x||y|$. So, $|x^2| = |x|^2 = x^2$.
3) $|x + y| \leq |x| + |y|$, (triangular inequality).
4) $||x| - |y|| \leq |x - y|$.
Proof. 1) evident. 2) if x and y have the same sign, then $|xy| = xy$. In the case where $x, y \in \mathbb{R}_+$, $|x| = x$ and $|y| = y$, and in the case where $x, y \in \mathbb{R}_-$, $|x| = -x$ and $|y| = -y$, so in both cases $|x||y| = xy$. If x and y are of different signs, then $|xy| = -(xy)$. In the case where for example $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_-$, $|x| = x$ and $|y| = -y$, then $|x||y| = x(-y) = -(xy)$. 3) Since, from, 2) $\forall z \in \mathbb{R}, |z|^2 = z^2$, then for any $x, y \in \mathbb{R}$, we have $|x + y|^2 = (x + y)^2 = |x|^2 + 2xy + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$, so $|x + y| \leq |x| + |y|$. 4) We demonstrate in the same way that: $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x + y|$, and by replacing y by $(-y)$ in the last inequality, we get the result.

Definition 1.4 Let E be a non-empty part of $\mathbb{R}$. We say that:
$i)$ E is bounded above, if there is a real number M such that, $\forall x \in E, x \leq M$, in this case M is called an upper bound of E.
$ii)$ E is bounded below, if there is a real number m such that, $\forall x \in E, m \leq x$, in this case m is called a lower bound of E.
$iii)$ E is bounded, if E is both bounded above and below. Equivalently: E is bounded$\Leftrightarrow$there exists $\alpha \in \mathbb{R}_{+}$, such that $\forall x \in E, |x| \leq \alpha$.

Remark 1.1

$a)$ If M is an upper bound of E, any element greater than M is also an upper bound of E. When E is bounded above, the least upper bound of E is called **the supremum** of E, and denoted by $supE$, or $maxE$ if it belongs to E. The $supE$ when it exists, it is unique.
$b)$ If m is a lower bound of E, any element less than m is also a lower bound of E. When E is bounded below, the first lower bound of E is called **the infimum** of E and denoted by $infE$, or $minE$ if it belongs to E. The infE when it exists, it is unique.
$c)$ In the case where a non-empty part E of $\mathbb{R}$ is bounded, $[infE, supE]$ is the smallest closed interval containing E.
Let us end the axiomatic definition of $\mathbb{R}$, by the following.

4-**Axiom of the upper bound**.

a_{16}) Any non empty, bounded above (respectively bounded below) part of $\mathbb{R}$, has an supremum (respectively an infimum).

Remark 1.2. If $x, y \in \mathbb{R}$ such that $x < y + \varepsilon, \forall \varepsilon > 0$, then $x \leq y$. Indeed, suppose that x>y then for $\varepsilon = x - y$, we have $x < y + x - y = x$, contradiction.

Proposition 1.2. Let E be a bounded part of $\mathbb{R}$, M_0 and m_0 two real numbers, then:

1) $M_0 = supE \iff \begin{cases} i)\ \forall x \in \mathbb{R}, x \leq M_0; \\ ii)\ \forall \varepsilon > 0, \text{there exists } x_\varepsilon \in E, \text{such that } M_0 - \varepsilon < x_\varepsilon. \end{cases}$

2) $m_0 = infE \iff \begin{cases} i) \forall x \in \mathbb{R}, m_0 \leq x; \\ ii)\ \forall \varepsilon > 0, \text{there exists } x_\varepsilon \in E, \text{such that } x_\varepsilon < m_0 + \varepsilon. \end{cases}$

Proof. 1) Since M_0 is the an upper bound of E, then $i)$ $\forall x \in E, x \leq M_0$. To demonstrate $ii)$, suppose that there exists $\varepsilon > 0$, such that $\forall x \in E, x \leq M_0 - \varepsilon$, that is $M_0 - \varepsilon$ is an upper bound of E less than M_0, contradiction with the definition of $supE$. Reciprocally $i)$ implies that M_0 is an upper bound of E.To demonstrate that M_0 is the least upper bound of E, suppose that there exists $M_0' < M_0$, such that $M_0' = supE$. According to $i)$ and) $\forall \varepsilon > 0$, there exists $x_\varepsilon \in E$, such that $M_0 - \varepsilon < x_\varepsilon \leq M_0' < M_0$, so $M_0 < M_0' + \varepsilon$, using the remark 1.2, we get $M_0' = M_0$. Property 2) is demonstrated in the same way.

Example 1.1.

$a)$ If, $E = \{-1,0,1\}$ then, $infE = minE = -1$ and $supE = maxE = 1$.
$b)$ If $E = [0,1]$ then, $infE = minE = 0$ and $supE = maxE = 1$.
$c)$ If $E = [0,1[$ then, $infE = minE = 0$ and $supE = 1$.
$d)$ If $E =]0,1]$ then, $infE = 0$ and $supE = maxE = 1$.
$e)$ If $E =]0,1[$ then, $infE = 0$ and $supE = 1$.
Let us demonstrate, for example that in) $supE = 1$. Using property $a)$ in Proposition 1.2, it is clear that $i)$ $\forall x \in E, x < 1$. To demonstrate $ii)$, let $\varepsilon > 0$, if $\varepsilon \leq 1$ then $0 \leq 1 - \varepsilon < 1$, as $\mathbb{R}$ is an interval, there exists $x_\varepsilon \in \mathbb{R}$ such that $1 - \varepsilon < x_\varepsilon < 1$, so $x_\varepsilon \in E$. If, $1 < \varepsilon$, then $1 - \varepsilon < 0 < x, \forall x \in E$.

Example 1.2. This is still a motivation to introduce the set $\mathbb{R}$. Let $E = \{r \in \mathbb{Q}_{+}, r^2 < 2\}$ be a part of $\mathbb{Q}$. As $0 \in E$, $E \neq \emptyset$ and as $\forall r \in E, 0 \leq r < \sqrt{2} < 2$, then E is bounded in $\mathbb{Q}$, and $minE = 0 \in \mathbb{Q}_{+}$. But $supE$ is not in $\mathbb{Q}$, which shows that the axiom a_{16} of the upper bound is not true in $\mathbb{Q}$. Let us prof that $supE \notin \mathbb{Q}$. Suppose that, there exists $p \in \mathbb{Z}, q \in \mathbb{N}^*$, such that

$supE = \frac{p}{q} = r$. In the case where $0 < 2 - r^2$, we have $s = \frac{2-r^2}{5} \in \mathbb{Q}_+^*$, so $s < 1$ and $(r + s)^2 = r^2 + 2rs + s^2 < r^2 + 5s = 2$, witch implies that $r + s \in E$ therefore $s \leq 0$, contradiction. In the case where $0 < r^2 - 2$, we have $s = \frac{r^2-2}{5} \in \mathbb{Q}_+^*$, so $s < 1$ and $(r - s)^2 > r^2 - 2rs > r^2 - 4s = \frac{r^2+8}{5} > 2$, it follows that $r - s \in \mathbb{Q}_+^*$ and $r - s$ is an upper bound of E, witch is less than r, contradiction.

1. 4-Archimed's axiom, everywhere density of $\mathbb{Q}$ in $\mathbb{R}$

In all of the following: $\boldsymbol{S^C}$ denotes the complement of any set S **iffy** means, if and only if, **i.e.** means, that is Δ and ∇ be any family of elements α (sets of indices).

Proposition 1.3 (Archimed's axiom). $\mathbb{R}$ is Archimedean, i.e: For every $x, y \in \mathbb{R}_+^*$ satisfying $x < y$ there exists $n \in \mathbb{N}^*$, such that $y \leq nx$.

Proof. Suppose that, there exist x_0 and y_0 in $\mathbb{R}$, $x_0 < y_0$ and for all $n \in \mathbb{N}^*, nx_0 < y_0$. Since a non empty part $E = \{nx_0;\ n \in \mathbb{N}^*\}$ is bounded above by y_0. For $M_0 = supE$ and $\varepsilon = \frac{M_0}{2} > 0$, there exists $n_0 \in \mathbb{N}^*$ such that, $M_0 - \frac{M_0}{2} < n_0x_0$, hence $M_0 < (2n_0)x_0$, as $2n_0 \in E$, contradiction.

Remark 1.3.

$a)$ The set $\mathbb{N}$ of natural numbers is unbounded above. That is for every $y \in \mathbb{R}_+^*$, there exists $n \in \mathbb{N}^*$, such that $y \leq n$. It suffices to take $x = 1$ in the proposition 1.2.

$b)$ The set $\mathbb{Z}$ of relative numbers is both unbounded above and below, since $(-\mathbb{N})$ is unbounded below.

Definition 1. 5 (everywhere dense part in $\mathbb{R}$). A non-empty part Ein $\mathbb{R}$, is said to be everywhere dense in $\mathbb{R}$ if, for all x, y in $\mathbb{R}$, $x < y$ there exists $z \in E$, such that $x < z < y$.

Proposition 1.4. $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.

Proof. Let x, y are in $\mathbb{R}$ with $x < y$. Let us prove that there exists r in $\mathbb{Q}$ such that: $x < r < y$. Since $z = \frac{1}{y-x} > 0$, there exists $n \in \mathbb{N}^*$ such that $z = \frac{1}{y-x} < n$, or $nx + 1 < ny$ $(*)$, likewise for $nx \in \mathbb{R}$, there exists $k \in \mathbb{N}^*$ such that $nx < k$. Let $E = \{k \in \mathbb{N}^*;\ nx < k\}$ and $F = \{nx \in \mathbb{R};\ z < n\}$, E and F are non-empty, and F is bounded above by the elements of E. Let $p = supF$, then $p \in E$ and for $\varepsilon = 1$, there exists $n \in \mathbb{N}^*$ such that $p - 1 < nx < p$, witch implies that $nx < p < nx + 1$, using $(*)$ we obtain $nx < p < ny$ or $x < \frac{p}{n} < y, (r = \frac{p}{n} \in \mathbb{Q})$.

Example 1.3.

$a)$ $\sqrt{2}$ is the supremum of $E = \{r \in \mathbb{Q}_+, r^2 < 2\}$. Indeed $i)$ $\forall r \in E, r < \sqrt{2}$, $ii)$ For $0 < \varepsilon \leq \sqrt{2}$, we have $0 \leq \sqrt{2} - \varepsilon < \sqrt{2}$, since $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$, there exists $r_\varepsilon \in \mathbb{Q}$ such that, $0 \leq \sqrt{2} - \varepsilon < r_\varepsilon < \sqrt{2}$ $(r_\varepsilon \in E)$. If, $\sqrt{2} < \varepsilon$, then $\sqrt{2} - \varepsilon < 0 \leq r, \forall r \in E$.

$b)$ The set $\mathbb{Q}^C$, of the irrational numbers is everywhere dense in $\mathbb{R}$. Note that, for every $\in \mathbb{Q}$ $(\beta \neq 0), \alpha + \beta\sqrt{2} \in \mathbb{Q}^C$. Then if $x, y \in \mathbb{R}, x < y$ there exists $r \in \mathbb{Q}$ such that $x < r < y$. Since $\frac{\sqrt{2}}{y-r} \in \mathbb{R}_+^*$, there exists $n \in \mathbb{N}^*$, such that $\frac{\sqrt{2}}{y-r} < n$, then $x < r + \frac{1}{n}\sqrt{2} < y$ $\left(r + \frac{1}{n}\sqrt{2} \in \mathcal{QC}.\right.$

2-The Euclidean Topology, Topological space

2.1-Introduction

Starting from the open intervals in $\mathbb{R}$. We will define, the notion of open set, i.e. the parts of $\mathbb{R}$ which are the union of open intervals. We will demonstrate that, the open sets are stable by any union, and stable by the finite intersection. Let $\{O_\alpha;\ \alpha \in \Delta\}$ be a family of sets in $\mathbb{R}$, where $\forall \alpha \in \Delta$, O_α is the union of the open intervals in $\mathbb{R}$, then, the family $\tau_u = \{\emptyset, O_\alpha; \alpha \in \Delta\}$ define in addition to the algebraic stricture, a topological structure on $\mathbb{R}$. Open sets and therefore open intervals play a fundamental role in real analysis, namely: the study of the limit of a sequence, the continuity of a function, the derivability,...etc. In its primitive form topology was called situation geometry, or analysis situs. It is therefore a specific mathematical domain of geometry, which interested in the qualitative properties of mathematical objects, independently of any measurement. The study of topology requires at first a certain act of faith, which will make the internal beauty of this theory easier.

2.2-Open sets, closed sets, neighborhoods
Definition 2.1.
$a)$ Let $x_0 \in \mathbb{R}$ and $\delta > 0$, the interval $I(x_0, \delta) =]x_0 - \delta, x_0 + \delta[$ is called the open interval centered in x_0 with radius δ.
$b)$ The non-empty set O in $\mathbb{R}$, is called the open set, if $\forall x \in O$, there exists $\delta > 0$, such that, $]x - \delta, x + \delta[\subset O$.
$c)$ The complement of any open set in $\mathbb{R}$ is called a closed set.
$d)$ Let $x_0 \in \mathbb{R}$, we say that, the set N is a neighborhood of x_0, if there exists an open set O containing x_0, and $O \subset N$.
Example 2.1.
$a)$ All open intervals in $\mathbb{R}$ is an open set. For example $\forall a, b \in \mathbb{R}$, the open intervals $]a, b[$ and $]-\infty, a[$ are open sets. Indeed for each $x \in]a, b[$, there exists $\delta = \frac{1}{2} min(x - a, b - x) > 0$, such that $]x - \delta, x + \delta[\subset]a, b[$ and, for $x \in]-\infty, a[$, there exists $\delta = \frac{x-a}{2} > 0$ such that $]x - \delta, x + \delta[\subset]-\infty, a[$.
$b)$ $\mathbb{R}$ is open, since $\forall x \in \mathbb{R}$ and $\forall \varepsilon > 0$, the open interval $]x - \varepsilon, x + \varepsilon[\subset \mathbb{R}$.
c) $\forall a, b \in \mathbb{R}$, the closed intervals $]-\infty, a]$ and $[b, +\infty[$ are closed sets, indeed, $]-\infty, a]^C =]a, +\infty[$, and $[b, +\infty[^C =]-\infty, b[$.
$d)$ The interval $[a, b[$ is not open, indeed $\forall \varepsilon > 0,]a - \varepsilon, a + \varepsilon[\not\subset [a, b[$, since between $a - \varepsilon$ and a there exits at last an number less than a. Also he interval $]a, b]$ is not open.
$e)$ $\emptyset$ is closed. Since $\emptyset^C = \mathbb{R}$.
The open sets in $\mathbb{R}$ satisfied the following properties:
Proposition 2.1.
O_1-The union of any family of open sets is open.
O_2-The intersection of any finite family of open sets is open.
Proof. O_1-Let $\{O_\alpha;\ \alpha \in \Delta\}$ be a family of the open sets in $\mathbb{R}$. Then for $x \in \cup_{\alpha\in\Delta} O_\alpha$, there exists $\alpha \in \Delta$ such that $x \in O_\alpha$, therefore, there exists $\delta > 0$ such that $]x - \delta, x + \delta[\subset O_\alpha \subset \cup_{\alpha\in\Delta} O_\alpha$, hence $\cup_{\alpha\in\Delta} O_\alpha$ is an open. O_2-Let $\{O_\alpha;\ \alpha = 1, \ldots, n\}$ be a finite family of the open sets in $\mathbb{R}$. Then for $x \in \cap_{\alpha=1}^{n} O_\alpha$, we have $x \in O_\alpha, \forall \alpha \in \{1, \ldots, n\}$ therefore there exists $\delta_\alpha > 0$, such that, $]x - \delta_\alpha, x + \delta_\alpha[\subset O_\alpha, \forall \alpha \in \{1, \ldots, n\}$ then for $\delta = min\{\delta_\alpha, \alpha = 1,\ldots,n\}$, $]x-\delta, x+\delta[\subset \cap_{\alpha=1}^{n} O_\alpha$, witch implies that $\cap_{\alpha=1}^{n} O_\alpha$ is an open set. The collection $\tau_u = \{\emptyset, \mathbb{R}\} \cup \{O_\alpha;\ \alpha \in \Delta\}$, where $\forall \alpha \in \Delta$, O_α is an open set of $\mathbb{R}$, is called the **Euclidean** (or **usual**, or **natural**) **topology** of $\mathbb{R}$, the couple $(\mathbb{R}, \tau_u)$ is called **Euclidean** (or **usual**, or **natural**) space.
A topology can be defined on any nonempty set as follows:
Definition 2.2. Let E be a nonempty set. A family τ of subsets of E is called a topology, if:
$i)$ $\emptyset$ and E are in τ.
$ii)$ The union of any collection of elements of τ is an element of τ, (τ is stable by the union).

iii) The intersection of any finite collection of elements of τ is an element of τ, (τ is stable by the finite intersection).
The elements of τ are called the **open sets**, and (E, τ) is called a **topological space**. If $O \in \tau$, O^C is called a **closed set** in E and a set $N \subset E$ is called a **neighborhood** of a non-empty part A of E, if there exists $O \in \tau$ such that $A \subset O \subset N$. When $A = \{x\}$, we say that N is a neighborhood of the point x.

In the sequel, we use indifferently, space and subspace (respectively space $\mathbb{R}$) instead of, topological space and topological subspace (respectively instead of $(\mathbb{R}, \tau_u)$).

Example 2.2. In the space $\mathbb{R}$

a) $\forall a \in \mathbb{R}$, the singleton $\{a\}$ is closed, indeed for any $a \in \mathbb{R}, \{a\}^C =]-\infty, a[\cup]a, +\infty[$ which is according to O_2 is an open.

b) $\mathbb{N}$ and $\mathbb{Z}$ are closed sets, since $\mathbb{N}^C = \cup_{n\in\mathbb{N}}]n, n+1[$ and $\mathbb{Z}^C = \cup_{n\in\mathbb{Z}}]n, n+1[$.

c) $\mathbb{Q}$ and $\mathbb{Q}^C$ are neither open nor closed. In deed, suppose that $\mathbb{Q}$ is open then $\forall r \in \mathbb{Q}$, there exists $\delta > 0$ such that $]r-\delta, r+\delta[\subset \mathbb{Q}$, as $\mathbb{Q}^C$ is everywhere dense in $\mathbb{R}$, there exists $\alpha \in \mathbb{Q}^C$, $\alpha \in]r-\delta, r+\delta[\subset \mathbb{Q}$ contradiction. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we deduce that $\mathbb{Q}^C$ is not open and hence $\mathbb{Q}$ and $\mathbb{Q}^C$ are not closed.

d) Only $\emptyset$ and $\mathbb{R}$ are both open and closed (**clopen**). Indeed, if a subset A of $\mathbb{R}$ is clopen, then $A = \emptyset$. Since if we suppose that $A \neq \emptyset$, then for $x \in A^C$, one of the tow subsets $A \cap]-\infty, x]$ and $A \cap [x, +\infty[$ is nonempty. Suppose that $B = A \cap [x, +\infty[\neq \emptyset$, witch is clopen, then B is closed and bounded bellow, therefore it has an minimum. Let $b = minB$, as also $B = A \cap]x, +\infty[$ and B is open for $b \in B$, there is $\delta > 0$, such that $]b-\delta, b[\subset]b-\delta, b+\delta[\subset B$,witch implies that b is not the minimum of B, contradiction.

Remark 2.1 The intersection of any family of open set is not always open. In the space $\mathbb{R}$ for example, the family of open intervals $\left\{I_n\left(0,\frac{1}{n}\right), n \in \mathbb{N}^*\right\}$, is such that $\cap_{n\in\mathbb{N}^*} I_n\left(0,\frac{1}{n}\right) = 0$ which is a closed set. It is clear that $0 \in I_n\left(0,\frac{1}{n}\right), \forall n \in \mathbb{N}^*$ then $0 \in \cap_{n\in\mathbb{N}^*} I_n\left(0,\frac{1}{n}\right)$ and if $x \in \cap_{n\in\mathbb{N}^*} I_n\left(0,\frac{1}{n}\right)$, i.e. $-\frac{1}{n} < x < \frac{1}{n}, \forall n \in \mathbb{N}^*$, hence when $n \to +\infty, x = 0$.

On a non-empty set E, we can define several topologies and a subset of the space E can be open, closed, open and closed (**clopen**), neither open nor closed. If τ and σ are topologies on E, τ is called **coarser** (or **weaker** or **smaller**) than σ, equivalently σ is called **finer** (or **stronger** or **larger**) than τ, if every element of τ is an element of σ, and the relationship is expressed as $\tau \subset \sigma$. Of course, as sets of sets, $\tau \subset \sigma$. The ordering ⊂ is only a partial ordering. We say that **τ is equal to σ** if, τ⊂σ and σ⊂τ.

Example 2.3. Let E be a non-empty set.

a) Let $\mathcal{P}(E)$ be the collection of all parts of E, then $\mathcal{P}(E)$ is a topology on E, denoted τ_{dis} called a **discrete topology** and E is called a discrete space. This topology is the finest topology for E, since any open set of other topology is an open set in this topology.

b) The family $\tau_{idis} = \{\emptyset, \mathbb{R}\}$ is a topology on E, called **indiscrete topology** and E is called **indiscrete space**. This topology is the coarset topology in E, since any open set of this topology is an open set in other topology.

c) The family $\tau = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, E\}$, where $E = \{0,1,2\}$ is a topology on $\mathbb{R}$.

d) The family $\tau = \{[a, b[, a, b \in \mathbb{R}\}\cup\{\emptyset, \mathbb{R}\}$ is a topology on $\mathbb{R}$.

e) The collection $\tau_{cof} = \{\emptyset, A^C\}$, where A is finite i.e., the collection consisting of the empty set and, those subsets of E whose complements are finite. τ_{cof} is a topology on E, called **cofinite topology** and E is called **cofinite space**. For the proof, it suffices to apply the De Morgan's Laws and to remark that, $\emptyset$ is finite, the arbitrary intersection of finite sets is finite

and the finite union of finite sets is finite. In $\mathbb{R}$, $\tau_{cof} \subset \tau_u$, since if $O \in \tau_{cof}$, there is a finite $A \subset \mathbb{R}$, such that $O = A^C$, since A is closed in the space $\mathbb{R}$, then $O \in \tau_u$.
$f)$ Let E an uncountable set $\tau_{coc} = \{\emptyset, A^C\}$, where A is countable i.e., the collection consisting of the empty set and, those subsets of E whose complements are countable. Then τ_{coc} is a topology in E, called **cocountable topology** and E is the **cocountable space**
$g)$ Let $\mathcal{N} = \{\emptyset,\ N_n = \{1,2,\dots,\mathrm{n}\};\ \mathrm{n} \in \mathbb{N}^*\}$ be the family consisting of subsets of $\mathbb{N}^*$, then $(\mathbb{N}^*, \mathcal{N})$ is a topology in $\mathbb{N}^*$. Indeed, $i)$ $\emptyset, \mathbb{N}^* \in \mathcal{N}$. $ii)$ If, $N_{\mathrm{n}_1} = \{1,2,\dots,\mathrm{n}_1\}$, $N_{\mathrm{n}_2} = \{1,2,\dots,\mathrm{n}_2\}, \dots \mathrm{N}_{\mathrm{n}_\mathrm{k}} = \{1,2,\dots,\mathrm{n}_\mathrm{k}\}$, any arbitrary collection of the elements of $\mathcal{N}$, then $\cup_{k\in\mathbb{N}^*} \mathrm{N}_{\mathrm{n}_\mathrm{k}} = \{1,2,\dots,\mathrm{j}\}$, where $j = sup\{\mathrm{n}_\mathrm{k}, k \in \mathbb{N}^*\}$ if it exists, if not $\cup_{k\in\mathbb{N}^*} \mathrm{N}_{\mathrm{n}_\mathrm{k}} = \mathbb{N}^*$, then $\cup_{k\in\mathbb{N}^*} \mathrm{N}_{\mathrm{n}_\mathrm{k}} \in \mathcal{N}$. $iii)$ If, $N_{\mathrm{n}_1} = \{1,2,\dots,\mathrm{n}_1\}$, $N_{\mathrm{n}_2} = \{1,2,\dots,\mathrm{n}_2\}, \dots, \mathrm{N}_{\mathrm{n}_\mathrm{k}} = \{1,2,\dots,\mathrm{n}_\mathrm{k}\}$ is a finite collection of the elements of $\mathcal{N}$, then $\cap_{i=1}^{k}, \mathrm{N}_{\mathrm{n}_\mathrm{i}} = \{1,2,\dots,\mathrm{l}\}$, where $l = min\{\mathrm{n}_\mathrm{i}, 1 \le i \le k\}$, then $\cap_{i=1}^{k}, \mathrm{N}_{\mathrm{n}_\mathrm{i}} \in \mathcal{N}$.

As a direct consequence of the proposition 2.1 and De Morgan's Laws, if $\{S_\alpha;\ \alpha \in \Delta\}$ is any family of sets in $\mathbb{R}$, then $\cup_{\alpha\in\Delta} S_\alpha^C = (\cap_{\alpha\in\Delta} S_\alpha)^C$, we have:

Corollary 2.1
C_1-The intersection of any collection of closed sets is closed.
C_2-The union of any finite family of closed sets is closed.

Remark 2.2. The union of any family of closed sets is not hallways closed. In the space $\mathbb{R}$, the countable family of closed intervals $I_n = \left[0,1-\frac{1}{\mathrm{n}}\right], \mathrm{n} \in \mathbb{N}^*$ is such that $\cup_{\mathrm{n}\in\mathbb{N}^*} I_n = [0,1[$ which is not closed. In fact, if $x \in \cup_{\mathrm{n}\in\mathbb{N}^*} I_n$, there exists $\mathrm{n}_0 \in \mathbb{N}^*$ such that $0 \le x \le 1 - \frac{1}{\mathrm{n}_0} < 1$, then $x \in [0,1[$. Conversely, if $0 \le x < 1$, then $\frac{1}{1-\mathrm{x}} > 0$, from Archimed's axiom, there exists $\mathrm{n}_0 \in \mathbb{N}^*$ such that $\frac{1}{1-\mathrm{x}} < n_0$ or $x < 1 - \frac{1}{\mathrm{n}_0}$, then $x \in \cup_{\mathrm{n}\in\mathbb{N}^*} I_n$.

In the sequel, $\mathcal{N}(x)$ denote the collection of any neighborhoods of x. Before giving neighborhood properties, note that in any topological space, we have the following useful property.

Proposition 2.2. The non-empty set, is open iffy it is a neighborhood of each of its points.
Proof. Let U a non-empty part of E, then U is open if there exists $O \in \tau$, such that $U = O$. If $x \in U$, since $x \in U \subset U$, then $U \in \mathcal{N}(x)$. Reciprocally, if $U \in \mathcal{N}(x)$ $(x \epsilon N)$ there exists $O_x \in \tau$ such that $x \in O_x \subset U$, then $\cup_{\mathrm{x}\in\mathrm{U}} O_x \subset U$ and since $U = \cup_{\mathrm{x}\in\mathrm{U}} \{x\} \subset \cup_{\mathrm{x}\in\mathrm{U}} O_x$, therefore $U = \cup_{\mathrm{x}\in\mathrm{U}} O_x$, by O_1 in proposition 2.1, $U \in \tau$.

Theorem 2.1. In any space (E,τ), $\mathcal{N}(x)$ have the following properties.
N_1-Any point x of E has at least one neighborhood, and $\forall N \in \mathcal{N}(x), x \in N$.
N_2-If, $\{N_\alpha;\ \alpha \in \Delta\}$ is a family of elements in $\mathcal{N}(x)$, then $\cup_{\alpha\in\Delta} N_\alpha \in \mathcal{N}(x)$, $(\mathcal{N}(x)$ is stable by the union)
N_3-If, $\{N_\alpha;\ \alpha = 1,\dots,n\}$ is a finite elements of $\mathcal{N}(x)$, then $\cap_{\alpha=1}^{n} N_\alpha \in \mathcal{N}(x)$, $(\mathcal{N}(x)$ is stable by a finite intersection).
N_4-If, there exists a set M of E containing $N \in \mathcal{N}(x)$, then $M \in \mathcal{N}(x)$, $(\mathcal{N}(x)$ is hereditary on the right) or (absorption property).
N_5-If, $N \in \mathcal{N}(x)$, there exists $M \in \mathcal{N}(x)$, such that $N \in \mathcal{N}(y)$, $\forall y \in M$ $(M \subset N)$.

Proof. N_1-Since $x \in E$ witch is an open, by proposition 2.2 $E \in \mathcal{N}(x)$, if $N \in \mathcal{N}(x)$, there exists $O \in \tau$ such that $x \in O \subset N$. N_2-Let $x \in \cup_{\alpha\in\Delta} N_\alpha$, there exists $\alpha_0 \in \Delta$, such that $x \in N_{\alpha_0}$, hence there exists $O \in \tau$ such that $x \in O \subset N_{\alpha_0} \subset \cup_{\alpha\in\Delta} N_\alpha$, then $\cup_{\alpha\in\Delta} N_\alpha \in \mathcal{N}(x)$. N_3- If, for $\alpha = 1,\dots,n$, $N_\alpha \in \mathcal{N}(x)$; there exists $O_\alpha \in \tau, \mathrm{x} \in O_\alpha \subset N_\alpha$ then $x \in O = \cap_{\alpha=1}^{n} O_\alpha \subset \cap_{\alpha=1}^{n} N_\alpha$, hence $\cap_{\alpha=1}^{n} N_\alpha \in \mathcal{N}(x)$. N_4- Since $N \in \mathcal{N}(x)$, there exists $O \in \tau$ such that $x \in O \subset N \subset M$, then $M \in \mathcal{N}(x)$. N_5-If, $N \in \mathcal{N}(x)$, there exists $O \in \tau$ such that $x \in O \subset N$, by proposition 2.2 $O \in \mathcal{N}(x)$, using N_3, $N \in \mathcal{N}(y)$ $\forall y \in O$, it suffices to take $M = O$.

Example 2.4.

a) In the space $\mathbb{R}$, for every a, b in $\mathbb{R}$, the intervals $]a, b]$ and $[a, b[$ are neither open nor closed.

b) Let $E = \{0,1,2\}$ and $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}, E\}$, then τ is a topology in E, $\mathcal{N}(\{0,1\}) = \{\{0,1\}, E\}$, $\mathcal{N}(\{0,2\}) = \{E\} = \mathcal{N}(\{1,2\}), \mathcal{N}(0) = \{\{0\}, \{0,1\}, \{0,2\}, E\}, \mathcal{N}(1) = \{\{1\}, \{0,1\}, \{1,2\}, E\}$ and $\mathcal{N}(2) = \{E\}$.

c) In the indiscrete space E, all parts are neither open nor closed and $\forall x \in E$, $\mathcal{N}(x) = E$.

d) In the discrete space E, all parts are both open and closed, and $\forall x \in E$, $\mathcal{N}(x)$ is the collection of all parts of E containing x.

Starting to the open sets, we have defined the closed sets and the neighborhoods. Another way is to define the neighborhoods and from the neighborhoods, we define the open sets and closed sets. The two paths are equivalent as it will be demonstrated in the following theorem.

Theorem 2.2. Let $x \in E$ and $\sigma(x)$ the family of the parts of E verifying $N_1, \dots, N_5$ in the theorem 2.1. Then, there exists in E an unique topology, whose $\sigma(x)$ constitutes, for each x in E, the family of neighborhoods of x, for this topology.

Proof. The idea of the construction of this topology comes from the fact that an open set is a neighborhood of each of its points (see, proposition 2.2). According to the definition 2.2, we will show that, the family τ=$\{\emptyset\}\cup\{$all parts $O \subset E$, such that: if $x \in O$ then $O \in \sigma(x)\}$ define a topology on E. i) Since for $x \in E$, $\{x\} \subset E$ and $x \in \{x\}$, then $\{x\} \in \sigma(x)$, by N_4, $E \in \sigma(x)$ therefore $E \in \tau$. ii) Let $\{O_\alpha;\ \alpha \in \Delta\}$ be a family of elements in τ. Since $\cup_{\alpha\in\Delta} O_\alpha \subset E$ and for $x \in \cup_{\alpha\in\Delta} O_\alpha$, there exists $\alpha_0 \in \Delta$, such that $x \in O_{\alpha_0} \in \sigma(x)$, as $O_{\alpha_0} \subset \cup_{\alpha\in\Delta} O_\alpha$ by N_4, $\cup_{\alpha\in\Delta} O_\alpha \in \sigma(x)$, therefore $\cup_{\alpha\in\Delta} O_\alpha \in \tau$. iii) Let $\{O_\alpha;\ \alpha = 1, \dots, n\}$ be a finite elements of τ, then $\cap_{\alpha=1}^{n} O_\alpha \subset E$ and for $x \in \cap_{\alpha=1}^{n} O_\alpha$, $\forall \alpha \in \{1, \dots, n\}, x \in O_\alpha \in \sigma(x)$, by N_3, $\cap_{\alpha=1}^{n} O_\alpha \in \sigma(x)$, so $\cap_{\alpha=1}^{n} O_\alpha \in \tau$.We then demonstrated that τ is a topology in E, it remains to demonstrate that if $\mathcal{N}(x)$ is the family of neighborhoods of $x \in E$ according to τ, then $\mathcal{N}(x) = \sigma(x)$. Let $N \in \mathcal{N}(x)$, there exists $O \in \tau$, $x \in O \subset N$ as $O \in \sigma(x)$ then by N_4, $N \in \sigma(x)$, so $\mathcal{N}(x) \subset \sigma(x)$. Conversely, if $N \in \sigma(x)$, by N_5 there exists $M \in \sigma(x)$, $M \subset N$ such that, $N \in \sigma(y)$ for every $y \in M$, then $M \in \tau$, so $N \in \mathcal{N}(x)$ and $\sigma(x) \subset \mathcal{N}(x)$. The uniqueness of τ comes from proposition 2.2.

2.3-Basis and subbasis of topology, basis of neighborhoods

The use of bases and subbases of topology (a parts of the topology), which we will introduce below, instead of the initial topology, is often more convenient and gives the same results as the initial topology.

Definition 2.3. Let (E, τ) be a topological space.

a) A family $\mathcal{B} \subset \tau$, is called a basis (or a base) of τ, if $\forall x \in O \in \tau$, there exists $B \in \mathcal{B}$ containing x and contained in O. Equivalently $O = \cup_{B\in\mathcal{B}} B$.

b) A family $\mathcal{B}(x) \subset \mathcal{N}(x)$, is called a basis of neighborhoods of x, or a fundamental system of neighborhoods of x, if $\forall N \in \mathcal{N}(x)$, there exists $B \in \mathcal{B}(x)$, such that $B \subset N$ i.e. $N = \cup_{B\in\mathcal{B}(x)} B$.

Remark 2.3.

a) The definition 2.3 means that, a topology τ is completely determined, by a given part of its elements.

b) It is clear that if, $\mathcal{B} \subset \tau$ is a basis and $\mathcal{B}'$ is a family of open sets containing $\mathcal{B}$, then $\mathcal{B}'$ is also a basis of τ. In particular τ is a basis of itself. Therefore a space E can have many basis.

Example 2.5.

a) Let $\tau = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, E\}$ be a topology in $E = \{0,1,2\}$, then $\mathcal{B} = \{\{0\}, \{1\}, \{2\}\}$ is a basis of a topology τ.

b) In the discrete space E, $\forall x \in E, \mathcal{B}(x) = \{x\}$ is a basis of neighborhoods of x, since $\mathcal{B}(x) \subset \mathcal{N}(x)$ and $\forall N \in \mathcal{N}(x), x \in N$ then $\{x\} \subset N$.

c) Let x be an element of the space $\mathbb{R}$, the collection $\mathcal{I}_n(x) = \left\{I_n\left(x, \frac{1}{n}\right) = \left]x - \frac{1}{n}, x + \frac{1}{n}\right[, n \in \mathbb{N}*\right.$ is a basis of neighborhoods of x. In fact, if $x \in E$ and $N \in \mathcal{N}x$ there exists $\delta > 0$, $I(x,\delta) \subset N$, by Archimed's axiom, there is $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \delta$, so $I_{n_0}(x, \frac{1}{n_0}) \subset N$.

Important. Let x be an element of the space $\mathbb{R}$, the open intervals containing x, constitute a basis of neighborhoods of x. In particular, the collection $\{I(x,\delta), \delta > 0\}$ constitute a basis of neighborhoods of x. Then, $N \in \mathcal{N}(x) \Leftrightarrow \exists \delta > 0, I(x,\delta) \subset N$. Indeed, $N \in \mathcal{N}(x) \Leftrightarrow \exists O \in \tau_u, x \in O \subset N \Leftrightarrow \exists \delta > 0, I(x,\delta) \subset O \subset N$. Therefore in the space $\mathbb{R}$, it suffices for neighborhoods of x to consider the collection of open intervals $\{I(x,\delta), \delta > 0\}$.

Theorem 2.3. Let (E,τ) be a topological space and $\mathcal{B}$ a family of parts of E, then: $\mathcal{B}$ is a basis of τ

$$\Leftrightarrow \begin{cases} b_1) \text{ Any element } x \text{ of } E \text{ belongs at least to } B \in \mathcal{B}. \\ b_2) \text{ If } B_1, B_2 \in \mathcal{B} \text{ and } x \in B_1 \cap B_2, \text{there exists } B \in \mathcal{B}, \text{such that } x \in B \subset B_1 \cap B_2. \end{cases}$$

Proof. Suppose that $\mathcal{B}$ is a basis of τ. b_1) Since $E \in \tau$ by definition 2.3, $E = \cup_{B \in \mathcal{B}} B$, then if $x \in E$, there exists $B \in \mathcal{B}$ such that $x \in B$. b_2) Let $B_1, B_2 \in \mathcal{B}$. Since $B_1 \cap B_2 \in \tau$ there exists a collection $\{B_\alpha;\ \alpha \in \Delta\}$ of the elements of $\mathcal{B}$ such that $B_1 \cap B_2 = \cup_{\alpha \in \Delta} B_\alpha$. So if $x \in B_1 \cap B_2$, there exists $\alpha_0 \in \Delta$ such that $x \in B_{\alpha_0} \subset B_1 \cap B_2$. Reciprocally, suppose that $\mathcal{B}$ is a family of parts of E satisfying b_1) and b_2) and demonstrate that $\mathcal{B}$ define a topology on E, therefore $\mathcal{B}$ is a basis. Let σ be a family of all parts $O \subset E$ defined by: $O \in$ σ if, there exists a collection $\{B_\alpha;\ \alpha \in \Delta\}$ of the elements of $\mathcal{B}$ such that $O = \cup_{\alpha \in \Delta} B_\alpha$. i) by b_1) $E = \cup_{B \in \mathcal{B}} B$ then $E \in \sigma$ and $\cup_{\alpha \in \Delta} \emptyset_\alpha = \emptyset$, then $\emptyset \in \sigma$. ii) Let $\{O_\alpha;\ \alpha \in \Delta\}$ be a collection of the elements of σ, then by b_1) for every $\alpha \in \Delta$, there exists a collection $\{B_{i_\alpha};\ i_\alpha \in I\}$, of the elements of $\mathcal{B}$ such that $O_\alpha = \cup_{i_\alpha \in I} B_{i_\alpha}$, so $\cup_{\alpha \in \Delta} O_\alpha = \cup_{\alpha \in \Delta}\left(\cup_{i_\alpha \in I} B_{i_\alpha}\right) = \cup_{(\alpha, i_\alpha) \in \Delta \times I} B_{i_\alpha} \in \sigma$.
iii) Let $\{O_\alpha;\ \alpha = 1, \dots, n\}$ be a finite family of σ, then by b_1) for every $\alpha \in \Delta$, there exists a collection $\{B_{i_\alpha};\ i_\alpha \in I\}$, of the elements of $\mathcal{B}$ such that $O_\alpha = \cup_{i_\alpha \in I} B_{i_\alpha}$, so $\cap_{\alpha=1}^n O_\alpha = \cap_{\alpha=1}^n \left(\cup_{i_\alpha \in I} B_{i_\alpha}\right) = \cup_{i_\alpha \in I}\left(\cap_{\alpha=1}^n B_{i_\alpha}\right)$, as from b_2) $\cap_{\alpha=1}^n B_{i_\alpha} \in \mathcal{B}$, then $\cap_{\alpha=1}^n O_\alpha \in \sigma$.

Now let τ be a topology on E and $\mathcal{B}$ a family of open sets in E satisfying b_1) and b_2) in theorem 2.3. $\mathcal{B}$ is a basis of a topology $\sigma \subset \tau$. So that $\mathcal{B}$ generates exactly τ, it must satisfy the conditions of the following.

Corollary 2.2. A family of open sets $\sigma \subset \tau$ is a basis of τ iffy for all open set $O \in \tau$ and for all x in O, there exists a set $U_x \in \sigma$ such that $x \in U_x \subset O$.

Proof. If σ is a basis of τ and $x \in O \in \tau$, there exists a collection $\{U_x;\ x \in O\}$ of the elements of σ, such that $O = \cup_{x \in O} U_x$, hence if $x \in O$, there exists $U_x \in \sigma$ such that $x \in U_x \subset O$. Reciprocally, if for all $x \in O \in \tau$, there exists the collection $\{U_x;\ x \in O\}$ of the elements of σ such that $O = \cup_{x \in O} U_x$, by definition 2.3 a) σ is a basis of τ.

Remark 2.4. Corollary 2.1, allows us to demonstrate that a family of open set of a given topology is a basis of this topology. For example in the space $\mathbb{R}$ the open intervals with rational extremities is the basis of τ_u. In fact, if $I(x,\delta)$ is an open interval centered in $x \in \mathbb{R}$, with radius $\delta > 0$, as $\mathbb{Q}$ is dance in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ between 0 and δ, therefore $I(x,r) \subset I(x,\delta)$.

Corollary 2.3. If $\mathcal{B}$ is a basis of the topology τ on E, and if σ is a collection of the elements of τ, such that any element of $\mathcal{B}$ is written as a union of elements of σ, then σ is also a basis of τ

Proof. Let $O \in \tau$, since $\mathcal{B}$ is a basis of the topology τ, by definition 2.3 a) there exists a collection $\{B_\alpha;\ \alpha \in \Delta\}$, of elements of $\mathcal{B}$ such that $O = \cup_{\alpha \in \Delta} B_\alpha$. By hypothesis $\forall \alpha \in \Delta$, there exists a collection $\left\{S_{\beta_\alpha}, \beta_\alpha \in \nabla\right\}$ of elements of σ such that $B_\alpha = \cup_{\beta_\alpha \in \nabla} S_{\beta_\alpha}$, so $O = \cup_{\alpha\in\Delta}\left(\cup_{\beta_\alpha\in\nabla} S_{\beta_\alpha}\right) = \cup_{(\alpha,\beta_\alpha)\in\Delta\times\nabla} S_{\beta_\alpha}$, i.e. σ is also a basis of τ.

Corollary 2.4. Let $\mathcal{B}$ be a family of elements of the topology τ on E. Then $\mathcal{B}$ is a basis of τ iffy, $\forall x \in E$, the family $\mathcal{B}(x) = \{B \in \mathcal{B}, x \in B\}$ is a fundamental system of neighborhoods of x.

Proof. Let $x \in E$ and $N \in \mathcal{N}(x)$, there exists $O \in \tau$, such that $x \in O \subset N$, since $\mathcal{B}$ is a basis of τ, there exists $B \in \mathcal{B}$, such that $x \in B \subset O$ i.e. $B \in \mathcal{B}(x)$ and $B \subset N$, by definition 2.3 $b)$ $\mathcal{B}(x)$ is a fundamental system of neighborhoods of x. Conversely, let $O \in \tau$ and $x \in O$ by proposition 2.1, $O \in \mathcal{N}(x)$ as $\mathcal{B}(x)$ is a fundamental system of neighborhoods of any $x \in E$, there exists $B \in \mathcal{B}(x)$ hence $B \in \mathcal{B}$, $x \in B$ and $B \subset O$ i.e. $\mathcal{B}$ is a basis of τ.

Definition 2.4. Let E be a topological space. The subset $S \subset \tau$ is said to be a subbase for the topology τ, if the collection of all finite intersects of sets in S forms a base of τ, i.e. the set $S\mathcal{B} = \{\cap_{\alpha=1}^{n} O_\alpha,\ O_\alpha \in S\}$, is a basis of τ.

Example 2.6.

$a)$ Let $E = \{a, b, c, d, e, f\}$ be, with the topology $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}, E\}$. Then the subset $S = \{\{a\}, \{a, c, d\}, \{b, c, d, e, f\}\} \subset \tau$ is a subbase of τ. Since, the collection of all finite intersections of elements from S is: $S\mathcal{B} = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Every set in τ is a trivial union of elements in $S\mathcal{B}$ and $E = \{a\} \cup \{b, c, d, e, f\}$, so $S\mathcal{B}$ is a base of τ and S is a subbase of τ.

$b)$ Let $E = \{a, b, c, d, e\}$ be, with the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}, \{a, b, c, d\}, E\}$. Then, the set $S = \{\{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, E\} \subset \tau$ is not a subbase of τ. The set of all finite intersects of sets from S is: $S\mathcal{B} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, E\}$. All sets except $\{b, d\}$ can be expressed as trivial intersections. However, $\{b, d\}$ cannot be expressed as a union of elements from $S\mathcal{B}$, so $S\mathcal{B}$ is not a base of τ and hence S is not a subbase of τ.

3-Topological parts, Weierstrass-Bolzano Theorem

Let (E, τ) be a topology space, A and B a non-empty subsets of E.

3.1-Closure, accumulation point, Weierstrass-Bolzano theorem

In addition to the open sets and the neighborhoods, which are introduced in section 2, in this section ,we will introduce other topological sets as well as their properties, among these sets, the closure, the interior, the set of accumulation points,...etc.

The adherence. We say that $x \in E$ is an adherent point of A, if every neighborhood of x contains at least one point of A, i.e. $\forall N \in \mathcal{N}(x), N \cap A \neq \emptyset$. The set of the adherent points of A, is called the adherence of A and it is noted $cl(A)$ or $\bar{A}$.

The closure. The intersection of all closed sets of E containing A, is called the closure of A and it is noted $cl(A)$. By C_1 in the corollary 2.1 $cl(A)$ is closed and it is the smallest one containing A.

Proposition 3.1. The closure of A is equal to its adherence i.e. $cl(A) = A$.

Proof. $cl(A) = A$ iffy $cl(A)^C = A^C$. If, $x \in cl(A)^C$, then $x \notin cl(A)$, there exists a closed set S containing A such that $x \notin S$, then $x \in S^C$ witch is an open set. By proposition 2.2, S^C is a neighborhood of x, since $S^C \cap A = \emptyset$ then $x \notin A$ i.e. $x \in A^C$. Inversely, if $x \in A^C$, then $x \notin A$, there exists $N \in \mathcal{N}(x),\ N \cap A = \emptyset$. Therefore ,there exists $O \in \tau, x \in O \subset N$ and $O \cap A = \emptyset$, so $A \subset O^C$, with is a closed set and $x \notin O^C$, then $x \notin$ cl(A) i.e. $x \in cl(A)^C$.

Proposition 3.2.

1) A is closed$\Leftrightarrow cl(A) = A$ then, $cl(cl(A)) = cl(A)$.
2) If, $A \subset B$, then $cl(A) \subset cl(B)$.

3) $cl(A \cup B) = cl(A) \cup cl(B)$.
4) $cl(A \cap B) \subset cl(A) \cap cl(B)$, the converse is not throw.

Proof. 1) Evident. 2) Since $B \subset cl(B)$ and $A \subset B$, then $A \subset cl(B)$, as cl(A) is the smallest closed containing A, then $cl(A) \subset cl(B)$. 3) Since, $A \subset A \cup B$ and $B \subset A \cup B$, by 2) $cl(A) \subset cl(A \cup B)$ and $cl(B) \subset cl(A \cup B)$, then $cl(A) \cup cl(B) \subset cl(A \cup B)$. Conversely, $A \subset cl(A)$ and $B \subset cl(B)$, then $A \cup B \subset cl(A) \cup cl(B)$, as $cl(A) \cup cl(B)$ is closed, 1) and 2) imply that $cl(A \cup B) \subset cl(A) \cup cl(B)$. 4) Since $A \cap B \subset A$ and $A \cap B \subset B$, then $cl(A \cap B) \subset cl(A)$ and $cl(A \cap B) \subset cl(B)$, so $cl(A \cap B) \subset cl(A) \cap cl(B)$. For the converse, it suffices to take in the space $\mathbb{R}$, $A = [0,1[\cup \{3\}, B = [1,2[$ then $cl(A) = [0,1] \cup \{3\}, cl(B) = [1,2], cl(A \cap B) = \emptyset,$ and $cl(A) \cap cl(B) = \{1\},$ therefore $cl(A) \cap cl(B) \not\subset cl(A \cap B)$.

Example 3.1. In the space $\mathbb{R}$.
$a)$ As $\forall a, b \in \mathbb{R}, [a, b]$ is closed then $cl([a, b]) = [a, b], \forall a, b \in \mathbb{R}$. Also, as $]-\infty, a]$ and $[b, +\infty[$ are closed $\forall a, b \in \mathbb{R},$ then $cl(]-\infty, a]) =]-\infty, a]$ and $cl([b, +\infty[) = [b, +\infty[$.
$b)$ $cl(]a, b[) = cl(]a, b]) = cl([a, b[) = [a, b], \forall a, b \in \mathbb{R}$. It suffices to prof that $cl(]a, b[) = [a, b], \forall a, b \in \mathbb{R},$ let $x \in cl(]a, b[),$ then $\forall \varepsilon > 0, I(x, \varepsilon) \cap]a, b[\neq \emptyset$ as $\forall \varepsilon > 0, I(x, \varepsilon) \cap]a, b[\subset I(x, \varepsilon) \cap [a, b],$ then $\forall \varepsilon > 0, I(x, \varepsilon) \cap [a, b] \neq \emptyset,$ so $x \in cl([a, b]) = [a, b]$.
$c)$ $cl(\mathbb{Q}) = \mathbb{R}$ and $cl(\mathbb{Q}^C) = \mathbb{R}$. Since $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ is closed then $cl(\mathbb{Q}) \subset cl(\mathbb{R}) = \mathbb{R}$. Conversely if $x \in \mathbb{R}$, then $\forall \varepsilon > 0$, there exists $r \in \mathbb{Q}$, such that $r \in I(x, \varepsilon)$ (we have used the density of $\mathbb{Q}$ in $\mathbb{R}$), so $I(x, \varepsilon) \cap \mathbb{Q} \neq \emptyset$ and $x \in cl(\mathbb{Q})$.
$d)$ As in $c)$ we use the density of $\mathbb{Q}^C$ in $\mathbb{R}$, to prof that $cl(\mathbb{Q}^C\}) = \mathbb{R}$.

Example 3.2.
$a)$ If, A is the part of the indiscrete space E, then $cl(A) = E$ if $A \neq \emptyset,$ or $cl(A) = \emptyset,$ if $A = \emptyset$.
$b)$ If, A is the part of the discrete space E, then $cl(A) = A$.
$c)$ Let $E = \{a, b, c, d\}$ provided with the topology, $\tau = \{\emptyset, \{b, c\}, \{b, c, d\}, E\}$. Then $cl(\{a\}) = \{a\}, cl(\{b\}) = E, cl(\{c\}) = E$ and $cl(\{d\}) = \{a, d\}$.

Accumulation point. We say that, $x \in E$ is an accumulation point of A, if every neighborhood of x contains at least one point of A other than x.The set of accumulation points of A is denoted by A'. Then: $x \in A' \iff \forall N \in \mathcal{N}(x), (N \backslash \{x\}) \cap A \neq \emptyset$.

Isolated point. We say that, $x \in A$ is an isolated point of A, if x is not an accumulation point of A.. The set of isolated points of A is denoted by A''. So, $x \in A'' \Leftrightarrow$there is $N \in \mathcal{N}(x)$, such that $N \cap A = \{x\}$.

Example 3.3. In the space $\mathbb{R}$, $\forall a, b \in \mathbb{R}$.
$a)$ All point of $A =]a, b[$ is an accumulation point and $A' = cl(A) = [a, b]$. Indeed, if $x \in]a, b[$, for every $\varepsilon > 0,$ if $x - \varepsilon \leq a < x,]x - \varepsilon, x[\cap A =]a, x[$ then $(I(x, \varepsilon) \backslash \{x\}) \cap A \neq \emptyset$, so x is an accumulation point of $]a, b[$; if $a < x - \varepsilon < x,]x - \varepsilon, x[\cap A =]x - \varepsilon, x[$ then $(I(x, \varepsilon) \backslash \{x\}) \cap A \neq \emptyset$, likewise in the cases, $x < x + \varepsilon \leq b$; $x < b < x + \varepsilon$ and $x - \varepsilon \leq a < b \leq x + \varepsilon$. In all cases x is an accumulation point of $]a, b[$. Also, a is an accumulation point of A, since for every $\varepsilon > 0$ if, $a + \varepsilon \leq b,]a, a + \varepsilon[\cap A =]a, a + \varepsilon[$ then $(I(a, \varepsilon) \backslash \{a\}) \cap A \neq \emptyset$ and if $b < a + \varepsilon$, then $(I(a, \varepsilon) \setminus \{a\}) =]a, b[\neq \emptyset$. Likewise, b is an accumulation point of $]a, b[$, so $A = cl(A) = [a, b]$.
$b)$ All point of $A =]-\infty, a[$ is an accumulation point and $A' =]-\infty, a] = cl(A)$. Indeed, if $x \in]-\infty, a[$, for every $\varepsilon > 0$, if, $x + \varepsilon \leq a, (I(x, \varepsilon) \setminus \{x\}) \cap A = [x - \varepsilon, x[\cup]x, x + \varepsilon]$, if $a < x + \varepsilon, (I(x, \varepsilon) \backslash \{x\}) \cap A = [x - \varepsilon, x[\cup]x, a[$, then x is an accumulation point of $]-\infty, a[$. we prove as in $a)$ that a is also an accumulation point of A. So $A' =]-\infty, a] = cl(A)$.
$c)$ Since, $\left(I\left(0, \frac{1}{2}\right) \setminus \{0\}\right) \cap A = \emptyset$ and $\left(I\left(1, \frac{1}{2}\right) \setminus \{1\}\right) \cap A = \emptyset$. Then, 0 and 1 are isolated points of $A = \{0,1\}$.

$d)$ All point of $A=\left\{\frac{1}{n}, n\in\mathbb{N}^*\right\}$ is an isolated point. Let us demonstrate for example that 1 is an isolated point of A. It suffices to take $\delta=\frac{1}{6}$, then $\left(I\left(1,\frac{1}{6}\right)\setminus\{1\}\right)\cap A=\emptyset$. As, if $x\in\left(I\left(1,\frac{1}{6}\right)\setminus\{1\}\right)\cap A=\emptyset$, there exists $n_0\in\mathbb{N}^*\setminus\{1\}$, such that $x=\frac{1}{n_0}$ and $1-\frac{1}{6}<\frac{1}{n_0}<1+\frac{1}{6}$, i.e. $5n_0<6<7n_0$ impossible, since $n_0\geq 2$. Also, $\frac{1}{2}$ is an isolated point of A, since $\left(I\left(\frac{1}{2},\frac{1}{8}\right)\setminus\right.$ $\left.12\right)\cap A=\emptyset$, if not there exists $n0\in\mathbb{N}*\setminus\{2\}$ such that $3n0<8<5n0$, impossible since $n0=1$ or $n_0\geq 3$. Note that 0 is an accumulation point of A, indeed $\forall\varepsilon>0$, by Archimedes axiom , there exists $n\in\mathbb{N}^*$, $\frac{1}{n}<\varepsilon$, so $(I(0,\varepsilon)\setminus\{0\})\cap A\neq\emptyset$.

$e)$ All points of the discrete space $\mathbb{R}$, are isolated points, since $\forall x\in\mathbb{R}$, the set $N=\{x\}\in\mathcal{N}(x)$ satisfies $(N\setminus\{x\})\cap N=\emptyset$.

Corollary 3.1.

$a)$ $cl(A)=A\cup A'$.

$b)$ A is closed$\Leftrightarrow A$ contains all its accumulation points.

Proof. $a)$ Let $x\in cl(A)$, since $A\subset cl(A)$, either x is in A or it isn't in A. If $x\in A$, then $x\in A\cup A'$. If, $x\notin A$ then $x\in A'$ since $\forall N\in\mathcal{N}(x)$, $(N\setminus\{x\})\cap A\neq\emptyset$, so $cl(A)\subset A\cup A'$. Inversely, if $x\in A\cup A'$ then $x\in A\subset cl(A)$ or $x\in A'$ then $\forall N\in\mathcal{N}(x)$, $(N\setminus\{x\})\cap A\neq\emptyset$, so $N\cap A\neq\emptyset$ and $x\in cl(A)$. $b)$ If A is closed, $A=cl(A)=A\cup A'$, then $A'\subset A$ i.e. A contains all its accumulation points. Now, if $A'\subset A$ as $cl(A)=A\cup A'$, then $cl(A)\subset A$, hence $cl(A)=A$ with implies that A is closed.

Starting from that, $\bigcap_{n\in\mathbb{N}^*} I_n=\emptyset$, where $I_n=\left]0,\frac{1}{n}\right[$, $n\in\mathbb{N}^*$, if not there exists $x\in\left]0,\frac{1}{n}\right[$, $\forall n\in\mathbb{N}^*$, then $n<\frac{1}{x}$, $\forall n\in\mathbb{N}^*$, i.e. $\mathbb{N}$ is bounded above, contradiction.

Question. Does there exists sequences of intervals of $\mathbb{R}$ whose intersection is not empty? The answer is given by the following Cantor principal.

Lemma 3.1. (Nested interval theorem or Cantor principal). The intersection of decreasing sequence of nonempty intervals $I_n=[a_n,b_n]$, $\forall n\in\mathbb{N}^*$, is not empty. And if, $inf_{n\in\mathbb{N}^*}\{b_n-an=0,$ then, the intersection is reduced to a single point.

Proof. Let $A=\{a_1,a_2,\ldots\}$ and $B=\{b_1,b_2,\ldots\}$, since $I_{n+1}\subset I_n$, $\forall n\in\mathbb{N}^*$, then A is bounded above, by the elements of B, and B is bounded below by the elements of A, so $\forall n\in\mathbb{N}^*$ $a_n\leq \sup A\leq b_n$, $infB\leq b_n$. Since $infB$, is the first lower bound of B then, $\forall n\in\mathbb{N}^*$ $a_n\leq\sup A\leq infB\leq b_n$ i.e. $\cap_{n\in\mathbb{N}^*} I_n=[supA,infB]\neq\emptyset$. Let for any $n\in\mathbb{N}^*$, $l_n=b_n-a_n$ and $L=\{l_n,n\in\mathbb{N}^*\}$, it is obvious that $\forall n\in\mathbb{N}^*$, $0\leq l_n$, by a_{16} chapter I $infL$ exists, if $infL=0$ then $supA=infB$, if not for $\varepsilon=\frac{infB-supA}{2}>0$, there exists $n_0\in\mathbb{N}^*$ such that $l_{n_0}<\frac{infB-supA}{2}\leq\frac{b_{n_0}-a_{n_0}}{2}=\frac{l_{n_0}}{2}$, contradiction. Therefore, the intersection is reduced to a single point.

Theorem 3.1. (Weierstrass-Bolzano theorem). Any infinite bounded part E of the space $\mathbb{R}$ has at last one accumulation point.

Proof. Let $I_1=[infE,supE]$, it is clear that $E\subset I_1$, then one of the tow intervals $\left[infE,\frac{infE+supE}{2}\right]$, $\left[\frac{infE+supE}{2},supE\right]$ contains an infinity points of E, if not E is finite. Let I_2 the infinite one, then $I_2\subset I_1$, $E\cap I_2$ is an infinite part of $\mathbb{R}$, by the same, we devise I_2 on two intervals where one of them say $I_3\subset I_2$ and $E\cap I_3$ is an infinite part of $\mathbb{R}$, by the same idea, we construct a sequence of intervals such that $I_{n+1}\subset I_n=[a_n,b_n]=\frac{\text{supE}-\text{infE}}{2^{n-1}}$, and $E\cap I_n$, $\forall n\in\mathbb{N}^*$ is infinite. By Cantor principle, $\bigcap_{n\in\mathbb{N}^*} I_n\neq\emptyset$, since $0=inf\{l_n=b_n-a_n,n\in\mathbb{N}^*\}$, then $supE=infE=a$, and $\bigcap_{n\in\mathbb{N}^*} I_n=\{a\}$. then, a is an accumulation point of E, since for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^*$ such that $I_{n_0} < \varepsilon$, then $a \in I_{n_0} \subset I(a,\varepsilon)$ so $(I(a,\varepsilon) \setminus \{a\}) \cap E \neq \emptyset$.

3.2-Interior, boundary, exterior

The interior. A point $x \in A$ is an interior point of A, if $A \in \mathcal{N}(x)$. The set of all interior points of A, is denoted by $int(A)$ or A^0, and is called, the interior of A.
Then $int(A) = \cup_{O\subset A} O$, i.e. $int(A)$, is the greatest open set contained in A. In deed, if $x \in int(A)$, there exists an open $O_x, x \in O_x \subset A$, then $int(A) \subset \cup_{x\in A} O_x$ and obviously, if $x \in \cup_{x\in A} O_x$, there exists an open O_x, $x \in O_x \subset A$, so $x \in int(A)$.
Example 3.4. in the space $\mathbb{R}$.
$a)$ $\forall a, b \in \mathbb{R}, int([a,b]) = int(]a,b[) = int(]a,b]) =]a,b[$ and $int(]-\infty,a]) =]-\infty,a[$.
$b)$ $\forall a \in \mathbb{R}, int(\{a\}) = \emptyset$.
$c)$ $int(\mathbb{N}) = int(\mathbb{Z}) = int(\mathbb{Q}) = int(\mathbb{Q}^C) = \emptyset$. Prof that for example $int(\mathbb{N}) = \emptyset$, suppose that, there exists $\delta > 0$ such that $]n-\delta, n+\delta[\subset \mathbb{N}$, so $\mathbb{N}$ contains an element of $\mathbb{Q}^C$, contradiction.
Example 3.5.
$a)$ If A is the part of the indiscrete space E, $int(A) = \begin{cases} A \text{ if } A = E; \\ \emptyset \text{ if } A = \emptyset. \end{cases}$
$b)$ If, A is the part of the discrete space E, $int(A) = A$.
$c)$ If, $E = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, E\}$ then $int(\{d\}) = int(\{c\}) = \emptyset$ and $int(\{a,c,d\}) = \{a\}$.
Corollary 3.2. $x \in int(A) \iff A \in \mathcal{N}(x)$.
Proof. If $x \in int(A)$, witch is open, by proposition 2.2 $int(A) \in \mathcal{N}(x)$, as $int(A) \subset A$ by N_4 in theorem 2.1, $A \in \mathcal{N}(x)$. Reciprocally, if $A \in \mathcal{N}(x)$, there exists $O \in \tau, x \in O \subset A$ then $x \in int(A)$.
Proposition 3.3.
1) A is open$\Leftrightarrow$ $int(A) = A$ then, $int(int(A)) = int(A)$.
2) If, $A \subset B$, then $int(A) \subset int(B)$.
3) $int(A \cap B) = int(A) \cap int(B)$.
4) $int(A) \cup int(B) \subset int(A \cup B)$, the converse is not throw.
Proof. 1) Evident. 2) If $A \subset B$, as $int(A) \subset A$, then $int(A) \subset B$, since $int(A)$ is open and $int(B)$ is the greatest open contained in B, then $int(A) \subset int(B)$. 3) It is clear that by 2) $int(A \cap B) \subset int(A)$ and $int(A \cap B) \subset int(B)$, so $int(A \cap B) \subset int(A) \cap int(B)$. Conversely, $int(A) \subset A$ and $int(B) \subset B$, then $int(A) \cap int(B) \subset A \cap B$ so, by definition of $int(A \cap B)$ and $int(A) \cap int(B)$ is open, we have $int(A) \cap int(B) \subset int(A \cap B)$. 4) It is clear that by 2) $int(A) \subset int(A \cup B)$, $int(B) \subset int(A \cup B)$, then $int(A) \cup int(B) \subset int(A \cup B)$. To prof that the inverse is not true. Let in the space $\mathbb{R}$, $A = [-1,0[$ and $B = [0,1[$ then $(A) =]-1,0[, int(B) =]0,1[,$ $int(A \cup B) =]-1,1[\not\subseteq]-1,0[\cup]0,1[= int(A) \cup int(B)$.

The duality properties in the sense of "complement" between the closure and the interior are given by the following corollary:
Corollary 3.3. $cl(A^C) = int(A)^C$ and $int(A^C) = cl(A)^C$.
Proof. If, $cl(A^C) = int(A)^C$ then $(int(A^C)^C = cl(A)$, so $int(A^C\}) = cl(A)^C$. It remains to prof that $cl(A^C) = int(A)^C$. Since, $int(A) = \cup_{\alpha\in\Delta} O_\alpha$, where $\forall\alpha\in\Delta$, O_α is open and $O_\alpha \subset A$, then $\forall\alpha \in\Delta, A^C \subset {O_\alpha}^C$ and $int(A)^C = \cap_{\alpha\in\Delta} {O_\alpha}^C = cl(A^C)$ (by definition of the closure of A^C).
The boundary. A point $x \in E$ is an boundary point of A, if $x \in cl(A) \cap cl(A^C)$. The set of all boundary points of A, is denoted by $bd(A)$ or $fr(A)$, and is called the boundary of A. Then $bd(A) = cl(A) \cap cl(A^C) = cl(A) \setminus int(A)$ is closed.
Corollary 3.4.

$a)$ A is closed$\Leftrightarrow$ $bd(A) \subset A$.
$b)$ A is open$\Leftrightarrow$ $bd(A) \cap A = \emptyset$.
$c)$ $bd(cl(A)) \subset bd(A)$ and $bd(int(A)) \subset bd(A)$.
Proof. $a)$ A is closed iffy, $cl(A) = A$, then $bd(A) = A \cap cl(A^C) \subset A$. Conversely, if $bd(A) \subset A$ and $cl(A) \not\subset A$, there exists $x \in cl(A)$ and $x \notin A$, i.e. $x \in A^C \subset cl(A^C)$ then $x \in bd(A) \subset A$, contradiction. $b)$ A is open iffy $int(A) = A$, since $cl(A^C) = int(A)^C = A^C$, then $A \cap bd(A) = [cl(A^C)^C \cap cl(A^C)] \cap cl(A) = \emptyset$. Conversely, if $bd(A) \cap A = \emptyset$ and $A \not\subset int(A)$, there exists $x \in A \subset cl(A)$ and $x \notin int(A)$, i.e. $x \in int(A)^C = cl(A^C)$, so $x \in bd(A) \cap A$, contradiction. $c)$ $bd(cl(A)) = cl(cl(A)) \cap cl(cl(A)^C) = cl(A) \cap cl(cl(A)^C) \subset cl(A) \cap cl(A^C) = bd(A)$ and $bd(int(A)) = cl(int(A)) \cap cl(int(A)^C) = cl(int(A)) \cap cl(cl(A^C)) = cl(int(A)) \cap cl(A^C) \subset cl(A) \cap cl(A^C) = bd(A)$.
It is obvious that.
Corollary 3.5. If A is closed then: $A = bd(A) \Leftrightarrow int(A) = \emptyset \Leftrightarrow cl(A^C) = E$.
The exterior. A point $x \in E$ is an exterior point of A, if $x \in int(A^C)$. The set of all exterior points of A, is denoted by $ext(A)$, and is called the exterior of A. Then $ext(A) = int(A^C)$ is open.
Corollary 3.6.
$a)$ $ext(A) = \varphi \Leftrightarrow cl(A) = E$.
$b)$ $ext(A) = ext(ext(A)^C) = ext(cl(A))$.
$c)$ $ext(A \cup B) = ext(A) \cap ext(B)$ and $ext(A) \cup ext(B) \subset ext(A \cap B)$.
Proof. $a)$ $ext(A) = \emptyset$ iffy $cl(A)^C = \emptyset$ iffy $cl(A) = E$. $b)$ $ext(A) = cl(A)^C$ iffy $ext(A)^C = cl(A)$ iffy $ext(ext(A)^C) = ext\big(cl(A)\big) = int(cl(A)^C) = \Big(cl\big(cl(A)\big)\Big)^C = cl(A)^C = int(A^C) = ext(A)$. $c)$ $ext(A \cup B) = int(A \cup B)^C = int(A^C \cap B^C) = int(A^C) \cap int(B^C\}) = ext(A) \cap ext(B)$ and $ext(A) \cup ext(B) = int(A^C) \cup int(B^C) \subset int(A^C \cup B^C) = int((A \cap B)^C) = ext(A \cap B)$.
Example 3.6. in the space $\mathbb{R}$.
$a)$ $\forall a, b \in \mathbb{R}, ext([a,b]) = ext(]a,b[) = ext(]a,b]) =]-\infty, a[\cup]b, +\infty[$ and $ext(]-\infty, a]) =]a, +\infty[$.
$b)$ $\forall a, b \in \mathbb{R}, bd([a,b]) = bd(]a,b[) = bd(]a,b]) = \{a,b\}$ and $bd(]-\infty, a]) = \{a\}$.
$c)$ $\forall a \in \mathbb{R}, ext(\{a\}) =]-\infty, a[\cup]a, +\infty[, bd(\{a\}) = \{a\}$.
$d)$ $ext(\mathbb{N}) =]-\infty, 0[\cup (\cup_{n\in\mathbb{N}}]n, n+1[), bd(\mathbb{N}) = \mathbb{N};$ $ext(\mathbb{Z}) = \cup_{n\in\mathbb{Z}}]n, n+1[, bd(\mathbb{Z}) = \mathbb{Z};$ $ext(\mathbb{Q}) = ext(\mathbb{Q}^C) = \emptyset, bd(\mathbb{Q}) = bd(\mathbb{Q}^C) = \mathbb{R}$.
Example 3.7.
$a)$ If A is the part of the indiscrete space E, $ext(A) = \begin{cases} A \text{ if } A = \emptyset; \\ \emptyset \text{ if } A = E. \end{cases}$
$b)$ In the discrete space E, $ext(\emptyset) = E$ and $ext(E) = \emptyset$.
$c)$ If, $E = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, E\}$ then $ext(\{d\}) = \{a,b\}$ and $ext(\{a,c,d\}) = \emptyset$.

4-Metric Space

In this chapter, we are interested in the definition, of one special kind of a topological space, called metric space, i.e. a space witch a metric is defined. This space is very useful, in the study of the Cauchy sequences, uniform continuity, and enjoying several properties, which are not valid in a general topological space. We present here, and in the chapter 13 and 14, the particularities of this space such as: open, neighborhood, limit, continuity,...,etc.

4.1 Definitions and examples

Definition 4.1. The application d from the set $E \times E$ into $\mathbb{R}_+$ is said to be a metric over E, if it satisfies for every $x, y, z \in E$, the following three axioms:
m_1) $d(x,y) = 0 \iff x = y$, (separation axiom).
m_2) $d(x,y) = d(y,x)$, (symmetry axiom).
m_3) $d(x,y) \leq d(x,z) + d(z,y)$, (triangular inequality axiom).
The couple (E,d) or simply E if no confusion (as in general, there are more than one metric defined on E)

Example 4.1.

a) The function $d_u: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by: $\forall x, y \in \mathbb{R}, d_u(x,y) = |x-y|$, where $|.|$ is the absolute value function on $\mathbb{R}$ is a metric on $\mathbb{R}$, called the usual metric and $(\mathbb{R}, d_u)$ is called usual metric space.

b) Let $\mathbb{R}^n$, $n \in \mathbb{N}^*$ be the n dimensional Euclidian space, and d_1, d_2, d_∞ the functions from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}_+$ defined respectively by :$\forall x, y \in \mathbb{R}^n$, $d_1(x,y) = \sum_{\alpha=1}^{n} |x_\alpha - y_\alpha|$; $d_2(x,y) = \sqrt{\sum_{\alpha=1}^{n} (x_\alpha - y_\alpha)^2}$; and $d_\infty(x,y) = max_{1\leq\alpha\leq n}|x_\alpha - y_\alpha|$, where $x = (x^1, \dots, x_\alpha, \dots, x_n), y = (y^1, \dots, y_\alpha, \dots, y_n)$, and $\forall 1 \leq \alpha \leq n, x_\alpha, y_\alpha \in \mathbb{R}$. Then $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ are metric spaces. To prove the triangular inequality axiom for d_2, we us the following Cauchy-Bouniakowsky inequality: $\forall x, y \in \mathbb{R}^n, \sqrt{\sum_{\alpha=1}^{n} x_\alpha y_\alpha} \leq \left(\sqrt{\sum_{\alpha=1}^{n} {x_\alpha}^2}\right)\left(\sqrt{\sum_{\alpha=1}^{n} {y_\alpha}^2}\right)$.

c) Let E be an ensemble, the function d from $E \times E$ to $\mathbb{R}_+$ defined by: $\forall x, y \in E, d(x,y) = \begin{cases} 1, \text{if } x \neq y \\ 0, \text{if } x = y, \end{cases}$ is a metric on E called a **discrete metric**, and the metric space (E,d) is called a **discrete metric space**.

d) Let $a, b \in \mathbb{R}$ be, and $E = C([a,b])$ the set of the continuous functions from $[a,b]$ to $\mathbb{R}$. The applications d_1, d_2, d_∞ from $E \times E$ to $\mathbb{R}_+$ defined respectively by: $\forall f, g \in E, d_1(f,g) = \int_a^b |f(t) - g(t)|dt$; $d_2(f,g) = \sqrt{\int_a^b (f(t) - g(t))^2 dt}$ and $d_\infty(f,g) = max_{a\leq t\leq b}|f(t) - g(t)|$ are the metrics on E, so (E, d_1), (E, d_2) and (E, d_∞) are metric spaces. To prove the separation axiom for d_1, we us the following result concerning the integral of the positive function: if the continuous function $h: [a,b] \to \mathbb{R}_+$ is such that $\int_a^b h(t)dt = 0$, then $h(t) = 0, \forall t \in [a,b]$.To prove the triangular inequality axiom of d_2, we us the following integral Cauchy-Bouniakowsky inequality:

$$\forall f, g \in E, \int_a^b f(t)g(t)dt \leq \left(\sqrt{\int_a^b f(t)^2 dt}\right)\left(\sqrt{\int_a^b g(t)^2 dt}\right).$$

Proposition 4.1. In a metric space (E,d), the following inequality holds: $\forall x, y, z \in E$; $|d(x,y) - d(x,z)| \leq d(y,z)$.

Proof. By, the symmetric and the triangular inequality property in the definition 4.1, $\forall x, y, z \in E, d(x,z) \leq d(x,y) + d(y,z)$ and $d(x,y) \leq d(x,z) + d(y,z)$, then $\forall x, y, z \in E, d(x,z) - d(x,y) \leq d(y,z)$ and $d(x,y) - d(x,z) \leq d(y,z)$, so $\forall x, y, z \in E, -d(y,z) \leq d(x,z) - d(x,y) \leq d(y,z)$, i.e. $\forall x, y, z \in E, |d(x,y) - d(x,z)| \leq d(y,z)$.
It's easy to check that:

Proposition 4.2. If, (E,d) is a metric space and $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function, which satisfies, for every $u, v \in \mathbb{R}_+$, $\phi(u+v) \leq \phi(u) + \phi(v)$ and $\phi(0) = 0$. Then, the composition function $\phi \circ d$ is a metric on E.

Example 4.2. Let (E,d) be a metric space, the functions $\delta, \delta', \delta'': E \times E \to \mathbb{R}_+$, defined respectively by: $\forall x, y \in E, \delta(x,y) = min(1, d(x,y))$; $\delta'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ and $\delta''(x,y) = ln(1 + d(x,y))$ are metrics. Indeed, the function $\phi, \psi, \theta: \mathbb{R}_+ \to \mathbb{R}_+$ defined respectively by, $\forall u \in \mathbb{R}^+, \phi(u) = min(1,u)$; $\psi(u) = \frac{u}{1+u}$ and $\theta(u) = ln(1+u)$, where ln is the Neperien logarithm function, satisfy the conditions of the proposition 4.2, since $\forall u, v \in \mathbb{R}^+$, $\phi(u+v) = min(1, u+v) \leq min(1,u) + min(1,v) = \phi(u) + \phi(v)$; $\psi(u+v) = \frac{u+v}{1+u+v} = \frac{u}{1+u+v} + \frac{v}{1+u+v} \leq \frac{u}{1+u} + \frac{v}{1+v} = \psi(u) + \psi(v)$ and $\theta(u+v) = ln(1+u+v) \leq ln(1+u)(1+v) = ln(1+u) + ln(1+v) = \theta(u) + \theta(v)$. The others conditions are obvious.

Definition 4.3. Let a part A, of the metric space E.

a) A is said to be bounded, if there is $k > 0$, such that $d(x,y) \leq k, \forall x, y \in A$.

b) When A is bounded, the real number $\delta(A) = sup_{x,y\in A} d(x,y)$ is called, the diameter of A.

It is clear that:

Corollary 4.1. If A and B are two subsets of the metric space (E,d). Then:

a) A is bounded$\Leftrightarrow$ $\delta(A) < +\infty$.

b) $\delta(A) = 0 \Leftrightarrow A = \{x\}$.

c) $\delta(A) = \delta(cl(A))$.

d) If $A \subset B \Longrightarrow \delta(A) \subset \delta(B)$.

e) $\delta(A \cup B)$= $\delta(A) + d(A,B) + \delta(B)$.

Example 4.3.

a) In the metric space E, $\forall x \in E$, the part $\{x\}$ is bounded and $\delta(\{x\}) = 0$.

b) In the usual metric space, $\delta([a,b]) = max_{x,y\in[a,b]}|x-y| = b-a$.

Definition 4.4. Let A and B are nonvoide parts of a metric space (E,d). We call distance between A and B, the real number $d(A,B) = min_{(x,y)\in A\times B} d(x,y)$, and for a fixed x in E, distance between $\{x\}$ and B, the real number $d(x,B) = min_{y\in B} d(x,y)$.

Remark 4.1. Since in the usual metric $\mathbb{R}$, for $A = \{0\}$ and $B = \frac{1}{n}, \forall n \in \mathbb{N}^*$ $d(A,B) = min\left\{\frac{1}{n}, \forall n \in \mathbb{N}^*\right\} = 0$, then the first axiom in the definition of the metric is not checked, then the distance between the parts of a metric space, is not a metric.

Definition 4.5. Let (E,d) be a metric space, $a \in E$ and $r > 0$.

a) The set $B(a,r) = \{x \in E, d(a,x) < r\}$, is called the open bull, with center a and radius r.

b) The set $\tilde{B}(a,r) = \{x \in E, d(a,x) \leq r\}$, is called the closed bull, with center a and radius r.

c) The set $S(a,r) = \{x \in E, d(a,x) = r\}$, is called the sphere, with center a and radius r.

Example 4.4.

a) In $(\mathbb{R}, d_u)$, if a$\in\mathbb{R}$; $B(a,r) = \{x \in \mathbb{R}, |a-x| < r\} =]a-r, a+r[$; $\tilde{B}(a,r) = \{x \in \mathbb{R}, |a-x| \leq r\} = [a-r, a+r]$; and $S(a,r) = \{x \in E, |a-x| = r\} = \{a-r, a+r\}$.

b) In $(\mathbb{R}^2, d_1), B(0,1) = \{(x,y) \in \mathbb{R}^2, |x| + |y| < 1\}$, witch is the surface of a lozenge.

c) In $(\mathbb{R}^2, d_2), \tilde{B}(0,1) = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$, witch is a disk.

d) In $(\mathbb{R}^2, d_\infty), S(0,1) = \{(x,y) \in \mathbb{R}^2, max(|x|,|y|) = 1\}$, witch is a square.

e) In a discrete metric space E.

If, $r < 1, B(a,r) = B(a,r) = \{a\}$ and $S(a,r) = \emptyset$.

If, $r = 1, B(a,r) = \{a\}$ and $B(a,r) = S(a,r) = E$.

If, $r > 1, B(a,r) = B(a,r) = E$ and $S(a,r) = \emptyset$.

Proposition 4.2. A part of a metric space, is bounded$\Leftrightarrow$it is contained, in an open or closed ball.

Proof. Let A a bounded part of a metric space E, then δ(A)<+∞ and $\forall x, y \in A, d(x,y) \leq \delta(A)$, so for $y = a$ (fixed), $\forall x \in A, d(a,x) \leq \delta(A)$, then $A \subset \tilde{B}(a, \delta(A))$. Inversely, suppose that there exist $a \in E$ and $r > 0$, such that $A \subset \tilde{B}(a,r)$, then $\forall x, y \in A, d(a,x) \leq r$ and $d(a,y) \leq r$ since, by the triangular inequality axiom, $d(x,y) \leq d(x,a) + d(a,y)$, then $\forall x, y \in A, d(x,y) \leq 2r$ and A is bounded.

4.2. Metric-induced topology

The topology on a metric space, is closely related to the open balls, defined by the metric of this space. Then, if (E,d) is a metric space, the collection $\tau_d \cup \{\emptyset\}$ of the subsets of E, defined by: $O\in\tau_d \Leftrightarrow$for every $x \in O$, there exists $r > 0$, such that $B(x,r) \subset O$, is a topology on E, called a **metric-induced topology**, or **associated topology**, or **adjacent topology** to the metric space E. Let us verify that $\tau_d \cup \{\emptyset\}$, satisfies the axioms of a topology:
O_1-It is clear that $\emptyset$ and E are in $\tau_d \cup \{\emptyset\}$.
O_2-If, $\{O_\alpha, \alpha \in\Delta\}$ is the collection of the elements of τ_d then, for $x \in\cup_{\alpha\in\Delta} O_\alpha$, there exist $\alpha_0 \in \Delta$ and $x \in O_{\alpha_0}$, so there exits r>0, such that $B(x,r) \subset O_{\alpha_0} \subset\cup_{\alpha\in\Delta} O_\alpha$, so $\cup_{\alpha\in\Delta} O_\alpha\in\tau_d$.
O_3-If, $\{O_\alpha, 1 \le \alpha \le n, n \in \mathbb{N}^*\}$ is the finite collection of the elements of τ_d then, for $x \in \cap_{\alpha=1}^{n} O_\alpha$, we have $x \in O_\alpha, \forall\alpha = 1,\dots,n$; then there exists $r_\alpha > 0$, such that $B(x,r_\alpha) \subset O_\alpha$, so for $r = min\{r_\alpha, 1 \le \alpha \le n\}, B(x,r) \subset\cap_{\alpha=1}^{n} O_\alpha$, therefore $\cap_{\alpha=1}^{n} O_\alpha \in \tau_d$.

Proposition 4.3.
$a)$ The open ball is open.
$b)$ The closed ball is closed.

Proof. $a)$ Let $x \in B(a,r)$, then for $\rho = r - d(a,x) > 0, B(x,\rho) \subset B(a,r)$, so $B(a,r)$ is open. Indeed, for $y \in B(x,\rho)$, $d(x,y) < \rho = r - d(a,x)$, then $d(a,y) \le d(a,x) + d(x,y) < r$, therefore $y \in B(a,r)$. $b)$ Let $x \in \tilde{B}(a,r)^C$, then $B(x,\rho) \subset \tilde{B}(a,r)^C$, where $\rho = d(a,x) - r > 0$. Indeed if $y \in B(x,\rho), d(x,y) < \rho = d(a,x) - r \le d(a,y) + d(y,x) - r$, implies that $r < d(a,y)$, so $y \in \tilde{B}(a,r)^C$ hence $\tilde{B}(a,r)^C$ is open , so $\tilde{B}(a,r)$ is closed.

Remark 4.2.
$a)$ All the definitions given in a general topological space, remain valid for a space associated to a metric such as: the neighborhoods, the closure, the interior, the boundary,...etc.
$b)$ Any open in τ_d is the union of the open balls, indeed if, $O \in \tau_d$, $O =\cup_{x\in O} \{x\} \subset \cup_{x\in O} B(x,r) \subset O$. So for every x in a metric space, the collection $\{B(x,r), r > 0\}$ of open balls, constitute a basis of neighborhoods of x. Also, $N \in \mathcal{N}(x) \Leftrightarrow \exists r > 0, B(x,r) \subset N$. Indeed: $N \in \mathcal{N}(x) \Leftrightarrow \exists O \in \tau_d, x \in O \subset N \Leftrightarrow \exists r > 0, B(x,r) \subset O \subset N$. Therefore in (E,τ_d), it suffices for neighborhoods of x to consider the collection of open bulls $\{B(x,r), r > 0\}$.
$c)$ It is clear that, in a metric space E, for $0 < \rho < r$ and $a \in E$, $B(a,\rho) \subset \tilde{B}(a,\rho) \subset B(a,r)$.
$d)$ Do not believe that, the interior of a closed ball is the open ball, and that, the closure of an open ball is the closed ball. In fact, in a discrete metric space, $B(a,r) = \{a\}$, since $clo(B(a,r)) = clo(\{a\}) = \{a\}$ (We will see in chapter 13, that in metric space the singleton is closed) and $\tilde{B}(a,r) = E$, then $int(\tilde{B}(a,r)) = int(E) = E$.

5-Densety, Countability and Separation Axioms

5.1-Densety, countability

A topological space, with the concepts defined in the preceding chapters, becomes more and more interesting, if it is reinforced by additional conditions, such as: countability, separability, compactness, ... etc. In the sequel, we will introduce the axioms of this series in the order of their successive reinforcement. Let E be a topological space, A and B a non-empty subsets of E.

Everywhere dense part. The part A is said to be everywhere dense in E, if $cl(A) = E$.
Dense part. The part A is said to be dense in E, if $int(cl(A)) \neq \emptyset$. Equivalently, the part A is said to be nowhere dense in E, if $int(cl(A)) = \emptyset$.

Example 5.1.
$a)$ $\mathbb{Q}$ and $\mathbb{Q}^C$ are everywhere dense in the space $\mathbb{R}$.

b) In the space $\mathbb{R}$, all intervals are dense, for example: $A = [0,1[$ is dense in $\mathbb{R}$, since $int(cl(A)) =]0,1[\neq \emptyset$.
c) In the indiscrete space, all part A is dense in E, since $int(cl(A)) = E \neq \emptyset$. The only nowhere dense subset is $\emptyset$.
d) In the discrete space E, all part A, is dense in itself, since $cl(A) = A$. The only nowhere dense subset is $\emptyset$.
It is obvious that.
Proposition 5.1.
1) If $cl(A) = E$ and $A \subset B$, then $cl(B) = E$.
2) A part A of the space E is nowhere dense⟺cl(A) is nowhere dense⟺ $cl(A)^C$ is everywhere dense.
3) If A or B are nowhere dense, then $A \cap B$ is nowhere dense.
The following corollaries give useful characterizations for the everywhere dense parts.
Corollary 5.1. $cl(A) = E$ ⟺for every non empty open $O, O \cap A \neq \emptyset$.
Proof. Since $\forall x \in O, x \in cl(A)$ and $O \in \mathcal{N}(x)$, then $O \cap A \neq \emptyset$. Inversely, let $x \in E$ and $N \in \mathcal{N}(x)$, there exists an open O, $x \in O \subset N$, such that $\emptyset \neq O \cap A \subset N \cap A$, so $x \in cl(A)$.
Corollary 5.2. If $cl(A) = E$ and O is an open, then $cl(O \cap A) = cl(O)$.
Proof. Let $x \in cl(O)$ and $N \in \mathcal{N}(x)$, there exists an open U, $x \in U \subset N$, as $U \in \mathcal{N}(x)$, then $U \cap O \neq \emptyset$, furthermore $U \cap O$ is open. By corollary 5.1, $U \cap (O \cap A) = (U \cap O) \cap A \neq \emptyset$, then $N \cap (O \cap A) \neq \emptyset$, so $x \in cl(O \cap A)$, the converse is obvious.
Remark 5.1. It may be that, the equality in corollary 5.2 is false, if O is not open, as shown in the following example. Let $E = \{1,2,3\}$, $\tau = \{\emptyset, \{1,2\}, E\}$, $A = \{1\}$ and $U = \{3\}$ no open, then $cl(A) = E$, $cl(A \cap U) = \emptyset$ and $cl(U) = \{3\}$, so $cl(A \cap U) \neq cl(U)$.
Corollary 5.3. Let U be a part, of a space E. Then, U is open⟺ $cl(A) \cap U \subset cl(A \cap U)$, for all $A \in \mathcal{P}(E)$.
Proof. Let $x \in cl(A) \cap U$ and $N \in \mathcal{N}(x)$, as $U \in N(x)$, then $N \cap U \in \mathcal{N}(x)$, so $N \cap (A \cap U) \neq \emptyset$, therefore $x \in cl(A \cap U)$. Reciprocally, if $cl(A) \cap U \subset cl(A \cap U)$, for all $A \in \mathcal{P}(E)$, then for $A = U^C$, $cl(U^C) \cap U \subset cl(U^C \cap U) = \emptyset$, so $cl(U^C) \cap U = int(U)^C \cap U = \emptyset$, which implies that $U \subset int(U)$, hence U is open.
Separable space. A topological space is called separable, if it has an everywhere dense countable subset.
First countability. A space E, is called first countable or 1D-space, if each point of E, has a countable basis of neighborhoods.
Second countability. A space E, is said to be, second countable or 2D-space, if the topology of E, has a countable basis.

Example 5.2.
i) The space $\mathbb{R}$ is:
a) Separable, since $\mathbb{Q}$ is countable and $cl(\mathbb{Q}) = \mathbb{R}$.
b) 1D-space, since $\forall x \in \mathbb{R}$, the countable family $\left\{\left]x - \frac{1}{n}, x + \frac{1}{n}\right[, n \in \mathbb{N}^*\right\}$ is a basis of neighborhoods of x.
c) 2D-space, since the countable family $\left\{\left]r - \frac{1}{n}, r + \frac{1}{n}\right[, n \in \mathbb{N}^* \text{and } r \in \mathbb{Q}\right\}$ is a basis of τ_u.
ii) The indiscrete space is separable, and 2D-space, since B=E.
iii) The discrete space is clearly 1D-space, but it is 2D-space, only if it is countable.
iv) The cofinite space is 2D-space, it is a discrete space if it is finite.
Proposition 5.2. Any 2D-space, is 1D-space and separable.

Proof. It is clear that 2D-space is 1D-space. Let $B = \{B_n, n \in \mathbb{N}\}$ a countable basis and $D = \{x_n, n \in \mathbb{N}\}, x_n \in B_n$ then $cl(D) = E$, if not the open set $O = cl(D)^C$ has no element of D, impossible since there exists $x_{n_0} \in B_{n_0} \subset O$.

Remark 5.2. In general, the separable space is not 2D-space, for example $\tau = \{\emptyset, \{[a, b[, a, b \in \mathbb{R}\}, \mathbb{R}\}$ is a separable topology on $\mathbb{R}$, which is not 2D-space.

Proposition 5.3. In the separable space, every disjoint collection of open sets is countable.

Proof. Let $\{O_\alpha, \alpha \in \Delta\}$ be a disjoint collection of open sets and D a countable subset of the space E, with $cl(D) = E$. Suppose that $\{O_\alpha, \alpha \in \Delta\}$ is uncountable. Since for every $x \in O_\alpha$ and every $\in \Delta$ $O_\alpha \cap D \neq \emptyset$, then $\cup_{\alpha\in\Delta} (O_\alpha \cap D) \neq \emptyset$ as $\{O_\alpha, \alpha \in \Delta\}$ is a disjoint collection, this union is uncountable, therefore D is uncountable, contradiction.

The collection $\{U_\alpha, \alpha \in \Delta\}$ of the subsets of E, is called **a cover** of E, if $E \subset \cup_{\alpha\in\Delta} U_\alpha$. We say that, $\{U_\alpha, \alpha \in \delta \subset \Delta\}$ is a **subcover** of $\{U_\alpha, \alpha \in \Delta\}$ if, it is a cover of E. A cover of a space, where the elements are open (respectively closed) is called **open cover** (respectively **closed cover**).

Proposition 5.4. In 2D-space, any open cover has a countable subcover.

Proof. Let $\{O_\alpha, \alpha \in \Delta\}$ be an open cover of a space E and $\{B_n, n \in \mathbb{N}\}$ a countable basis of τ, then for $x \in E$, there exists $(\alpha_0, n_0) \in \Delta \times \mathbb{N}$, such that $x \in B_{n_0} \subset O_{\alpha_0}$. The collection $\{B_n, n \in N \subset \mathbb{N}\}$ whitch contains B_{n_0} is then finite or countable and the collection $\{O_{\alpha_n}, n \in N\}$ cover E, since $E \subset \cup_{n\in N} O_{\alpha_n}$.

5-2. First variation of the separation axioms

Another important type, of additional conditions, on a topological space, is provided by the separations axioms i.e. distinct points or disjoint closed sets, may be separated by disjoint open sets. In addition to the open sets, the separation axioms are required to complete the structure of the topology. We will give these axioms, according to the increasing degree of separation.

A space E is:

T_0 (or T_0-space, or Kolmogorov space), if x, y are distinct points of E, there exists an open set O, which contains one of the points but not the other.

T_1 (or T_1-space, or Fréchet space), if x, y are distinct points of E, there exists: an open O_x which contains x but not y and, an open O_y which contains y but not x, (O_x and O_y are not necessarily disjoint).

T_2 (or T_2-space, or Hausdorff space, or separate space), if x, y are distinct points of E, there exist disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$.

Regular (or Regular space), if for a closed set F in E, and $x \in F^C$, there exist disjoint open sets O_F and O_x such $F \subset O_F$ and $x \in O_x$.

T_3 (or T_3-space), if it is both T_1 and regular or T_2 and regular.

Normal (or normal space), if F and G are disjoint closed sets, there exist disjoint open sets O_F and O_G such that $F \subset O_F$ and $G \subset O_G$.

T_4 or (or T_4-space), if it is both T_1 and normal or T_2 and normal.

Completely normal, if A and B are disjoint, there exist disjoint open sets O_A and O_B such that $A \subset O_A$ and $B \subset O_B$.

T_5, (or T_5-space), if it is both T_1 and completely normal or T_2 and completely normal.

Example 5.3.

$a)$ $(\mathbb{N}^*, \mathcal{N})$ defined in example 2.3 $g)$, is T_0-space. Indeed, if $m, n \in \mathbb{N}^*$, m<n then, the open set $N_m = \{1,2, \dots, m\}$ containing m and do not containing n. In the other hand, since if $m, n \in \mathbb{N}^*, m < n$, any open $N_n = \{1,2, \dots, n\}$ containing both n and m, so $(\mathbb{N}^*, \mathcal{N})$ is not T_1-space. Then $T_0 \nRightarrow T_1$.

$b)$ The space $\mathbb{R}$ is Hausdorff. In fact, if $x, y \in \mathbb{R}, x \neq y$ $(x < y)$ there exists $z \in]x, y[$, such that, $\left]x - \frac{1}{2}, z\right[\cap \left]z, y + \frac{1}{2}\right[= \emptyset$.
$c)$ The discrete space E, is T_0, T_1 and T_2-space. Indeed, if $x, y, z \in E, x \neq y \neq z$, the open $\{x\}$ not containing y, then E, is T_0 the open $\{x, z\}$ not containing y and the open $\{y, z\}$ not containing x, then E, is T_1. As, the open $\{x\}$ and $\{y\}$ are disjoint, E is T_2.
$d)$ Let $E = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, \{b, c\}, E\}$, where its elements are clopen. Then $a \notin \{b, c\}, b \notin \{a\}, c \notin \{a\}$ and $\{a\} \cap \{b, c\} = \emptyset$, so E is regular, since there is not open sets containing b but not c and no open set containing c but not b. Therefore, E is not T_0-space, so it is not T_1-space and it is not T_3. The only disjoint clopen sets are $\{a\}$ and $\{b, c\}$, then E is normal but it is not T_4-space.

The T_i-space (i=0,...,5) are characterized by:

Proposition 5.5. Let E be a space. The following assertions are equivalent:
a) E is T_0.
$b)$ If, x, y are distinct points of E then $cl(\{x\}) \neq cl(\{y\})$.
$c)$ If, x, y are distinct points of E, then x and y are not accumulation points of $A = \{x, y\}$.

Proof. $a) \Longrightarrow b)$. As, E is T_0 and x, y are distinct points of E, there exists an open $O \ni x$, such that $\{y\} \subset O^C$ which is closed, then $cl(\{y\}) \subset O^C$, so $x \in cl(\{x\})$ and $x \notin cl(\{y\})$ then $cl(\{x\}) \nsubseteq cl(\{y\})$. $b) \Longrightarrow c)$. Since x, y are distinct points of E, by $b)$ $x \notin cl(\{y\})$ and $y \notin cl(\{x\})$, there exist an open $O \ni x$ and an open $U \ni y$ such that: $O \cap \{y\} = \emptyset$ and $U \cap \{x\} = \emptyset$, so $O \cap (A \backslash \{x\}) = \emptyset$ and $U \cap (A \backslash \{y\}) = \emptyset$, then x and y are not accumulation points of $A = \{x, y\}$. $c) \Longrightarrow a)$. If x is not an accumulation point of A, there exists $N \in \mathcal{N}(x)$ such that $(N \backslash \{x\}) \cap A = \emptyset$, therefore there exists an open sets O_x, $x \in O_x \subset N$, then $(O_x \setminus \{x\}) \cap A = \emptyset$, so $y \notin O_x$,. The same proof, for y, it suffices to replace x by y, hence E is T_0.

Proposition 5.6. E is $T_1 \Leftrightarrow \forall x \in E$, the singleton $\{x\}$ is closed.

Proof. let $x \in E$, suppose that, there exists $y \in cl(\{x\})$ and $y \notin \{x\}$ then, there exists an open $O_y \ni y$, and $x \notin O_y$, such that $O_y \cap \{x\} = \{x\}$, contradiction. Conversely, let $x, y \in E, x \neq y$, then $y \notin cl(\{x\}) = \{x\}$, so there exists an open $O_y \ni y$, such that $O_y \cap \{x\} = \emptyset$, hence $x \notin O_y$, so E is T_1.

Corollary 5.4. In a T_1-space E. The point $x \in E$, is an accumulation point of a infinite part $A \subset E$ $\Leftrightarrow$every neighborhood of x, contains infinite points of A.

Proof. Let x be an accumulation point of a part A. Suppose that, there exists $N \in \mathcal{N}(x)$ contains, a finite number of points of A, and let $F = \{x_1, \ldots, x_n\}$ be this number, without x (if $x \in A$), as E is a T_1-space, by the proposition 5.6, the singleton is closed, then $F = \cup_{i=1}^{n} \{x_i\}$ is closed, F^C is open, $F^C \in \mathcal{N}(x)$, since $N \cap F^C \in \mathcal{N}(x)$, and $((N \cap F^C) \setminus \{x\}) \cap A = N \cap F^C \cap \{x\}^C \cap A = (F \cap F^C) \cap \{x\}^C = \emptyset$, contradiction with $((N \cap F^C) \setminus \{x\}) \cap A \neq \emptyset$. It is clear that, without any condition on the space E, if $N \in \mathcal{N}(x)$, N contains infinite points of A, then $(N \setminus \{x\}) \cap A \neq \emptyset$, so x is an accumulation point of a part A.

Remark 5.3. By proposition 5 2 and corollary 3.1, the finite part of a T_1-space, is closed, and does not have accumulation points.

Proposition 5.8. E is Hausdorff$\Leftrightarrow$the intersection of the closed neighborhoods of any $x \in E$, is reduced to the singleton $\{x\}$.

Proof. Let E be a Hausdorff space, $x \in E$ and $\mathcal{N}(x)$ the collection of the closed neighborhoods of x. Let's demonstrate that $\cap_{N \in \mathcal{N}(x)} N = \{x\}$. Suppose that, there exists $y \in \cap_{N \in \mathcal{N}(x)} N$, and $y \neq x$, by the assumption, there exist disjoint open sets $O_x \ni x,, O_y \ni y$, witch implies that $O_x \subset {O_y}^C$, therefore ${O_y}^C \in \mathcal{N}(x)$ (absorption property N_4, theorem 2.1), since $y \in \cap_{N \in \mathcal{N}(x)} N$, then ${O_y}^C \in \mathcal{N}(x)$ contradiction $(O_y \cap {O_y}^C = \emptyset)$. If now, $\forall x \in E$, $\cap_{N \in \mathcal{N}(x)} N = \{x\}$ then for $x \neq y$, there exists $N \in \mathcal{N}(x)$, such that $y \notin N$, therefore there exists an open $O, x \in O \subset N$, then $y \notin O, x \in O \subset cl(O) \subset cl(N) = N$, so $y \in cl(O)^C$ and $O \cap cl(O)^C = \emptyset$, E is then Hausdorff.

Example 5.4.

$a)$ If, E is an infinite set, then cofinite space E is a T_1-space, but it is not Hausdroff space. Indeed, $\forall x \in E, \{x\}^C$ is open then {x} is closed, by proposition 5.6 E is a T_1-space. If, now x, y are distinct points of E, and O is any open, such that $x \in O$ and $y \notin O$, then $y \in O^C$ witch is finite and closed. Thus, there are no nonempty open sets disjoint with O. Therefore E is not Hausdorff. Then $T_1 \not\Rightarrow T_2$.

$b)$ Let E be a non countable set, it is easy to verify that the collection τ_{coc} of all subsets of E, with countable complements, union $\emptyset$ is a topology in E, called **cocountable topology** and (E, τ_{coc}) is called **cocountable space.** E is T_1-space but not Hausdorff. Indeed if, $x, y \in E$ and $x \neq y$, as $x \in \{y\}^C \in \tau_{coc}$, we can consider an open O containing x but not y.As O^C is countable and contains no nonvoide open sets, then E is not Hausdorff.

Proposition 5.9. Let E be a space. The following assertions are equivalent:

$a)$ E is regular.

$b)$ Every element of E, has a basis of closed neighborhoods.

$c)$ If $x \in E$ and, O is an open containing x, O contains a closed neighborhood of x.

$d)$ For every closed part $A \subset E$, the intersection of all closed neighborhoods of A is reduced to A.

Proof. $a) \Longrightarrow b)$. If $x \in E$, and $N \in \mathcal{N}(x)$ a neighborhood of x, there exists an open $O, x \in O \subset N$, since $x \notin O^C$ witch is closed, by $a)$ there exist an open $O_x \ni x$, and an open $U \supset O^C$ such that $O_x \cap U = \emptyset$, then $O_x \subset U^C \subset O$, so $x \in O_x \subset U^C \subset N$, U^C is a closed neighborhoods of x. $b) \Longrightarrow c)$. Let $x \in E$, and O an open containing x, since $O \in \mathcal{N}(x)$, by $b)$ there exists a closed neighborhoods F of x, contained in O so, there exists an open $O_x, x \in O_x \subset F \subset O$, then $O_x \subset cl(O_x) \subset F \subset O$. $c) \Longrightarrow d)$. Let A be a closed set in E, suppose that the intersection of all closed neighborhoods of A is not contained in A, then there is x in this intersection witch is not in A, so $x \in O = A^C$ witch is open, by $c)$ there exists an open $O_x \ni x$, such that $x \in O_x \subset cl(O_x) \subset O = A^C$, then $A \subset cl(O_x)^C \subset {O_x}^C$, so ${O_x}^C$ is a closed neighborhood of A which does not contain x, contradiction. Since A is in every closed neighborhood of A, A is in their intersection. $d) \Longrightarrow a)$ Let A be a closed set in E and x an element of E, witch is not in A, since by $d)$ A is the intersection of all closed neighborhoods of A, there exists a closed neighborhood N of A such that $x \notin N$ i.e. $x \in N^C = U$ witch is open. Therefore, there exists an open O such that $A \subset O \subset N$, since $O \cap U = \emptyset$, then E is regulat.

Example 5.5. Example of a space which is Hausdorff, but not regular, and hence not T_3. Let $S = \{(x, y) \in \mathbb{R}^2, y \geq 0\}$ be the subset of the Euclidian plane $(\mathbb{R}^2, \|.\|)$, $L = \mathbb{R} \times \{0\} = \{(x, y) \in \mathbb{R}^2, y = 0\}$ the subset of S, and for each $(a, r) \in S \times \mathbb{R}_+^*, B_r(a) = \{b \in \mathbb{R}^2 : \|a - b\| < r\}$, the open balls of center a and radius r and let

$$O_r(a) = \begin{cases} B_r(a) \cap S \text{ if } a \in S \setminus L; \\ \big(B_r(a) \cap (S \setminus L)\big) \cup \{a\} \text{ if } a \in L. \end{cases}$$

Then, the collection $\mathcal{U} = \{O_r(a), (a, r) \in S \times \mathbb{R}_+^*\}$, is a topology on S. If a and b are distinct points of S, let $\|a - b\| = 2r$, then $O_r(a)$ and $O_r(b)$ are disjoint open subsets of S containing a and b, respectively, hence S is a Hausdorff. Consider the point $a = (0,0)$, the open set $O_r(a)$, $a \in S$ and the open subsets of $O_r(a)$ which contain a sets of the form $O_s(a)$, where 0<s<r. Let $b \in \{b \in L, \|a - b\| < s\}$. Open sets containing b contain sets of the form $O_t(b)$, $t > 0$. Since $\|a - b\| < s, O_t(b) \cap O_s(a) \neq \emptyset$ and $b \in cl(O_s(a))$ but $b \notin O_r(a)$, thus $cl(O_s(a))$ is not a subset of $O_r(a)$, therefore S is not regular.

Example 5.6. Let $E = \{a, b, c\}$ and the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, E\}$, the closed nonempty sets in E are $\{b, c\}, \{a, c\}, \{c\}$ and E, since there are no nonempty disjoints closed sets, then E is not normal. Because, for a and $\{b, c\}$, there is no open set containing $\{b, c\}$, but not containing a, then E is not regular.
Example 5.7. Example of a space which is T_1 and regular, hence T_3, but not normal, hence not T_4. Let $S = \{(x, y) \in \mathbb{R}^2, y \geq 0\}$ be the subset of the Cartesian plane $(\mathbb{R}^2, \|.\|)$, $L = \{(x, y) \in \mathbb{R}^2, y = 0\}$ the subset of S, and for each $(a, r) \in S \times \mathbb{R}_+^*$, $B_r(a) = \{b \in \mathbb{R}^2 : \|a - b\| < r\}$, the open balls of center a and radius r and let $O_r(a) = \begin{cases} B_r(a) \cap S \text{ if } a \in S \setminus L; \\ B_r(b) \cup \{a\} \text{ if } a \in L. \end{cases}$
Then, the collection $\mathcal{V} = \{O_r(a), (a, r) \in S \times \mathbb{R}_+^*\}$, is a topology on S. Let a, b in S, and $0 < r \leq \|a - b\|$, then $a \notin O_r(b)$ and $b \notin O_r(a)$, so S is T_1. Let O any nonempty open in S, then for every $\xi \in O$, there exists $r > 0$, such that $\xi \in O_{\frac{r}{2}}(\xi) \subset O$. Since $cl\left(O_{\frac{r}{2}}(\xi)\right) = \left(b \in \mathbb{R}^2, \|\xi - b\| \leq \frac{r}{2}\right) \cap S \subset O_r(\xi)$, if $\xi \in S \setminus L$ and $O_{\frac{r}{2}}(\eta) = \left\{b \in \mathbb{R}^2, \|\eta - b\| \leq \frac{r}{2}\right\}$, where $\eta = \left(x, \frac{r}{2}\right)$ when $\xi=(x,0)\in L$. Thus, $cl\left(O_{\frac{r}{2}}(\xi)\right) \subset O_r(\xi)$. Therefore, $O_{\frac{r}{2}}(\xi) \subset cl\left(O_{\frac{r}{2}}(\xi)\right) \subset O_r(\xi) \subset O$, for every $\xi \in S$, by proposition 5.9 $c)$ S is regular, then it is T_3. To demonstrate that, S is not T_4, therefore it is not normal. Since, any subset of L is closed, the two subsets $F = \{(x, 0), x \in \mathbb{Q}\}$, $G = \{(x, 0), x \in \mathbb{Q}^C\}$ are nonvoide disjoint closed in S. If, S is T_4, there exist two disjoint open $O_F \supset F$ and $O_G \supset G$. For each $a = (x, 0) \in G$, there exists $r_a > 0$, such that $O_{r_a}(a) \subset O_G$. Let $\{G_n\}$ be, the sequence of the subsets of G defined by: $\forall n \in \mathbb{N}^*$ $G_n = \left\{(x, 0) \in G, r_a \geq \frac{1}{n}\right\}$. Now, it will be shown that, for some interval $I \subset L$, every point of I is arbitrarily close to G_n. Suppose that, for each interval I and for each $n \in \mathbb{N}^*$, there exists a subinterval J of I, such that $J \cap G_n = \emptyset$. Let the rational numbers be ordred in a single sequence $(u_0, u_1, \dots, u_n, \dots)$. We then construct a sequence of closed intervals $I_n \subset I$, such that $I_{n+1} \subset I_n$, $u_n \notin I_n$ and $I_n \cap G_n = \emptyset$. By the Cantor principal (see lemma 3.1), there exists $s \in \mathbb{R}$ such that, $\forall n \in \mathbb{N}^*$, $s \in I_n$. Since $u_n \notin I_n$, then s is not rational, so for sufficiently large n, $s \in G_n$, therefore $I_n \cap G_n \neq \emptyset$, contradiction. Hence, for some $n\in\mathbb{N}^*$ there exists an interval I, such that every subinterval of I contains points of G_n. Consequently, there are points of G_n, arbitrarily close to $(x', 0) \in F \subset O \subset O_F$, then there exists $r > 0$ such that $O_r((x', 0)) \subset O_F$. On the other hand, for each $a = (x, 0) \in G_n$ there is a set $O_r((x, 0)) \subset O_G$, with $r \geq \frac{1}{n}$. Hence if, x is sufficiently close to x', the sets $O_r((x', 0))$ and $O_r((x, 0))$ intersect, thus O_F and O_G are not disjoint, therefore S is not T_4 and not normal.
Proposition 5.10. E is normal$\Leftrightarrow$every open set O contains a closed neighborhood of each closed set F.
Proof. Since E is normal, $G = O^C$and F are closed, there exist disjoint open $O_G \supset G$, $O_F \supset F$, $F\subset O_F \subset {O_G}^C \subset O$, then O contains a closed neighborhoods ${O_G}^C$ of each F. Inversely, let F and G are disjoint closed sets, since $F \subset O = G^C$, then O contains a closed neighborhoods S of F, so there is an open W such that $F \subset W \subset S \subset O$. Let $U = S^C$, then $G \subset U$, as $W \cap U \subset S \cap U = U^C \cap U = \emptyset$, therefore $W \cap U = \emptyset$.
Corollary 5.5. Let A be a closed part of a normal space E, contained in an open O of E. Then, there exists an open U, such that $A \subset U \subset cl(U) \subset O$.
Proof. Since A and $B = O^C$ are two disjoint closed in a normal space E, there are two disjoint open $U \supset A, W \supset B$, or $U \supset A, O \supset W^C$, since $U \subset W^C$ then $A \subset U \subset cl(U) \subset O$.
Corollary 5.6. Let A be a closed part of a T_4-space (E, τ) and let $\mathcal{B}$ be a basis of τ. Then for all $x \in A^C$, there exist $B, B' \in \mathcal{B}$ such that $x \in B \subset cl(B) \subset B' \subset A^C$.

Proof. As $A^C \in \tau$, there is $B' \in \mathcal{B}$ such that $x \in B' \subset A^C$ as E is T_4-space, then $\{x\}$ is closed. As, $\{x\} \subset B'$, which is open by the corollary 5.5, there is $U \in \tau$ such that $x \in U \subset cl(U) \subset B'$. Then, there is $B \in \mathcal{B}$ such that $x \in B \subset U \subset cl(U) \subset B' \subset A^C$. Therefore $x \in B \subset cl(B) \subset B' \subset A^C$.

Proposition 5.11. E is completely normal$\Leftrightarrow$ every subset $S \subset E$, contains a closed neighborhood of each $A \subset int(S)$, where $cl(A) \subset S$.

6-Topological Subspace, Product Topological Space

6.1. Topological subspace

The notion of, topological subspace, is a convenient way, to define and study new topological spaces. Let A be a nonempty part of the space (E, τ), the collection τ_A defined by the part $O_A \subset E$ is an element of τ_A iffy, there exists $O \in \tau$ such that $O_A = A \cap O$ is a topology, in fact:

O_{A1}- As, $\emptyset, E \in \tau$, then $A \cap \emptyset = \emptyset \in \tau_A$ and $A \cap E = A \in \tau_A$.

O_{A2}-As the family $\{O_\alpha, \alpha \in \Delta\}$ of open sets in E, satisfies $\cup_{\alpha\in\Delta} O_\alpha$ is also an open in E and $\cup_{\alpha\in\Delta} (A \cap O_\alpha) = A \cap (\cup_{\alpha\in\Delta} O_\alpha)$ then $A \cap (\cup_{\alpha\in\Delta} O_\alpha) \in \tau_A$.

O_{A3}-As the finite family $\{O_\alpha, \alpha = 1, \dots, n\}$ of open sets in E, satisfies $\cap_{\alpha=1}^{n} O_\alpha$ is also an open in E and $\cap_{\alpha=1}^{n} (A \cap O_\alpha) = A \cap (\cap_{\alpha=1}^{n} O_\alpha)$ then $A \cap (\cap_{\alpha=1}^{n} O_\alpha) \in \tau_A$.

The pair (A, τ_A) is called a **subspace** of (E, τ), and τ_A is called the **induced** (or **relative**, or **trace**) **topology** for A. It is clear that τ_A is closely related to τ, i.e. τ_A changes if τ changes.

Example 6.1 In the space $\mathbb{R}$

$a)$ If $A = [0,1[$, then, $A \cap \left]-\frac{1}{2}, \frac{1}{2}\right[= \left[0, \frac{1}{2}\right[\in \tau_A$, but $\left[0, \frac{1}{2}\right[\notin \tau$. So an open in A, is not necessary an open in E and an open in E, is not necessary an open in A.

$b)$ If $A = \mathbb{N}$, $\forall n \in \mathbb{N}, \mathbb{N} \cap]n-1, n+1[= \{n\}$ are open sets in $\mathbb{N}$, while it is closed in the space $\mathbb{R}$.

Proposition 6.1. Let (A, τ_A) be, a subspace of a space (E, τ), then

$a)$ Every closed in A has the form $A \cap F$, where F is a closed in E.

$b)$ Every neighborhood of $x \in A$, has the form $A \cap N$, where N is a neighborhood of x for τ.

Proof. $a)$ Let G be a closed in A, then its complementary G^{C_A} in A is open in A, therefore there exists an open O in E, such that $G^{C_A} = A \cap O$, so $G = (A \cap O)^{C_A} = A^{C_A} \cup O^{C_A} = \emptyset \cup O^{C_A} = O^{C_A} = A \cap O^{C_E} = A \cap F$, where $F = O^{C_E}$ is closed. $b)$ Let N_A be a neighborhood in A of $x \in A$, then there is an open O_A in A, such that $x \in O_A \subset N_A$, therefore there exists O in E such that, $O_A = A \cap O$, let $N = N_A \cup O$, as $N_A \subset N$ by N_4 in theorem 2.1, $N \in \mathcal{N}(x)$ in E and $N_A \subset A \cap N = A \cap (N_A \cup O) = (A \cap N_A) \cup (A \cap O) = N_A \cup O_A \subset N_A$, so $N_A = A \cap N$.

Proposition 6.2. An open (respectively closed) in (A, τ_A) is an open (respectively closed) in (E, τ), iffy A is an open (respectively closed) in (E, τ).

Proof. It suffices to demonstrate for open ones, that for closed ones is similar. If $O_A \in \tau_A$, there is $O \in \tau$, such that $O_A = A \cap O$, as $A \in \tau$, then $O_A \in \tau$. Reciprocally if, for every $O \in \tau, A \cap O \in \tau$, then $A \cap E = A \in \tau$.

Let x in (E, τ), $\mathcal{N}(x)$ the collection of the neighborhoods of x, then $\mathcal{N}_A(x)$={ $N_A = A \cap N$, $N \in \mathcal{N}(x)\}$ is the collection of the neighborhoods of $x \in A$. Indeed:

N_{A1}-Let $x \in A$, if $N_A \in \mathcal{N}_A(x)$, there exists $N \in \mathcal{N}(x)$, such that $N_A = A \cap N$, since by N_1 in theorem 2.1, $x \in N$ then $x \in N_A$.

N_{A2}-Let $\{N_{A_\alpha};\ \alpha \in\Delta\}$ be a family of elements in $\mathcal{N}_A(x)$, then for any α∈Δ, there is $N_\alpha \in \mathcal{N}(x)$, such that $N_{A_\alpha} = A \cap N_\alpha$ so $\cup_{\alpha\in\Delta} N_{A_\alpha} = A \cap (\cup_{\alpha\in\Delta} N_\alpha)$, as by N_2 in theorem 2.1, $\cup_{\alpha\in\Delta} N_\alpha \in \mathcal{N}(x)$, then $\cup_{\alpha\in\Delta} N_{A_\alpha} \in \mathcal{N}_A(x)$.

N_{A3}-Let $\{N_{A_\alpha};\ \alpha = 1, \dots, n\}$ be a finite elements of $\mathcal{N}_A(x)$, then for any $\alpha \in \{1, \dots, n\}$, there exists $N_\alpha \in \mathcal{N}(x)$, such that $N_{A_\alpha} = A \cap N_\alpha$ so $\cap_{\alpha=1}^{n} N_{A_\alpha} = A \cap (\cap_{\alpha=1}^{n} N_\alpha)$ as by N_3 in theorem 2.1, $\cap_{\alpha=1}^{n} N_\alpha \in \mathcal{N}(x)$, then $\cap_{\alpha=1}^{n} N_{A_\alpha} \in \mathcal{N}_A(x)$.
N_{A4}-Let $M_A \subset A$, containing $N_A \in \mathcal{N}_A(x)$, then there is $N \in \mathcal{N}(x)$, such that $M_A = A \cap N$. Let $M = M_A \cup N$, then $N \subset M$, by N_4 in theorem 2.1 $M \in \mathcal{N}(x)$, as $A \cap M = A \cap (M_A \cup N) = (A \cap M_A) \cup (A \cap N) = M_A \cup M_A = M_A$, then $M_A \in \mathcal{N}_A(x)$.
N_{A5}-Let $x \in A$ and $N_A \in \mathcal{N}_A(x)$, there is $N \in \mathcal{N}(x)$ such that $N_A = A \cap N$, by N_5 in theorem 2.1, there exists $M \in \mathcal{N}(x)$ $(M \subset N)$, such that $N \in \mathcal{N}(y)$, $\forall y \in M$. so $M_A = A \cap M \in \mathcal{N}_A(x)$, $M_A \subset N_A$ then $N_A \in \mathcal{N}_A(y)$, for any $y \in M_A$.

Let (A, τ_A) be the subspace of a space (E, τ) and $B \subset A$, then:

Proposition 6.3.
$a)$ If τ_B is the topology in B induced by τ and σ_B is the topology in B induced by τ_A. Then τ_B=σ_B.
$b)$ If $\mathcal{B}$ is a basis of τ, then the collection $\mathcal{B}_A = \{A \cap B, B \in \mathcal{B}\}$ is the basis of τ_A.

Proof. $a)$ If $O_B \in \tau_B$, there exists $O \in \tau$, such that $O_B = B \cap O \subset A \cap O \in \tau_A$, then $\tau_B \subset \tau_A$. Inversely, if $U_B \in \sigma_B$, there exists $O_A \in \tau_A$, such that U_B=B$\cap O_A$, as there is $O \in \tau$, such that $O_A = A \cap O$, then $U_B = B \cap A \cap O = B \cap O \in \tau_B$, then $\tau_A \subset \tau_B$. $b)$ If $O_A \in \tau_A$, there is $O \in \tau$, such that $O_A = A \cap O$, since there is $B \in \mathcal{B}$, $B \subset O$, then $B_A = A \cap B \subset O_A$.

Proposition 6.4.
$a)$ $cl_{\tau_A}(B) = A \cap cl(B)$, where $cl_{\tau_A}(B)$ is the closure of B in A.
$b)$ $int_{\tau_A}(B) \supset A \cap int(B)$, where $int_{\tau_A}(B)$ is the interior of B in A.
$c)$ $cl_{\tau_A}(B) = cl(B) \Leftrightarrow$A is closed.

Proof. $a)$ Let $x \in cl_{\tau_A}(B) \subset A$, then $\forall N_A \in \mathcal{N}_A(x), N_A \cap B \neq \emptyset$, since there exists $N \in \mathcal{N}(x)$, such that $N_A = A \cap N$, then $\emptyset \neq A \cap N \cap B \subset N \cap B$. So $x \in cl(B)$,.therefore $x \in A \cap cl(B)$. If now, $x \in A \cap cl(B)$., and $N_A \in \mathcal{N}_A(x)$, there exists $N \in \mathcal{N}(x)$, such that $N_A = A \cap N$, as $N \cap B \neq \emptyset$, and $N_A \cap B = N \cap (A \cap B) = N \cap B$, then $N_A \cap B \neq \emptyset$, so $x \in cl_{\tau_A}(B)$. $b)$ As, $A \cap int(B)$ is an open in A, containing in B, and $int_{\tau_A}(B)$ is is the greatest open in A, containing in B, then $int_{\tau_A}(B) \supset A \cap int(B)$. $c)$ Since by $a)$ $cl_{\tau_A}(B) = A \cap cl(B) = cl(B)$ witch is closed in E, proposition 6.2 implies that A is closed in E. Inversely, if A is closed in E, then $cl_{\tau_A}(B) = A \cap cl(B)$ is a closed set in E, containing B, then $cl(B) \subset cl_{\tau_A}(B) \subset cl(B)$, therefore $cl_{\tau_A}(B) = cl(B)$.

A set $B \subset A$ is said, to have **a particular property** relative to A, if B has the property in the subspace (A, τ_A). A set A is said to have **a property** which has been defined only for topological space, if it has the property when considered as a subspace. If for a particular property, every subspace has the property whenever a space does, the property is said to be **hereditary**. If every closed subset when considered as a subspace has a property whenever the space has property, that property is said to be **weakly hereditary**. Then we have the following hereditary properties.

Proposition 6.5. Let (A, τ_A) be the subspace of the space (E, τ). Then, if E is:
$a)$ First countable, then A is first countable.
$b)$ Separable and A is open, then A is separable.
$c)$ Hausdorff, then A is Hausdorff.
$d)$ Regular, then A is regular.

Proof. $a)$ Let $x \in A$ and let $\mathcal{N}_n(x) = \{N_n, n \in \mathbb{N}\}$ be a countable basis of neighborhoods of x in E. Since for $N_A \in \mathcal{N}_A(x)$, there exists $N \in \mathcal{N}(x)$, such that $N_A = A \cap N$ thus, there exists $n_0 \in \mathbb{N}$, such that $N_{n_0} \subset N$, then $N_{An_0} = A \cap N_{n_0} \subset N_A$, therefore the countable family $\mathcal{N}_{An}(x) = \{N_{An} = A \cap N_n, n \in \mathbb{N}\}$ is a basis of neighborhoods of $x \in A$. $b)$ Let D be a countable part of E, such that $cl(D) = E$, then $A \cap D$ is a countable part of A, such that $cl(A \cap D) = A$. Since, if $x \in A$ (which is open), and $N \in \mathcal{N}(x)$ then $A \cap N \in \mathcal{N}(x)$, so $(A \cap N) \cap D = N \cap (A \cap D) \neq \emptyset$, therefore $x \in cl(A \cap D)$. $c)$ Let $x, y \in A, x \neq y$, there are two disjoint open $O \ni x$, $U \ni y$, therefore, there are in A two open $O_A = A \cap O \ni x$ and $U_A = A \cap U \ni y$ such that $O_A \cap U_A = A \cap (O \cap U) = \emptyset$. $d)$ Let $x \in A$ and F_A a closed in A, with $x \notin F_A$, then there is a closed F in E, such that $F_A = A \cap F$, so $x \notin F$, by regularity of E, there are two disjoint open in E, $O \ni x$, $U \supset F$, therefore there are in A, two open $O_A = A \cap O \ni x$, $U_A = A \cap U \supset A \cap F = F_A$ such that $O_A \cap U_A = A \cap (O \cap U) = \emptyset$.

Remark 6.1. In general, separability is not hereditary. Indeed, let (E, τ) be a topological space, where τ is the family of all parts of E, containing a fixed point $a \in E$ and $\emptyset$. Since, if $x \in E, N \in N(x), N \cap \{a\} \neq \emptyset$ then $cl(\{a\}) = E$, therefore E is separable, while $A = \{a\}^C$ witch has $\tau_A = \mathcal{P}(A)$ is not separable. But, the separability is hereditary, when A is open in E, in fact if D is a countable subset of E, with $cl(D) = E$ then $A \cap D$ is a countable subset of A, as for $x \in A$ and $N_A \in \mathcal{N}_A(x)$, there exists $N \in \mathcal{N}(x)$, such that $N_A = A \cap N$, since A is open then $A \in N(x)$, then $A \cap N \in \mathcal{N}(x)$, so $(A \cap N) \cap D \neq \emptyset$, so $(A \cap N) \cap (A \cap D) = N_A \cap (A \cap D) \neq \emptyset$ then $x \in cl(A \cap D)$ and $cl(A \cap D) = A$.

6.2. Product topological space

Let $\{(E_\alpha, \tau_\alpha), 1 \leq \alpha \leq n\}$ be a finite collection of topological spaces, and let $E = \Pi_{\alpha=1}^n E_\alpha$ be the finite product space, that is $x \in E$, if $x = (x_1, \ldots, x_\alpha, \ldots, x_n)$, where for every $\alpha \in \{1, \ldots, n\}$ $x_\alpha \in E_\alpha$. If, $y = (y_1, \ldots, y_\alpha, \ldots, y_n)$ then $x = y$ if, for every $\alpha \in \{1, \ldots, n\}, x_\alpha = y_\alpha$. For any $O_\alpha \in \tau_\alpha$: The part $E_1 \times \ldots \times E_{\alpha-1} \times O_\alpha \times E_{\alpha+1} \times \ldots \times E_n$ of E is called **open elementary cylinder** of basis O_α and the part $\Pi_{\alpha=1}^n O_\alpha$, of E, is called **open cylinder** or **open paving** or **elementary open set** of E. Then we have:

$a)$ An open elementary cylinder of basis $O_\alpha \in \tau_\alpha$, is an open cylinder, where $O_\alpha = E_\alpha$ for every $\alpha \in \{1, \ldots, n\}$.

$b)$ An open cylinder is the intersection of the following open elementary cylinder: $O_1 \times E_2 \times \ldots \times E_n$,...,$E_1 \times \ldots \times E_{\alpha-1} \times O_\alpha \times E_{\alpha+1} \times \ldots \times E_n$,...,$E_1 \times \ldots \times E_{n-1} \times O_n$. In fact, if $x \in \cap_{\alpha=1}^n (E_1 \times \ldots \times E_{\alpha-1} \times O_\alpha \times E_{\alpha+1} \times \ldots \times E_n)$, then, for every $\alpha \in \{1, \ldots, n\}$, $x \in E_1 \times \ldots \times E_{\alpha-1} \times O_\alpha \times E_{\alpha+1} \times \ldots \times E_n$, so for every $\alpha \in \{1, \ldots, n\}$, $x_\alpha \in O_\alpha$ i.e. $x \in \Pi_{\alpha=1}^n O_\alpha$. The reverse is clear.

$c)$ The intersection of two open cylinder is an open cylinder, since for every $\alpha \in \{1, \ldots, n\}$, if $O_\alpha, U_\alpha \in \tau_\alpha$, then $O_\alpha \cap U_\alpha \in \tau_\alpha$ therefore $(\Pi_{\alpha=1}^n O_\alpha) \cap (\Pi_{\alpha=1}^n U_\alpha) = \Pi_{\alpha=1}^n (O_\alpha \cap U_\alpha)$.

$d)$ Since, for every $\alpha \in \{1, \ldots, n\}$, $E_\alpha \in \tau_\alpha$, then $E = \Pi_{\alpha=1}^n E_\alpha$ is an open cylinder.

$e)$ If, there exists $\alpha_0 \in \{1, \ldots, n\}$ such that, $O_{\alpha_0} = \emptyset$, then $\Pi_{\alpha \in \Delta} O_\alpha = \emptyset$.

Let τ be the collection of parts of E, defined by: $\Omega \in \tau$ if, for every $x \in E$, there exists an open cylinder containing x and contained in Ω. That is the elements of τ, are the union of any open cylinder. Then τ is a topology on E, called **the finite product topology** on E and the pair (E, τ) is called the **finite product space** of the topological spaces E_α, $1 \leq \alpha \leq n$. Let us prof that τ is a topology.

O_1-Since, for every $\alpha \in \{1, \ldots, n\}$, $\emptyset_\alpha, E_\alpha \in \tau_\alpha$, then $\emptyset = \Pi_{\alpha=1}^n \emptyset_\alpha, E = \Pi_{\alpha=1}^n E_\alpha \in \tau$.

O_2-Let ∇ be any index set, $\{\Omega_\beta, \beta \in \nabla\}$ a collection of the elements of τ, for every $\beta \in \nabla$, every $x \in \Omega_\beta$, there exists an open paving P_β, such that, $x \in P_\beta \subset \Omega_\beta$, so $x \in \cup_{\beta \in \nabla} P_\beta \subset \cup_{\beta \in \nabla} \Omega_\beta$, as $\cup_{\beta \in \nabla} P_\beta = \cup_{\beta \in \nabla} \left(\Pi_{\alpha=1}^n O_{\alpha,\beta}\right) = \Pi_{\alpha \in \Delta}\left(\cup_{\beta \in \nabla} O_{\alpha,\beta}\right)$ and $\cup_{\beta \in \nabla} O_{\alpha,\beta} \in \tau_\alpha$, for every $\alpha \in \{1, \ldots, n\}$, then $\cup_{\beta \in \nabla} \Omega_\beta \in \tau$.

O_3-Let $\{\Omega_\beta, \beta \in \{1, \ldots, k\}\}$ be a finite collection of the elements of τ, then for every $\beta \in \{1, \ldots, k\}$, every $x \in \Omega_\beta$, there is an open paving P_β such that, $x \in P_\beta \subset \Omega_\beta$, so $x \in$

$\cap_{\beta=1}^{k} P_\beta \subset \cap_{\beta=1}^{k} \Omega_\beta$, as $\cap_{\beta=1}^{k} P_\beta = \cap_{\beta=1}^{k} \left(\Pi_{\alpha=1}^{n} O_{\alpha,\beta}\right) = \Pi_{\alpha=1}^{n}\left(\cap_{\beta=1}^{k} O_{\alpha,\beta}\right)$ and $\cap_{\beta=1}^{k} O_{\alpha,\beta} \in \tau_\alpha$, then $\cap_{\beta=1}^{n} \Omega_\beta \in \tau$.

Example 6.2.

a) Let the space $\mathbb{R}$ be, the product topology on $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$, ($n$ time) is defined by the elements which are the union of the open paving $\Pi_{\alpha=1}^{n} I_\alpha$ where, for every $\alpha \in \{1, \ldots, n\}$, I_α is the open interval in $\mathbb{R}$.

b) If, $\{(E_\alpha, \tau_\alpha), 1 \leq \alpha \leq n\}$, is the finite collection, of the discrete topological spaces, then the product space, $E = \Pi_{\alpha=1}^{n} E_\alpha$ is also, a discrete space, since, $P = \Pi_{\alpha=1}^{n} O_\alpha$ with $1 \leq \alpha \leq n$ $O_\alpha \in \tau_\alpha$ then if, for $\alpha \in \{1, \ldots, n\}$, $O_\alpha = E_\alpha$, $P = E$, and if there is some $\alpha \in \{1, \ldots, n\}$, such that $O_\alpha = \emptyset$, $P = \emptyset$, then the open set is E or $\emptyset$, so $\tau = \{\emptyset, E\}$.

Remark 6.1.

a) When, $\{(E_\alpha, \tau_\alpha), \alpha \in \Delta\}$ is an arbitrarily collection, the **box topology**, is a topology where, its elements are union of the part $\Pi_{\alpha\in\Delta} O_\alpha$ of $E = \Pi_{\alpha\in\Delta} E_\alpha$, where, for every $\alpha \in \Delta, O_\alpha \in \tau_\alpha$.

b) The family where, its elements are union of elementary open sets: $\Pi_{\alpha\in\Delta} O_\alpha = \{O_{\alpha_1} \times \ldots \times O_{\alpha_n}\} \times \Pi_{\alpha\notin\{\alpha_1,\ldots,\alpha_n\}} E_\alpha$ (only a finite number $O_\alpha \neq E_\alpha$), $O_\alpha \in \tau_\alpha$ is called the **product topology**.

Let (E, τ) be, a finite product topology space, $x \in E$, $\mathcal{N}(x)$ the collection of neighborhoods of x, and for every $\alpha \in \{1, \ldots, n\}$ $\mathcal{N}_\alpha(x_\alpha)$, the neighborhoods of x_α related to the topology τ_α. Then, the family $\mathcal{B}(x) = \{N = \Pi_{\alpha=1}^{n} N_\alpha, \text{where } \forall\alpha \in \{1, \ldots, n\}, N_\alpha \in \mathcal{N}_\alpha(x_\alpha)\}$, is a fundamental system of neighborhoods of x. Indeed, if $N \in \mathcal{B}(x)$, there are $N_1, \ldots, N_\alpha, \ldots, N_n$, where for every $\alpha \in \{1, \ldots, n\}$, $N_\alpha \in \mathcal{N}_\alpha(x_\alpha)$ such that, $N = \Pi_{\alpha=1}^{n} N_\alpha$. Therefore, there is $O_1, \ldots, O_\alpha, \ldots, O_n$ where for every $\alpha \in \{1, \ldots, n\}$, $O_\alpha \in \tau_\alpha$ such that $x_\alpha \in O_\alpha \subset N_\alpha$, so $x \in O = \Pi_{\alpha=1}^{n} O_\alpha \subset N$, as $O \in \tau$ then $N \in \mathcal{N}(x)$ which implies that $\mathcal{B}(x) \subset \mathcal{N}(x)$. Inversely, let $N \in \mathcal{N}(x)$, there exists $O \in \tau$, $x \in O \subset N$ therefore there are $O_1, \ldots, O_\alpha, \ldots, O_n$ where, for every $\alpha \in \{1, \ldots, n\}, O_\alpha \in \tau_\alpha$ such that $x \in O = \Pi_{\alpha=1}^{n} O_\alpha \subset N$, since $O_\alpha \in \mathcal{N}_\alpha(x_\alpha)$, then $\Pi_{\alpha=1}^{n} O_\alpha \in \mathcal{B}(x)$.

Note that for a finite produced space, several notions mentioned previously can be introduced. We will introduce some one, let $E = \Pi_{\alpha=1}^{n} E_\alpha$ be, the finite product space and let $A = \Pi_{\alpha=1}^{n} A_\alpha$ be the part of E, where for every $\alpha \in \{1, \ldots, n\}$, A_α is the part of E_α.

Proposition 6.6.

a) $cl(A) = \Pi_{\alpha=1}^{n} cl(A_\alpha)$.

b) $int(A) = \Pi_{\alpha=1}^{n} int(A_\alpha)$.

c) A is closed$\Leftrightarrow$ $\forall\alpha \in \{1, \ldots, n\}, A_\alpha$ is closed.

d) If, $\forall\alpha \in \{1, \ldots, n\}$, A_α is a subspace of E_α, then $A = \Pi_{\alpha=1}^{n} A_\alpha$ is a subspace of E.

Proof. a) If $x \in cl(A)$ and $N \in \mathcal{N}(x)$, there are $N_1 \in \mathcal{N}(x_1), \ldots, N_\alpha \in \mathcal{N}(x_\alpha), \ldots, N_n \in \mathcal{N}(x_\alpha)$ such that $N = \Pi_{\alpha=1}^{n} N_\alpha$ and $N \cap A = \Pi_{\alpha=1}^{n}(N_\alpha \cap A_\alpha) \neq \emptyset$, then, $\forall\alpha \in \{1, \ldots, n\}$,.$N_\alpha \cap A_\alpha \neq \emptyset$, so $x_\alpha \in cl(A_\alpha), \forall\alpha \in \{1, \ldots, n\}$, then $x \in \Pi_{\alpha=1}^{n} cl(A_\alpha)$ Conversely, if $x \in \Pi_{\alpha=1}^{n} cl(A_\alpha)$ and $N \in \mathcal{N}(x)$, there are $N_1 \in \mathcal{N}(x_1), \ldots, N_\alpha \in \mathcal{N}(x_\alpha), \ldots, N_n \in \mathcal{N}(x_\alpha)$ such that $N = \Pi_{\alpha=1}^{n} N_\alpha$, as $\forall\alpha \in \{1, \ldots, n\}, x_\alpha \in cl(A_\alpha)$, then $\forall\alpha \in \{1, \ldots, n\}, N_\alpha \cap A_\alpha \neq \emptyset$, so $N \cap A = \Pi_{\alpha=1}^{n}(N_\alpha \cap A_\alpha) \neq \emptyset$ so $x \in cl(A)$. b) It suffices to demonstrate that, if $A = A_1 \times A_2$, $int(A_1 \times A_2) = int(A_1) \times int(A_2)$ equivalently $(int(A_1 \times A_2))^C$=$(int(A_1) \times int(A_2))^C$. Since $(int(A_1 \times A_2))^C = cl((A_1 \times A_2)^C) = cl[(A_1{}^C \times E_2) \cup (E_1 \times A_2{}^C)] = [cl(A_1{}^C) \times E_2)] \cup [(E_1 \times cl(A_2{}^C)]$ =then $(int(A_1 \times A_2)C=intA_1C\times E_2)\cup(E_1\times intA_2C=(int(A_1)\times int(A_2))C$. $c)$ A is closed iffy $cl(A) = \Pi_{\alpha=1}^{n} cl(A_\alpha) = \Pi_{\alpha=1}^{n} A_\alpha = A$ iffy, $\forall\alpha \in \{1, \ldots, n\}, A_\alpha = cl(A_\alpha)$. d) Let $O \in \tau$, there are $O_1\in\tau_1$,...,$O_\alpha\in\tau_\alpha$,..., $O_n\in\tau_n$ such that $O = \Pi_{\alpha=1}^{n} O_\alpha \in \tau$, and $A_1 \cap O_1 \in \tau_{A_1}, \ldots, A_\alpha \cap O_\alpha \in \tau_{A_\alpha}, \ldots, A_n \cap O_n \in \tau_{A_n}$, as $\Pi_{\alpha=1}^{n}(A_\alpha \cap O_\alpha)$=$(\Pi_{\alpha=1}^{n} A_\alpha) \cap (\Pi_{\alpha=1}^{n} O_\alpha) = A \cap O$, then A is a subspace of E.

Proposition 6.7. The product space $E = \Pi_{\alpha=1}^{n} E_\alpha$ is Hausdorff$\Leftrightarrow$ $\forall\alpha \in \{1,\dots,n\}$, the space E_α is Hausdorff.
Proof. Let for $\alpha \in \{1,\dots,n\}$, $x_\alpha, y_\alpha \in E_\alpha$, $x_\alpha \neq y_\alpha$, then $x = (x_1,\dots,x_{\alpha-1},x_\alpha,\ \ x_{\alpha+1},\dots,x_n) \neq (x_1,\dots,x_{\alpha-1},y_\alpha,\ \ x_{\alpha+1},\dots,x_n) = y$ since $E$is Hausdorff there are two disjoint open sets $O \ni x$ and $O' \ni y$, therefore there are $O_1, O_1^{'} \in \tau_1,\dots,O_\alpha, O_\alpha^{'} \in \tau_\alpha,\dots,O_n, O_\alpha^{'} \in \tau_n$ such that $O = \Pi_{\alpha=1}^{n} O_\alpha$ and $O' = \Pi_{\alpha=1}^{n} O_\alpha^{'}$, as $\Pi_{\alpha=1}^{n} O_\alpha \cap \Pi_{\alpha=1}^{n} O_\alpha^{'} = \Pi_{\alpha=1}^{n}(O_\alpha \cap O_\alpha^{'}) = \emptyset$, and $\forall\alpha \in \{1,\dots,n\}$, $x_\alpha \in O_\alpha, y_\alpha \in O_\alpha^{'}$ then: for $\alpha = 1, x_1 \in O_1, y_1 \in O_1^{'}$ and $\forall\alpha \neq 1, x_\alpha \in O_\alpha \cap O_\alpha^{'} \neq \emptyset$, so $O_1 \cap O_1^{'}=\emptyset$ witch implies that E_1 is Hausdorff, for $\alpha = 2, x_2 \in O_2, y_2 \in O_2^{'}$ and $\forall\alpha \neq 2, x_\alpha \in O_\alpha \cap O_\alpha^{'} \neq \emptyset$, so $O_2 \cap O_2' = \emptyset$ witch implies that E_2 is Hausdorff,...,for $\alpha = n$, $x_n \in O_n$, $y_n \in O_n^{'}$ and $\forall\alpha \neq n$, $x_\alpha \in O_\alpha \cap O_\alpha^{'} \neq \emptyset$, so $O_n \cap O_n^{'}=\emptyset$ witch implies that E_n is Hausdorff. Inversely, let $x, y \in E, x \neq y$ there are $x_\alpha, y_\alpha \in E_\alpha$, with $x_\alpha \neq y_\alpha$ since E_α is Hausdorff, there are two disjoint open sets $O_\alpha \ni x_\alpha$ and $O_\alpha^{'} \ni y_\alpha$ so $x \in O = E_1 \times\dots\times E_{\alpha-1} \times O_\alpha \times E_{\alpha+1} \times\dots\times E_n$, $y \in O' = E_1 \times\dots\times E_{\alpha-1} \times O_\alpha^{'} \times E_{\alpha+1} \times\dots\times E_n$ witch are two open sets in E, and $O \cap O' = E_1 \times\dots\times E_{\alpha-1} \times (O_\alpha \cap O_\alpha^{'}) \times E_{\alpha+1} \times\dots\times E_n = \emptyset$.
Proposition 6.8. Let (E,τ) be a topological space. The diagonal $\Delta = \{(x,y) \in E^2, y = x\}$ is closed$\Leftrightarrow$$E$ is Hausdorff.
Proof. If $x, y \in E$, $x \neq y$ then $(x,y) \notin \Delta^C$ witch is a open in E^2, there are two open sets in E, $O \ni x, O' \ni y$ such that $\Delta^C = O \times O'$, so $O \cap O^{'} = \emptyset$, thus E is Hausdorff. Inversely, let $(x,y) \in \Delta^C$ then $x \neq y$, since E is Hausdorff there are two disjoint open in E, $O \ni x, O' \ni y$, then the open $O \times O'$ of E^2 satisfies $(x,y) \in O \times O' \subset \Delta^C$, so Δ^C is a neighborhood of all its elements, therefore Δ^C is open and Δ is closed.
Example 6.3.
$a)$ In $(\mathbb{R}^n,\tau)$, where τ is the product topology of usual topology in ℝ. For $n \in \mathbb{N}^*$, the sphere, $S_{n-1} = \{x \in \mathbb{R}^n, \sum_{\alpha=1}^{n} {x_\alpha}^2 = 1\}$ is a subspace of the finite product usual space $\mathbb{R}^n$.
$b)$ The cylinder $S_1 \times \mathbb{R}$ is a subspace of the space $(\mathbb{R}^2,\tau)$, where τ is the product topology of usual topology in ℝ.
$c)$ The n-dimensional torus ${S_1}^n = S_1 \times\dots\times S_1$, ($n$ times) is a topological subspace of $(\mathbb{R}^n,\tau)$,where τ is the product topology of usual topology in ℝ.
$d)$ Since the space ℝ, is Hausdorff, then $\mathbb{R}^n$ is Hausdorff.

7-Sequences, Limits and Continuity.

7.1 Sequences

A **sequence** of points, of a nonempty set E, is an map $x: n \in N \subseteq \mathbb{N} \mapsto x_n \in E$, denoted $\{x_n;\ n \in \mathbb{N}\}$; $(x_n)_{n\in\mathbb{N}}$ or simply $\{x_n\}$.
Definition 7.1. We say that, a sequence $\{x_n\}$ of a space E, converges to a point $x \in E$, or that x is a limit of the sequence $\{x_n\}$ if, for every $N \in \mathcal{N}(x), x_n \in N$ except, for a finite number of indices. In other words, for every $N \in \mathcal{N}(x)$, there is $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n > n_0, x_n \in N$. We then write: $lim_{n\to\infty} x_n = x$ or $x_n \to x$ when $n \to \infty$ or simply $x_n \to x$. A sequence which is not convergent is said to be divergent.
Proposition 7.1. Let E be a topological space, $x \in E$, $\mathcal{B}(x)$ a basis of neighborhoods of x. Then: $lim_{n\to\infty} x_n = x$ $\Leftrightarrow$for every $B \in \mathcal{B}(x)$, there is $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n > n_0, x_n \in B$.
Proof. If, $lim_{n\to\infty} x_n = x$ then, for $B \in \mathcal{B}(x) \subset \mathcal{N}(x)$, there is $n_0 \in \mathbb{N}$, such that for every $n > n_0$, $x_n \in B$. Reciprocally, if $\mathcal{N} \in N(x)$, there is $B \in \mathcal{B}(x)$, such that $B \subset N$, therefore, there is $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n > n_0, x_n \in B \subset V$, then $lim_{n\to\infty} x_n = x$.

Note that, n_0 in the definition 5.1, depends of the neighborhood N of x, and that, a limit of a sequence, in an arbitrary space, may not be unique. Also, the definition 7.1 remains true when $n \geq n_0$.

Proposition 7.2. In Hausdorff space, when the limit of the sequence exists, it is unique.

Proof. If the sequence $\{x_n\}$ has two limits x and y in the space E, such that $x \neq y$, there is two disjoint open $O \ni x, U \ni y$, therefore there are $n_1, n_2 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n \geq n_1, x_n \in O$ and, for every $n \in \mathbb{N}, n \geq n_2, \{x_n\} \in U$, then for every $n \in \mathbb{N}, n \geq max(n_1, n_2), x_n \in O \cap U = \emptyset$, contradiction.

The converse of the proposition 7.2 is not true. There are spaces which are not Hausdorff in which every convergence sequence has a unique limit.

Example 7.1. In the cocountable space E, which is not Hausdorff, the stationary sequence $\{x_n\}$ (i.e. there exists $n_0 \in \mathbb{N}$, such that $x_n = x_{n_0} = x, \forall n > n_0$) has only one limit. Indeed if, there are two limits $x, y \in E$ such that $x \neq y$, since $\{x\}^C$ is a neighborhood of y, there exists $n_0^{'} \in \mathbb{N}$, such that $\forall n \geq n_0^{'}, x_n \in \{x\}^C$, so $\forall n \geq max(n_0, n_0^{'}), x_n = x \in \{x\}^C$ contradiction. In the case when $\{x_n\}$ is not stationary, for any $x \in E$, the set $N = (\{x\} \cup (\cup_{i=1}^{n} x_n))^C$ is a neighborhood of x, witch not contains x_n, therefore $\{x_n\}$ is not convergent in E.

Although a space in which sequences have unique limits is not necessarily Hausdorff, it is must be T_1.

Theorem 7.1. If, in a space E, every sequence has at most one limit, then E is T_1.

Proof. If E is not T_1, there are $x, y \in E, x \neq y$ such that evey open O containing x, contains also y. Since the constante sequence $\{y\}$, converges to y, also converges to x as the limit is unique then $x = y$, contradiction, thus $y \notin O$ and E is T_1.

It is possible to have T_1-space, in which sequences do not have unique limit.

Example 7.2. The sequence $\{n\}$ in the cofinite space $\mathbb{N}^*$ which is T_1-space, converges to any $p \in \mathbb{N}^*$, indeed if $N \in \mathcal{N}(p)$, N^C contains a finite elements of $\mathbb{N}^*$ say n_0 then $\forall n > n_0, n \in N$.

Example 7.3.

$a)$ If,for every $n \in \mathbb{N}$, $x_n = x$, i.e. the sequence $\{x_n\}$ is constant, then $lim_{n\to\infty} x_n = x$. Since, for every $N \in \mathcal{N}(x), x \in N$.

$b)$ In the indiscrete space E, any sequence $\{x_n\}$, converges to any element $x \in E$. Indeed, the only neighborhood of any point x is E.

$c)$ In the discrete space E, the sequence $\{x_n\}$, converges to $x \in E \Leftrightarrow \{x_n\}$ is stationary. Indeed, if $x \in E$, any part A of E, containing x is a neighborhood of x, since there is $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n \geq n_0, x_n = x$, then there is $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n \geq n_0, x_n \in A$, so $lim_{n\to\infty} x_n = x$. Now, if $lim_{n\to\infty} x_n = x$, because, $\{x\} \in \mathcal{N}(x)$, there exists $n_0 \in \mathbb{N}$, such that, for every $n \in \mathbb{N}$, $n \geq n_0, x_n \in \{x\}$, so there is $n_0 \in \mathbb{N}$, such that, for every $n \in \mathbb{N}, n \geq n_0, x_n = x$, i.e. the sequence $\{x_n\}$ is stationary.

$d)$ In the space $\mathbb{R}$, the sequence, $\left\{\frac{1}{n}\right\}$ converges to the unique limite 0. In fact, for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}^*$, such that, $\frac{1}{n_0} < \varepsilon$, so $\forall n > n_0, \frac{1}{n} < \frac{1}{n_0} < \varepsilon$, equivalently $\forall I(0, \varepsilon)$, there is $n_0 \in \mathbb{N}^*$, such that, for every $n \in \mathbb{N}^*, n > n_0, \frac{1}{n} \in I(0, \varepsilon)$, i.e. $lim_{n\to\infty} \frac{1}{n} = 0$, since the space $\mathbb{R}$ is Hausdroff, the limit is unique.

$e)$ The sequence, $\left\{\frac{1}{n}\right\}$ diverges in the discrete space $\mathbb{R}$. Because, if, there is $x \in \mathbb{R}$, such that $lim_{n\to\infty} \frac{1}{n} = x$, then for $\{x\} \in \mathcal{N}(x)$, there is $n_0 \in \mathbb{N}^*$, such that, for every $n \in \mathbb{N}^*, n > n_0, \frac{1}{n} \in \{x\}$, so $\frac{1}{n_0} = \frac{1}{n_0+1}$, then 0=1, contradiction.

Definition 7.2. A subsequence of a sequence $\{x_n\}$ in the set E, is the sequence $\{x_{\varphi(n)}\}\subset\{x_n\}$where the function, $\varphi: n \in \mathbb{N} \mapsto \varphi(n) \in \mathbb{N}$, is strictly increasing.
It is clear that $\varphi(n) \geq n, \forall n \in \mathbb{N}$. then:
Proposition 7.3. Any subsequence of a convergent sequence, is convergent and, has the same limit of the sequence.
Proof. let $\{x_{\varphi(n)}\}$ be a subsequence, of a convergent sequence $\{x_n\}$, in a space E, if $x \in E$ is a limit of $\{x_n\}$, for every $N \in \mathcal{N}(x)$, there is $n_0 \in \mathbb{N}$, such that, for every $n \in \mathbb{N}, n > n_0, x_n \in N$, as $\varphi(n) > \varphi(n_0) \geq n_0$, then $x_{\varphi(n)} \in N$ so $x_{\varphi(n)} \longrightarrow x$.
Remark 7.1. The proposition 7.3 indicates that, if a sequence has two subsequences, which converge towards two different limits, then the sequence is divergent. For example in the space $\mathbb{R}$, which is Hausdorff, the sequence $\{(-1)^n\}$, has tow subsequence $\{1\}$ and $\{-1\}$, which converge to 1 if n is even and to -1.if n is odd. Then $\{(-1)^n\}$ is divergent.
Proposition 7.4. If all subsequences, of a given sequence converge and they have the same limit, then, the sequence converges towards this limit.
Proof. Let $x$be the common limit of all subsequences, of a given sequence x_n in the space E. If, x is not the limit of the sequence$\{x_n\}$, there is $N \in \mathcal{N}(x)$, such that for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$, $m \geq n$, and $x_m \notin N$, as the function $\varphi: n \in \mathbb{N} \mapsto \varphi(n) = m \in \mathbb{N}$ is strictly increasing, so the subsequence $\{x_{\varphi(n)}\}$ satisfies, the following: there is $N \in \mathcal{N}(x)$, such that for every $n \in \mathbb{N}$, there is $\varphi(n) \in \mathbb{N}$, $\varphi(n) \geq n$, and $x_{\varphi(n)} \notin N$, i.e. x is not the limit of the subsequence $\{x_{\varphi(n)}\}$, contradiction.

Let A be a part of the space E. If the sequence $\{x_n\} \subset A$ has a limit x, then for every $N \in \mathcal{N}(x)$, there is $n_0 \in \mathbb{N}$, such that, for every $n \in \mathbb{N}, n > n_0, \ x_n \in N$ so $(N \setminus \{x\}) \cap A \neq \emptyset$, i.e. $x \in A'$ the set of accumulation points of A, as $cl(A) = A \cup A'$ therefore $x \in cl(A)$. If now $x \in cl(A)$, is there a sequence in A, which converges towards x?. The answer is given by the following proposition, whose proof, is based on the fact that, if $\{N_n, n \in \mathbb{N}\}$ is a countable basis of neighborhoods of $x \in E$, then the collection $\{B_n, n \in \mathbb{N}\}$ where B_n=$\cap_{i=1}^{n} N_i$ is also, a countable basis of decreasing neighborhoods of x.
Proposition 7.5. In 1D-space E, if $x \in cl(A)$, there is a sequence $\{x_n\} \subset A$, witch converges to x.
Proof. Let $x \in cl(A)$ and $\{B_n, n \in \mathbb{N}\}$ a countable basis of decreasing neighborhoods of x. Since, for every $n \in \mathbb{N}, B_n \cap A \neq \emptyset$, there is a sequence $\{x_n\} \subset B_n \cap A$, which converges to$x$. Indeed, if $N \in \mathcal{N}(x)$, there is some $n_0 \in \mathbb{N}$, such that $B_{n_0} \subset N$, since, for every $n \in \mathbb{N}, n > n_0$, $B_n \subset B_{n_0}$, then for every $n \in \mathbb{N}, n > n_0, B_n \subset N$, then for every $n \in \mathbb{N}, n > n_0, x_n \in B_n \subset N$, so $x_n \longrightarrow x$.

7.2-The adherent value

Let $\{x_n\}$ be a sequenc in the space E. For any $n \in \mathbb{N}$, let $A_n = \{x_k, k \geq n\}$ be. An element x, of the space E, is called an **adherent value** (or **a limit point**) of the sequence $\{x_n\}$, if $\forall N \in \mathcal{N}(x)$, and $\forall n \in \mathbb{N}$, $N \cap A_n \neq \emptyset$, equivalentely, if $\forall N \in \mathcal{N}(x)$ and $\forall n \in \mathbb{N}$, there is $k \geq n$ such that, $x_k \in N$.

If $\{x_n\}$ converges to x, then x is an adherent value of $\{x_n\}$. Indeed for $N \in \mathcal{N}(x)$, there is $n_0 \in \mathbb{N}, \forall p \in \mathbb{N}, p > n_0, x_p \in N$, then for every $n \in \mathbb{N}$, there exists $k = p + n \in \mathbb{N}, k \geq n \ (k \geq p > n_0)$, such that x_k∈N. If, x is an adherent value of $\{x_n\}$, then $x \in cl(\{x_n\})$, since $\forall N \in \mathcal{N}(x), N \cap \{x_n\} \neq \emptyset$. But, an adherent element is not necessary an adherent value, as shown in the following example.
Example 7.4. In the space $\mathbb{R}$:
$a)$ The sequence $\left\{\frac{1}{n}\right\}$ has 0 as an unique adherent value (its limit) and $\forall n \in \mathbb{N}^*, \frac{1}{n}$ are the adherent elements witch are not the adherent values.

b) The sequence $\{(-1)^n\}$ is divergent, but it has tow adherent values 1 and -1.
Proposition 7.6. In a Hausdorff space, the only adherent value of a convergent sequence, is its limit.
Proof. If, there exists an adherent value $y \in E$ of the sequence $\{x_n\}, y \neq x = lim_{n\to\infty} x_n$, there are tow disjoint open $O \ni x, U \ni y$, then for every $n \in \mathbb{N}$, there exists $(k,l) \in \mathbb{N}^2$ such that for $m = max(k,l) > n$, $x_m \in O \cap U$, contradiction.
As a direct consequence of the proposition 7.5, we have.
Proposition 7.7. In 1D-space, if x is an adherent value of the sequence $\{x_n\}$, there exists a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ witch converges to x.
Proposition 7.8. The set of the adherent values, of the sequence $\{x_n\}$, in the arbitrary space E, is the closed $A = \cap_{n\geq 0} cl(A_n)$.
Proof. Let $x \in A$, an adherent value of the sequence $\{x_n\}$ and $N \in \mathcal{N}(x)$, then for every n∈N, there exists $k \in \mathbb{N}, k \geq n$ such that $x_k \in N$, so for every $n \in \mathbb{N}$, $N \cap A_n \neq \emptyset$, so $x \in cl(A_n)$, therefore $x \in \cap_{n\geq 0} cl(A_n)$. Inversely, if $x \in \cap_{n\geq 0} cl(A_n)$, $x \in cl(A_n)$, $\forall n \in \mathbb{N}$, so $\forall N \in \mathcal{N}(x), \forall n \in \mathbb{N}, N \cap A_n \neq \emptyset$, therefore there exists $k \in \mathbb{N}$, $k \geq n$ and $x_k \in A_n$, then $x \in A$.

7.3 Limit and Continuity

Functions on spaces are important tools for studying properties of spaces and for constructing new spaces previously existing ones. Let (E,τ), (F,σ) are tow topological spaces, x_0 an accumulation point of E, $l \in F$, and the map $f: E \to F$.
Definition 7.3. l is called, a limit of $f(x)$ when x tends to x_0, and we write $lim_{x\to x_0} f(x) = l$, or $f(x) \to l$, when $x \to x_0$ if, for every neighborhood V of l, there exists $N \in \mathcal{N}(x_0)$ such that $f(N) \subset V$, or equivalently for every open U containing l, there exists $O \in \tau$, $x_0 \in O$such that $f(O) \subset U$.
Proposition 7.9. In Hausdorff space, the limit when it exists is unique.
Proof. If f has in $x_0 \in E$, two limits $l, l' \in F$, and $l \neq l'$, there are two disjoint open in F, $U \ni l, U' \ni l'$, thus there are two open $O, O' \in \tau$, $x_0 \in O \cap O'$ such that, $f(O) \subset U$ and $f(O') \subset U'$, then $f(O \cap O') \subset f(O) \cap f(O') \subset U \cap U' = \emptyset$, so $f(O \cap O') = \emptyset$, therefore $O \cap O' = \emptyset$, contradiction.
Definition 7.4. f is said to be continuous at a point $x_0 \in E$, and we write $lim_{x\to x_0} f(x) = f(x_0)$, or $f(x) \to f(x_0)$, when $x \to x_0$, if for any neighborhood V of $f(x_0)$, there exists $N \in \mathcal{N}(x_0)$ such that $f(N) \subset V$, that is to say that $f(x_0) \in F$, is a limit of $f(x)$ when x tends to x_0.
Since $N \subset f^{-1}(f(N)) \subset f^{-1}(V)$, then $f^{-1}(V) \in \mathcal{N}(x_0)$, so $lim_{x\to x_0} f(x) = f(x_0)$, if for each neighborhood V of $f(x_0)$, $f^{-1}(V) \in \mathcal{N}(x_0)$. If, this property holds for each point $x \in E$, f is called continuous on E, or simply continuous.
Proposition 7.10. Let $(E,\tau), (F,\sigma), (G,\rho)$ are topological spaces, if the map $f: (E,\tau) \to (F,\sigma)$ is continuous in $x_0 \in E$ and the map $g: (F,\sigma) \to (G,\rho)$ is continuous in $f(x_0) \in F$, then the composition map $g \circ f: (E,\tau) \to (G,\rho)$ is continuous in x_0.
Proof. Let W be a neighborhood of $(g \circ f)(x_0) = g(f(x_0))$, since g is continuous in $f(x_0)$, then $g^{-1}(W)$ is a neighborhood of $f(x_0)$, as f is continuous in x_0, then $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W) \in \mathcal{N}(x_0)$.
Theorem 7.2. The following assertions are equivalent:
1) f is continuous.
2) The inverse image of open sets is open.
3) The inverse image of closed sets is closed
4) For any $A \subset E$, $f(cl(A)) \subset cl(f(A))$.

Proof. 1) $\Longrightarrow$ 2) Let $U \in \sigma$ and $x \in f^{-1}(U)$, then $f(x) \in U$ witch is a neighborhood of $f(x)$, since f is continuous in x there is $N \in \mathcal{N}(x)$ such that $f(N) \subset U$. Therefore, there is $O \in \tau, x \in O \subset N \subset f^{-1}(f(N)) \subset f^{-1}(U)$, so $f^{-1}(U) \in \tau$. 2) $\Longrightarrow$ 3) Let S be a closed in F, then S^C is an open, by 2) $f^{-1}(S^C) = f^{-1}(S)^C \in \tau$, so $f^{-1}(S)$ is closed in E. 3) $\Longrightarrow$ 1) Let $x \in E$, V a neighborhood of $f(x)$, there exists $U \in \sigma, f(x) \in U \subset V$, as U^C is closed, by 3) $f^{-1}(U^C) = f^{-1}(U)^C$ is closed in E, so $f^{-1}(U) \in \tau$ and $x \in f^{-1}(U) \subset f^{-1}(V)$, then $f^{-1}(V) \in \mathcal{N}(x)$, since x is arbitrary then f is continuous.The demonstration will be closed, if we demonstrate 3) $\Leftrightarrow$ 4). 3) $\Longrightarrow$ 4) Since, $cl(f(A))$ is closed in F, by 3) $f^{-1}(cl(f(A)))$ is closed in E, as $f(A) \subset cl(f(A))$, then $A \subset f^{-1}(f(A)) \subset f^{-1}(cl(f(A)))$, it follows that $f(cl(A)) \subset f(f^{-1}(cl(f(A)))) \subset cl(f(A))$. 4) $\Longrightarrow$ 3) Let S be a closed in F, since $f(cl(f^{-1}(S))) \subset cl(f(f^{-1}(S))) \subset cl(S) = S$, then $f^{-1}(f(cl(f^{-1}(S)))) \subset f^{-1}(cl(f(f^{-1}(S)))) \subset f^{-1}(cl(S)) = f^{-1}(S)$, so $cl(f^{-1}(S)) \subset f^{-1}(S) \subset cl(f^{-1}(S))$, so $f^{-1}(S)$ is closed.

Proposition 7.11. If f is a continuous function from a space E into the space $\mathbb{R}$. Then, for every $z \in E$ and every $\varepsilon > 0$, there is an open O in E containing z, such that, for every $x, y \in O$, $|f(x) - f(y)| < \varepsilon$.

Proof. As f is continuous then, for all $z \in E$ and $\varepsilon > 0$, $f^{-1}\left(I\left(f(z), \frac{\varepsilon}{2}\right)\right) = O \subset E$ is open and containing z. Using the definition of the continuity of f at the point z, we have for every $x, y \in O$, $f(x), f(y) \in I\left(f(z), \frac{\varepsilon}{2}\right)$. Therefore, $|f(x) - f(y)| \le |f(x) - f(z)| + |f(x) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

The relation between, the sequences and the continuity in a 1D-space, is given by the following theorem:

Theorem 7.3. If E is 1D-space, then: $lim_{x \to x_0} f(x) = f(x_0)$ $\Leftrightarrow$for every sequence $\{x_n\} \subset E$, converging to x_0, $f(x_n)$ converges to $f(x_0)$.

Proof. Let $\{x_n\} \subset E$ be a sequence witch converges to x_0, and U an open in F containing $f(x_0)$, since $lim_{x \to x_0} f(x) = f(x_0)$, there exists an open $O \ni x_0$, such that $f(O) \subset U$, therefore there exists $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n > n_0, x_n \in O$ then $f(x_n) \in U$. Conversely,
if f is not continuous, there is some open U such that $f^{-1}(U)$ is not open in E. Then, $(f^{-1}(U))^C$ is not closed in E, so there is some x in $cl((f^{-1}(U))^C)$ which is not in $(f^{-1}(U))^C$. By proposition 7.5, there is a sequence $\{x_n\}$ in $(f^{-1}(U))^C$ witch converges to x. As, $x \in f^{-1}(U)$, thus $f(x) \in U$, because for every $n \in \mathbb{N}$, $x_n \in (f^{-1}(U))^C$, then x_n is not in $f^{-1}(U)$, so $f(x_n)$ is not in U. Therefore the sequence $\{f(x_n)\}$ not converges to $f(x)$, contradiction.

As a direct consequence of the theorem 7.3. We have

Corollary 7.1. If E is 1D-space, then: $lim_{x \to x_0} f(x) = l$ $\Leftrightarrow$for every sequence $\{x_n\} \subset E$, such that $lim_{n \to \infty} x_n = x_0$, we have $lim_{n \to \infty} f(x_n) = l$.

Note that, the notion of continuity is closely related to the topologies defined on E and F. A map may be continuous for one topology, and not continuous for another, as shown in the following example:

Example 7.5.

a) The function $f:(\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$, defined by $f(x) = x, \forall x \in \mathbb{R}$ is continuous on $\mathbb{R}$, it is also continuous from $(\mathbb{R}, \tau_{dis}\})$ to the space $(\mathbb{R}, \tau_u)$ but f is not continuous from $(\mathbb{R}, \tau_u)$ into $(\mathbb{R}, \tau_{dis}\})$, since for the neighborhood $V = \{f(x)\} = \{x\}$ of $f(x)$, $f^{-1}(V) = \{x\}$, which is not an usual neighborhood of x.

b) We can demonstrate that the collection $\tau = \{\emptyset, \mathbb{R}\} \cup \{]a, +\infty[, a \in \mathbb{R}\}$ is a topology on $\mathbb{R}$. The function $f: (\mathbb{R}, \tau) \to (\mathbb{R}_+, \tau_u)$ defined by $f(x) = x^2, \forall x \in \mathbb{R}$ is not continuous on $\mathbb{R}$, since it is not continuous in 0, since there is $\delta > 0$ such that a neighborhood $I(0,\delta)$ of $f(0) = 0$ don't contains any image of any neighborhoods N of 0. As, we suppose there is $N \in \mathcal{N}(0)$, such that $f(N) \subset I(0,\delta)$ then there exists $a \in \mathbb{R}$, such that $0 \in]a,+\infty[$, and $f(]a,+\infty[) = \mathbb{R}_+ \subset f(N) \subset]-\delta,\delta[$ impossible.

Example 7.6.

$a)$ Let τ and τ' tow topologies on E, if $\tau \subset \tau'$ then, the identity map $i:(E,\tau') \to (E,\tau)$ defined by: $\forall x \in E, i(x) = x$ is continuous on E. In fact, if $O \in \tau, i^{-1}(O) = O \in \tau \subset \tau'$.

$b)$ Let $f:(E,\tau_{dis}) \to (F,\sigma)$ be a map and σ any topology on F, then f is continuous on E, in fact $\forall U \in \sigma, f^{-1}(U) \in \tau_{dis}$.

$c)$ Let $f:(E,\tau) \to (F,\tau_{idis})$ be a map, where τ is any topology on E, then f is continuous on E, in fact $\forall U \in \tau_{dis}, U = \emptyset$ or F then $f^{-1}(U) = \emptyset$ or E so $f^{-1}(U) \in \tau$.

$d)$ Let $f:(E,\tau) \to (\mathbb{R},\tau_u)$ be a continuous function on E, since $\{0\}$, $]-\infty,0]$ are closed and $]0,+\infty[$ is open in the space $\mathbb{R}$, then: $A = \{x \in E, f(x) = 0\} = f^{-1}(\{0\})$; $B = \{x \in E, f(x) \leq 0\} = f^{-1}(]-\infty,0])$ are closed in E and $C = \{x \in E, f(x) > 0\} = f^{-1}(]0,+\infty[)$ is open in E.

$d)$ Let E be a space and F a Hausdorff space. Then, the set $A = \{x \in E, f(x) = g(x)\}$, where $f,g: E \longrightarrow F$ are continuous is closed. Indeed, for all $x \in A^C$ $f(x) \neq g(x)$, as F is Hausdorff, there are $U \in \mathcal{N}(f(x))$ and $W \in \mathcal{N}(g(x))$ such that $U \cap W = \emptyset$. Therefore, there are $N \in \mathcal{N}(x)$ and $V \in \mathcal{N}(x)$ such that $f(N) \subset U$ and $f(V) \subset W$. As for any $y \in N \cap V \in \mathcal{N}(x)$, $f(y) \in$U and $g(y) \in W$, then $f(y) \neq g(y)$, so $y \in A^C$. Hence, $N \cap V \subset A^C$, so $A^C \in \mathcal{N}(x)$ thus it is open.

Proposition 7.12. Let (E,τ) and (F,σ) are two topological spaces, if τ' is a topology on E, such that $\tau\subset\tau'$, and the map $f:(E,\tau) \to (F,\sigma)$ is continuous on E, then $f:(E,\tau') \to (F,\sigma)$ is continuous on E.

Proof. Let $U \in \sigma$, since $f:(E,\tau') \to (F,\sigma)$ is continuous on E, then for $U \in \sigma$, $f^{-1}(U) \in \tau$, since $\tau \subset \tau'$, then $f^{-1}(U) \in \tau'$.

Lemma 7.1. Let (A,τ_A) be a subspace of a space (E,τ), then the canonical injection $j: A \to E$, defined by $\forall x \in A, j(x) = x$ is continuous on A.

Proof. Let $O \in \tau$, since $j^{-1}(O) = \{x \in A, j(x) = x \in O\} = A \cap O \in \tau_A$ then j is continuous on A.

Remark 7.2. The lemma 7.1 allows us, to find the topology of a subset A of a space E, by saying that the topology τ_A is the coarser topology on A, which that the canonical injection j is continuous.

Lemma 7.2. Let (A,τ_A) be a subspace of a space (E,τ), if $f:(E,\tau) \to (F,\sigma)$ is continuous on E, then the restriction of f to A that is $g:(A,\tau_A) \to (F,\sigma)$ is continuous in A.

Proof. It suffices to remark that: $g = f \circ j$ and use proposition 7.10.

Lemma 7.3. Let (E,τ) be, the finite product space, i.e. $E = \Pi_{\alpha=1}^{n} E_\alpha$. For every $\alpha \in \{1,\ldots,n\}$, the coordinate projection, $\pi_\alpha:(E,\tau) \to (E_\alpha,\tau_\alpha)$, defined by: $\forall x = (x_1,\ldots,x_\alpha,\ldots,x_n) \in E$, $\pi_\alpha(x) = x_\alpha$ are continuous and surjective.

Proof. Let $\alpha \in \{1,\ldots,n\}$, $O_\alpha \in \tau_\alpha$, then $\pi_\alpha^{-1}(O_\alpha) = \{x \in E, \pi_\alpha(x) = x_\alpha \in O_\alpha\} = E_1 \times\ldots\times E_{\alpha-1} \times O_\alpha \times E_{\alpha+1} \times\ldots\times E_n \in \tau$. So, for every $\alpha \in \{1,\ldots,n\}$, π_α is continuous. As $\pi_\alpha(E) = \{\pi_\alpha(x) \in E_\alpha, x \in E\} = E_\alpha$, for all $\alpha \in \{1,\ldots,n\}$, then π_α is surjective for all $\alpha \in \{1,\ldots,n\}$.

Remark 7.3. The lemma 7.3 allows us, to find the product topology on E, by saying that the topology τ is the coarser topology on E, such that all of the coordinate projection π_α are continuous.

Let (E,τ) and (F,σ) are topological spaces, where $F = \Pi_{\alpha=1}^{n} F_\alpha$ is a finite product space of to spaces (F_α,σ_α), $\alpha \in \{1,\ldots,n\}$. Then, the map $f: E \to F$ has n components $(f_1,\ldots,f_\alpha,\ldots,f_n)$, where $\forall \alpha \in \{1,\ldots,n\}$, $f_\alpha:(E,\tau) \to (F_\alpha,\sigma_\alpha)$, that is $\forall x \in E$, $f(x) = (f_1(x),\ldots,f_\alpha(x),\ldots,f_n(x))$ and:

Lemma 7.4. f is continuous on $E \Leftrightarrow \forall \alpha \in \{1,\ldots,n\}$, f_α is continuous on E.

Proof. Since $\forall \alpha \in \{1, \dots, n\}$, $f_\alpha = \pi_\alpha \circ f$ where $\pi_\alpha : (F, \sigma) \to (F_\alpha, \sigma_\alpha)$ are the continuous coordinate projections, if f is continuous on E by proposition 7.10 $\forall \alpha \in \{1, \dots, n\}$, f_α is continuous on E. Inversely, let $U \in \sigma$ and $x \in f^{-1}(U)$, so $f(x) \in U$, there is an open cylinder $P = \Pi_{\alpha=1}^{n} U_\alpha$, where $\forall \alpha \in \{1, \dots, n\}$, $U_\alpha \in \sigma_\alpha$, such that $f(x) \in P \subset U$ i.e. $x \in f^{-1}(P) \subset f^{-1}(U)$, since $f^{-1}(P) = f^{-1}(\Pi_{\alpha=1}^{n} U_\alpha)$ then $\forall \alpha \in \{1, \dots, n\}$, $f_\alpha(x) \in U_\alpha$ or $\forall \alpha \in \{1, \dots, n\}$, $x \in f_\alpha^{-1}(U_\alpha)$, then $f^{-1}(P) = \cap_{\alpha=1}^{n} f_\alpha^{-1}(U_\alpha)$, as $\forall \alpha \in \{1, \dots, n\}$, f_α is continuous on E, $f_\alpha^{-1}(U_\alpha) \in \tau$ therefore $f^{-1}(P) \in \tau$, so $f^{-1}(U) \in \mathcal{N}(x), \forall x \in f^{-1}(U)$, it follows that $f^{-1}(U) \in \tau$.

8-Homeomorphism, Open and Closed Maps, Urysohn Lemma

8.1-Homeomorphism, Open and Closed Maps

The notion of homeomorphism is fundamental in topology, a homeomorphism is an isomorphism of topological structures. When two topological spaces are homeomorphic any property true for one is true for the other.

Definition 8.1. Let $(E, \tau), (F, \sigma)$ are two topological spaces. The map $f: E \to F$ is said to be an homeomorphism if, f is **biunivoque** and **bicontinuous** i.e. f is bijective and both f and its inverse map $f^{-1}: F \to E$ are continuous. When there exists an homeomorphism betwin E and F, we say that E and F are **topologically equivalent** (or **homeomorphic**).

It is clear that:

Lemma 8.1. The composition of two homeomorphisms is a homeomorphism.

Example 8.1.

a) The function $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = ax + b, \forall x \in R$, where $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ are two given constants, is an homeomorphism.

b) The function $f: \mathbb{R} \to]-1,1[$ defined by $f(x) = \frac{x}{1+|x|}, \forall x \in]-1,1[$, is an homeomorphism.

c) The exponential function $f: (\mathbb{R}, \tau_u) \to (\mathbb{R}_+^*, \tau_u)$, defined by $f(x) = e^x, \forall x \in \mathbb{R}$, is an homeomorphism.

d) The function $f:]0,1[\to]a, b[$, defined by $f(x) = (a-b)x + b, \forall x \in]0,1[$, where a and b are two given constants, is an homeomorphism i.e. $]0,1[$ and $]a, b[$ are homeomorphic.

e) Any bijective map, on a discrete space into a discrete one, is an homeomorphism.

Remark 8.1. It is not true that, the bijective continuous map is an homeomorphism. Indeed, the function $f: (\mathbb{R}, \tau_{dis}) \to (\mathbb{R}, \tau_u)$, defined by $f(x) = x, \forall x \in \mathbb{R}$ is one to one, continuous but $f^{-1}: (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_{dis})$ is not continuous.

Proposition 8.1. Let $E_1 \times E_2$ be a product space and $(a_1, a_2) \in E_1 \times E_2$. The two maps $g_1: E_1 \to E_1 \times \{a_2\}$ defined by $g_1(x) = (h_1(x), h_2(x)) = (x, a_2), \forall x \in E_1$ and $g_2: E_2 \to \{a_1\} \times E_2$ defined by $g_2(y) = (a_1, y), \forall y \in E_2$ are two homeomorphisms.

Proof. g_1 is an homeomorphism. since, the components h_1, h_2 of g_1 are bijective and continuous and its inverse $g_1^{-1} = \pi_1: E_1 \times \{a_2\} \to E_1, (x, a_2) \mapsto \pi_1(x, a_2) = x$ is continuous. By the same g_2 is an homeomorphism.

Corollary 8.1. Let $E_1 \times E_2$ be a product space, F a topological space, $(a_1, a_2) \in E_1 \times E_2$. If, the map $f: E_1 \times E_2 \to F$ is continuous, then the tow partial maps $f_1: E_1 \to F$, defined by $f_1(x) = f(x, a_2), \forall x \in E_1$ and $f_2: E_2 \to F$, defined by $f_2(y) = f(a_1, y), \forall y \in E_2$ are continuous.

Proof. It suffices to note that: $f_1 = f \circ g_1$ and $f_2 = f \circ g_2$ and apply proposition 8.1 and proposition 7.10.

Remark 8.2.

a) Corollary 8.1 is true for a finite product topological spaces.

b) The inverse of the corollary 8.1 is not true. Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \frac{xy}{x^2+y^2}$, if $(x,y) \neq (0,0)$ and $f(0,0) = 0$, since $f(x,x) = \frac{1}{2} \neq f(0,0)$, the function f is not continuous in $(0,0)$, then it is not continuous on $\mathbb{R}^2$, but $f_1(x) = f(x,a_2) = \frac{a_2x}{x^2+{a_2}^2}$, if $x \neq 0$, $f_1(0) = 0$ and $f_2(y) = f(a_1,y) = \frac{a_1y}{{a_1}^2+y^2}$, if $y \neq 0$, $f_2(0) = 0$ are continuous on $\mathbb{R}$.

The introduction of the open map, (respectively of the closed map), is motivated by the fact that, the image of an open set (respectively of a closed set), by a continuous map not always an open set (respectively a closed set), as shown in the following example: the function $f: (\mathbb{R},\tau_u) \to (\mathbb{R}_+,\tau_u)$, defined by $f(x) = x^2, \forall x \in \mathbb{R}$, is continuous, but $f(]-1,1[) = [0,1[$, by the same the exponential function $g: (\mathbb{R},\tau_u) \to (\mathbb{R}_+^*,\tau_u)$, defined by $g(x) = e^x, \forall x \in \mathbb{R}$, is continuous but $g(]-\infty,0]) =]0,1]$.

Definition 8.2. Let E and F are two topological spaces. The map $f: E \to F$ is called:
a) Open if for any open O in E, $f(O)$ is an open in F.
b) Closed if for any closed set C in E, $f(C)$ is a closed set in F.

Example 8.2.
a) If A is an open subspace (respectively a closed subspace) in a space E, the canonical injection $j: A \to E$ is open (respectively closed).
b) The homeomorphism is both open and closed.

Corollary 8.2. Let $E = \Pi_{\alpha=1}^n E_\alpha$ be the finite product space. For every $\alpha \in \{1,\dots,n\}$, the coordinate projections $\pi_\alpha: E \longrightarrow E_\alpha; x \longmapsto \pi_\alpha(x) = x_\alpha$ are open.

Proof. If O is an open in E, there are $O_1 \in \tau_1,\dots,O_\alpha \in \tau_\alpha,\dots, O_n \in \tau_n$ such that $O = \Pi_{\alpha=1}^n O_\alpha \in \tau$ then $\pi_\alpha(O) = \{\pi_\alpha(x) \in E_\alpha, x \in O\} = E_\alpha \cap O_\alpha = O_\alpha \in \tau_\alpha$. Then π_α is open for all $\alpha \in \{1,\dots,n\}$.

Remark 8.3. It is not true that, the coordinate projection is closed, indeed the set $C = \cup_{n\geq 1}\left(\left[n, n+\frac{1}{2}\right] \times \left[0,1-\frac{1}{n}\right]\right)$ is a closed in the space $\mathbb{R}^2$, and $\pi_2(C) = \{\pi_2(x,y), (x,y) \in C\} = \{y \in \mathbb{R}, (x,y) \in C\} = \left\{y \in \mathbb{R}, y \in \cup_{n\geq 1}\left[0,1-\frac{1}{n}\right]\right\} = \cup_{n\geq 1}\left[0,1-\frac{1}{n}\right] = [0,1[$, so $\pi_2: \mathbb{R}^2 \to \mathbb{R}$ is not closed.

Corollary 8.3. If a map f over a space E, into a Hausdorff space F is continuous, the graph $G_f = \{(x,y) \in E \times F, y = f(x)\}$ is closed.

Proof. Since, the map $h: E \times F \to F \times F$ defined by, $h(x,y) = (h_1(x,y), h_2(x,y)) = (f(x),y)$, $\forall (x,y) \in E \times F$ is continuous, since its components f and π_2 are continuous, as the diagonal $\Delta \subset F \times F$ is closed, then $h^{-1}(\Delta)$ is closed, as $h^{-1}(\Delta) = \{(x,y) \in E \times F, (f(x),y) \in \Delta\} = \{(x,y) \in E \times F, y = f(x)\} = G_f$, thus G_f is closed.

Remark 8.4. In general, the converse of the corollary 8.3 is false. Consider the function $f: (\mathbb{R},\tau_u) \to (\mathbb{R},\tau_u)$ defined by $f(x) = \frac{1}{x}$, if $x \neq 0$, $f(0) = 0$, the graph $G_f = \{(x,y) \in \mathbb{R}^2, xy = 1\} \cup (\{0,0\})$, since the function $h: \mathbb{R}^2 \to \mathbb{R}$, defined by $h(x,y) = xy, \forall (x,y) \in \mathbb{R}^2$ is continuous on $\mathbb{R}^2$, then $G_f = h^{-1}(\{1\}) \cup \{(0,0)\}$ is closed, but f is not continuous on 0, therefore it is not continuous in $\mathbb{R}$.

Corollary 8.4. A sequence $\{x_n\} = \{(x_1^n,\dots,x_\alpha^n,\dots,x_n^n)\}$ of 1D-finite product space $E = \Pi_{\alpha=1}^n E_\alpha$ converges to $x = (x_1,\dots,x_\alpha,\dots,x_n) \in E \Leftrightarrow \forall \alpha \in \{1,\dots,n\}$, the component sequence $\{x_\alpha^n\}$ converges to x_α in E_α.

Proof. Since by the lemma 7.3, $\forall \alpha \in \{1,\dots,n\}$, the coordinate projections π_α are continuous. If $\{x_n\}$ converges to x then $\forall \alpha \in \{1,\dots,n\}$, the sequence $\{\pi_\alpha(x_n)\} = \{x_\alpha^n\}$ converges to $\pi_\alpha(x) = x_\alpha$. Inversely, if $N \in \mathcal{N}(x)$, there are $N_1 \in \mathcal{N}(x_1),\dots,N_\alpha \in \mathcal{N}(x_\alpha),\dots,N_n \in \mathcal{N}(x_n)$, such that $N = \Pi_{\alpha=1}^n N_\alpha$, since $\forall \alpha \in \{1,\dots,n\}$, the sequence $\{x_\alpha^n\}$ converges to x_α, there is some $n_\alpha \in \mathbb{N}$, such that $\forall n \in \mathbb{N}, n > n_\alpha$ we have $x_\alpha^n \in N_\alpha$, so there is $n_0 = max\{n_\alpha, 1 \leq \alpha \leq n\}$ such that $\forall n \in \mathbb{N}, n > n_0, x_n \in N$, therefore $\{x_n\}$ converges to x.

Note that, several results obtained for a finite product space remain valid for the product space, whose proofs of someone are not too far from those obtained for a finite product spaces: Let $\{(E_\alpha, \tau_\alpha), \alpha \in \Delta\}$ be a collection of the topological spaces and let $E = \Pi_{\alpha\in\Delta} E_\alpha$ be the induced product space. Then we have:

a) If, for every $\alpha \in \Delta$,

i) E_α is 1D-space, then E is 1D-space.

ii) E_α is 2D-space, then E is 2D-space.

iii) E_α is Separable, then E is separable.

b) A sequence $\{x_n\}$ of the product space E converges to $x \in E$ if, for every $\alpha \in \Delta$, the component sequence $\{x_\alpha^n\}$ converges to x_α in E_α.

c) For all $\alpha \in \Delta$, the projection map $\pi_\alpha : x \in E \mapsto x_\alpha \in E_\alpha$ is continuous open and surjective.

d) The map f from a topological space F into E, is continuous$\Leftrightarrow$ for every $\alpha \in \Delta$, the map $f_\alpha = \pi_\alpha \circ f$ from F into E_α is continuous.

Let $\{(E, \tau);\ (F_\alpha, \tau_\alpha), \alpha \in \Delta\}$ be a family of the spaces , $F = \Pi_{\alpha\in\Delta} F_\alpha$ and let $\mathcal{T} = \{f_\alpha : E \longrightarrow F_\alpha;\ \alpha \in \Delta\}$ be a family of mappings.

Definition 8.3. We say that:

a) $\mathcal{T}$ separates points, if for every $x, y \in E,\ x \neq y$, there is some $\alpha \in \Delta$, such that $f_\alpha(x) \neq f_\alpha(y)$ in F_α.

b) $\mathcal{T}$ separates points and closed sets, if for every closed part $A \subset E$ and every $x \in A^C$, there is some $\alpha \in \Delta$, such that $f_\alpha(x) \notin cl\big(f_\alpha(A)\big)$.

Definition 8.4. The map $e: E \longrightarrow F$ defined by: for all $x \in E$, $e(x) = \Pi_{\alpha\in\Delta} f_\alpha(x)$ i.e. $f_\alpha(x) \in F_\alpha$ for all $\alpha \in \Delta$, is said to be the evaluation map.

Lemma 8.2. (Embidding Lemma**).** If, for all $f \in \mathcal{T}$, f is continuous and if, $\mathcal{T}$ separates points and separates points and closed sets. Then e is an embedding i.e. e is a homeomorphism betwin (E, τ) and the subspace $(e(E), \sigma)$ of F.

Proof. It is obvious that, e is onto, as $\mathcal{T}$ separates points then e is also one-to-one. Since, $f_\alpha = \pi_\alpha \circ e$ and f_α is continuous for all $\alpha \in \Delta$, then e is continuous. It remains to prove that, for all $O \in \tau$, the image $e(O)$ is a neighborhood of each of its points, therefore it is open. Let $y \in e(O)$ be, there exists $x \in O$ such that $y = e(x)$, as $x \notin O^C$ which is closed in E, because $\mathcal{T}$ separates points and closed sets, there is some $i \in \Delta$, such that $f_i(x) \notin cl\big(f_i(O^C)\big)$. As $\Big(cl\big(f_i(O^C)\big)\Big)^C$ is open in F_i, and $\pi_i : F \longrightarrow F_i$ is continuous then $\pi_i^{-1}\left(\Big(cl\big(f_i(O^C)\big)\Big)^C\right)$ is open in F, therefore $U = e(E) \cap \pi_i^{-1}\left(\Big(cl\big(f_i(O^C)\big)\Big)^C\right) \in \sigma$. Since, $(\pi_i \circ e)(x) = f_i(x) \in$ $clfiOCC$ then, $y{=}ex{\in}\pi i{-}1clfiOCC$, therefore $y{\in}U$. It remains to prove that $U{\subset}eO$. Let $z{\in}U$ be, there is $x' \in E$, such that $z = e(x') \in \pi_i^{-1}\left(\Big(cl\big(f_i(O^C)\big)\Big)^C\right)$, as $f_i(x') \in \Big(cl\big(f_i(O^C)\big)\Big)^C$, then $f_i(x') \notin f_i(O^C)$ and $x' \notin O^C$, thus $x' \in O$, which implies that $z = e(x') \in e(O)$. It follows that $e(O) \in \mathcal{N}(y)$, for all $y \in e(O)$, thus $e(O) \in \sigma$.

8.2. Second Variation of the Separation Axioms, Urysohn Lemma

In this section, we introduce two new types of separation axioms (the stronger separation properties). The first involves to use of closed neighborhoods in place of open sets in axioms T_2, the second concerns the existence of the **Urysohn function** for a subset A and B of the

space$E$, i.e. a continuous function $f: E \to [0.1]$, such that $f(A) = 0$ and $f(B) = 1$. A space E is said to be:
$T_{2\frac{1}{2}}$ (or $T_{2\frac{1}{2}}$-space or **completely Hausdorff** space), if for $x, y \in E$ $(x \neq y)$, there exist open sets O, O' containing x and y respectively, such that $cl(O) \cap cl(O') = \emptyset$.
$T_{3\frac{1}{2}}$ (or $T_{3\frac{1}{2}}$-space), if A is a closed subset of E and y is an element of A^C, there is a Urysohn function for A and $\{y\}$. E is said to be **Completely regular** space, or **Tychonoff** if, E is T_0 and $T_{3\frac{1}{2}}$.

The following implications, specifies the relationships between both first and second variation separation axioms:
$\mathbf{T_5 \Longrightarrow T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0}$
$\mathbf{T_5 \Longrightarrow}$ **completely normal.**
$\mathbf{T_4 \Longrightarrow}$ **normal**.
$\mathbf{T_3 \Longrightarrow}$ **regular**.
Note that the implications are not reversible.

A property is said to be, a **topological property** (or **topological invariant**), if whenever one space possesses a given property, any space homeomorphic to it, also possesses the same property. Similarly, a property is called a **continuous, open**, or **closed invariant** if any continuous (respectively open, closed) image of a space possessing the property also possesses the property. All the separation properties are topological properties. However, certain of the properties are preserved under less restrictive maps if. $(E, \tau), (F, \sigma)$ are two topological spaces and $f: E \to F$ is closed one to one, and E is T_0, T_1, Hausdorff, or completely Hausdorff, then F is T_0, T_1, Hausdorff, or completely Hausdorff. In particular if $\tau \subset \tau'$ are topologies for E, that is τ' is an expansion of τ, the identity map $i: (E, \tau) \to (E, \tau')$ is closed bijective and continuous, therefore it is an homeomorphism, then if (E, τ) is T_0, T_1, Hausdorff, or completely Hausdorff, (E, τ') is also T_0, T_1, Hausdorff, or completely Hausdorff. The stronger separation properties are not in general, preserved under expansion.

Every subspace of a T_0, T_1, Hausdorff, completely Hausdorff, regular, completely regular or completely normal space is T_0, T_1, Hausdorff, completely Hausdorff, regular, completely regular or completely normal. But only closed subspace of normal space need be normal.

Most separation properties are, however, preserved under products. Let $\{(E_\alpha, \tau_\alpha), \alpha \in \Delta\}$ be a collection of the topological spaces and let $E = \Pi_{\alpha\in\Delta} E_\alpha$be the induced product space. Then

a) E is T_0, T_1, Hausdorff, completely Hausdorff, regular or completely regular space, iffy for every $\alpha \in \Delta$, E_α is T_0, T_1, Hausdorff, completely Hausdorff, regular or completely regular space.

b) If E is normal or completely normal, each E_α is normal or completely normal, but the converse does not hold.

Theorem 8.1 (Urysohn Lemma). If A and **B** are two disjoint closed sets, in a normal space E, there is a Urysohn function for A and B. i.e. the normal space is completely regular.
Proof. Since $A \subset B^C = O_1$ which is an open, by corollary 5.5, there exists an open $O_{\frac{1}{2}}$ such that $A \subset O_{\frac{1}{2}} \subset cl\left(O_{\frac{1}{2}}\right) \subset O_1$, by the same, for $A \subset O_{\frac{1}{2}}$ there is an open $A \subset O_{\frac{1}{4}}$, such that $A \subset O_{\frac{1}{4}} \subset cl(O_{\frac{1}{4}}) \subset O_{\frac{1}{2}}$ and for $cl(O_{\frac{1}{2}}) \subset O_1$ there is an open$O_{\frac{3}{4}}$ such that $cl(O_{\frac{1}{2}}) \subset O_{\frac{3}{4}} \subset cl(O_{\frac{3}{4}}) \subset O_1$, so $A \subset O_{\frac{1}{4}} \subset cl(O_{\frac{1}{4}}) \subset O_{\frac{1}{2}} \subset cl(O_{\frac{1}{2}}) \subset O_{\frac{3}{4}} \subset cl(O_{\frac{3}{4}}) \subset O_1$. By iteration, $\forall \alpha \in \{1, \ldots, 2^n\}$, $n \in \mathbb{N}^*$, there exist opens $O_{\frac{\alpha}{2^n}}$ such that for $0 < r < s$, $O_r \subset cl(O_r) \subset O_s$. Let's define, for all $x \in E$, the function $f(x) = 1$, if $x \in B$ and $f(x) = inf\{r, x \in O_r$ and

$x \notin B\}$, then if $x \in A \subset O_{\frac{\alpha}{2^n}}$, $f(x) = inf\left\{\frac{\alpha}{2^n}, n \in \mathbb{N}^*\right\} = 0$, since $0 < \frac{\alpha}{2^n} \leq 1, \forall n \in \mathbb{N}^*$, $\forall \alpha \in \{1,\dots,2^n\}$, then $0 \leq f(x) \leq 1$, $\forall x \in E$. It remains to demonstrate that, $f: E \to [0,1]$ is continuous. Let U an open in the subspace $[0,1]$ of the space $\mathbb{R}$, there exists $]a, b[\subset \mathbb{R}$, such that $U = [0,1] \cap]a, b[=]a, 1]$ if $0 < a \leq 1 < b$, or $U = [0, b[$, if $a < 0 \leq b < 1$. Let $x \in f^{-1}(]a, 1])$, then $a < f(x) \leq 1$, so there is $r_0 > a$, such that $x \notin O_{r_0}$. For $s_0 \in]a, r_0[$, $cl(O_{s_0}) \subset O_{r_0}$ then $x \notin cl(O_{s_0})$ or $x \in \left(cl(O_{s_0})\right)^C$, so $f^{-1}(]a, 1]) = \cup_{s>a} (cl(O_s))^C$ witch is an open in E. If now, $x \in f^{-1}([0, b[)$, then $0 \leq f(x) < b$, so there exists $n_0 \in \mathbb{N}^*$, such that $0 \leq f(x) \leq \frac{1}{2^{n_0}} = r_0 < \frac{1}{n_0} < b$, then $x \in O_{r_0}$ and $f^{-1}([0, b[) = \cup_{r<b} O_r$ witch is an open in E.

Remark 8.5.

a) The Urysohn lemma is true in any space homeomorphic to $[0,1]$, in particular in any interval $[a, b]$. That is for two disjoint closed A and B in a normal space E, there is a continuous function $g: E \to [a, b]$, such that $g(A) = a$ and $g(B) = b$. It suffices to take $g = min\{b, max\{a, f\}\}$, where $f: E \to [0,1]$ is a continuous function.

b) If, for two disjoint closed parts A and B of the space E, there is an Urysohn continuous function f. Then E is normal. In fact, for any open U in $[0,1]$, $f^{-1}\left(\left[0, \frac{1}{4}\right[\cap U\right)$ is an open in $[0,1]$, containing A, and for any open V in $[0,1]$, $f^{-1}\left(\left]\frac{3}{4}, 1\right] \cap V\right)$ is an open in $[0,1]$, containing B. As $f^{-1}\left(\left[0, \frac{1}{4}\right[\cap U\right) \cap f^{-1}\left(\left]\frac{3}{4}, 1\right] \cap V\right) = \emptyset$, then E

We need the following lemma, to demonstrate the extension theorem of continuous functions defined on closed part of a normal space (Tietze-Hurysohn Theorem).

Lemma 8.3. If A is a closed part of the normal space E. The following properties are equivalent:

a) The bounded continuous function $f: A \to \mathbb{R}$, has a bounded extension continuous function over E.

b) The continuous function $f: A \to \mathbb{R}$, has an extension continuous function over E.

Proof. $a) \Longrightarrow b)$. Let $f: A \to \mathbb{R}_+$ be a continuous function. Then, the function $\varphi: A \to [0,1]$ defined by $\varphi = \frac{f}{1+f}$, is bounded continuous on A, by a) there is an extended continuous function $g: E \to [0,1]$, i.e. $g = \varphi$, on A. Then $B = g^{-1}(\{1\})$ is closed in E, and $A \cap B = \emptyset$, if not there is some $x_0 \in A \cap B$, then $g(x_0) = 1 = \frac{f(x_0)}{1+f(x_0)} < 1$, contradiction. By the Hurysohn lemma, there is a continuous function $h: E \to [0,1]$, such that $h(A) = 1$ and $h(B) = 0$. Since, the continuous product function $\psi = gh: E \to [0,1]$, satisfies $\psi(A) = g(A)h(A) = g(A)$ then ψ is also an extended continuous of φ. Let $\tilde{f}: E \to \mathbb{R}_+$, defined by $\tilde{f} = \frac{\psi}{1-\psi}$ if $\psi \neq 1$, and $\tilde{f} = 0$, if $\psi = 1$, since $\tilde{f}(A) = \frac{\psi(A)}{1-\psi(A)} = \frac{g(A)}{1-g(A)} = \frac{\varphi(A)}{1-\varphi(A)} = \frac{f(A)}{1+f(A)}(1 + f(A)) = f(A)$, then $\tilde{f}$ is an extended continuous function of f. In the case where $f: A \to \mathbb{R}_-^*$, we will do the same demonstration by considering the function $-f: E \to \mathbb{R}_+$. If, now $f: A \to \mathbb{R}$, we use the same argument for the composed continuous function $|.| \circ f: A \to \mathbb{R}_+$, where $|.|$ is the absolute value function on $\mathbb{R}$. $b) \Longrightarrow a)$. Let $f: A \to [0,1]$ be a bounded continuous function. By b) the continuous function $\phi: A \to \mathbb{R}_+$, defined by $\phi = \frac{f}{1-f}$, $0 \leq f(x) < 1$, has an extended continuous function $g: E \to \mathbb{R}_+$, Since $B = g^{-1}(\{0\})$ is closed in E and $A \cap B = \emptyset$, by the Hurysohn lemma, there is a continuous function $h: E \to [0,1]$, such that $h(A) = 1$ and $h(B) = 0$. As the continuous product function $\psi = gh: E \to \mathbb{R}_+$ is also an extended continuous of ϕ. Let $\tilde{f}: E \to]0,1[$, defined by $\tilde{f} = \frac{\psi}{1+\psi}$, then $\tilde{f}$ is an extended continuous of f to E.

Remark 8.6. Since in $a)$ the function is bounded then, there is $a, b \in \mathbb{R}$, such that $f(x) \in [a, b]$, who is homeomorphic to $[0,1]$, so the proof remains valid for $[a, b]$ and since $]0,1[$ is homeomorphic to $\mathbb{R}$, the conclusion remains valid for $\mathbb{R}$.

Before stating Tietz's theorem, let us recall that if $C_b(E)$ denotes the space of functions defined on the space E which are bounded and continuous, the map $f \in C_b(E) \mapsto \|f\| = sup_{x\in E}|f(x)| \in (\mathbb{R}_+, |.|)$; is a norme i.e. for every $f, g \in C_b(E)$ and every $\lambda \in \mathbb{R}$: $\|f\| = 0 \Leftrightarrow f = 0$; $\|\lambda f\| = |\lambda|\|f\|$ and $\|f + g\| \leq \|f\| + \|g\|$. The restriction of the map $\|.\|$, to the subspace $A \subset C_b(E)$ is also a norme. We come back in detail to this notion in the chapter on normed spaces later.

Theorem 8.2 (Tietze-Hurysohn theorem). If A is a closed part of the normal space E. Then any bounded continuous function $f: A \to \mathbb{R}$, has a unique extension continuous function over E.

Proof. We can obviously take, $f: A \to [-1,1]$. Because the two disjoint parts $A_0 = \left\{x \in A, f(x) \leq -\frac{1}{3}\right\}$ and $B_0 = \left\{x \in A, f(x) \geq \frac{1}{3}\right\}$ are closed in E (see, proposition 6.2) and E is normal then it is completely regular, by Hurysohn lemma there is $f_0 \in C_b(E)$ such that $f_0(A_0) = -\frac{1}{3}$, $f_0(B_0) = \frac{1}{3}$ and $-\frac{1}{3} \leq f_0(x) \leq \frac{1}{3}$, for every $x \in E$, so $\|f_0\| = \frac{1}{3}$. It follows that $\|f - f_0\|_A = sup_{x\in A}|(f - f_0)(x)| \leq \frac{2}{3}$. Applying the same argument, for the function $\frac{3}{2}(f - f_0): A \to [-1,1]$, there is $f_1 \in C_b(E)$, $\|f_1\| = \frac{1}{3}$ and $\left\|\frac{3}{2}(f - f_0) - f_1\right\|_A \leq \frac{2}{3}$ or $\left\|(f - f_0) - \frac{2}{3}f_1\right\|_A \leq \left(\frac{2}{3}\right)^2$. By iteration up to $n \in \mathbb{N}$, there is $f_n \in C_b(E)$, $\|f_n\| = \frac{1}{3}$ and $\left\|(f - f_0) - \frac{2}{3}f_1 - \left(\frac{2}{3}\right)^2 f_2 - \cdots - \left(\frac{2}{3}\right)^n f_n\right\|_A \leq \left(\frac{2}{3}\right)^{n+1}$ $(*)$ Because, the functions series $\sum_{k=0}^{\infty} g_k(x)$ where $g_k(x) = \left(\frac{2}{3}\right)^k f_k(x)$, $\forall x \in E$, satisfies: $|g_k(x)| = \left|\left(\frac{2}{3}\right)^k f_k(x)\right| \leq \left(\frac{2}{3}\right)^k |f_k(x)| \leq \left(\frac{2}{3}\right)^k \|f_k\| = \frac{1}{3}\left(\frac{2}{3}\right)^k$, for every $x \in E$, then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent to the continuous function g, defined en E, therefore it is simply convergent to the continuous function g defined on E. As from $(*)$, $\|f - g\|_A$=0, then the restriction of g into A is f.

9-Connectedness

9.1-Connected space

In this chapter, we introduce the idea of connectedness, which is a topological property related to the separation axioms, it examines the structure of topological spaces from the opposite point of view. Intuitively, a topological space is connected if it is all in one piece. To make this precise, the parts A and B of a space E, are said to be **a separation** of E, if $E = A \cup B$ and $A \cap B = \emptyset$. Then:

Definition 9.1. A space E, is said to be connected, if there is no separation of E in two nonempty open sets. E is said to be disconnected if, it is not connected i.e. E can be written as the union of two disjoint nonempty open subsets.

Proposition 9.1. Let E be a topological space. The following assertions are equivalent:

$a)$ E is connected.

$b)$ E has no separation by two nonempty closed sets.

$c)$ E has nontrivial two open separation, i.e. the only separation of E is $\emptyset$ and E.

$d)$ E does not have any nontrivial clopen sets, i.e. the only clopens of E are $\emptyset$ and E.

e) Any nontrivial part A of E, has its boundary $bd(A) \neq \emptyset$.
f) There is no surjective continuous function, from E to a discrete two point space.
g) Any continuous function from E to a discrete two point space is constant.
Proof. $a) \Longrightarrow b)$. If, there are two nonempty closed sets D and G in E, such that $D \cap G = \emptyset$ and $E = D \cup G$ then D^C, G^C are two disjoint open sets in E and $D^C \cup G^C = E$, contradiction with a). $b) \Longrightarrow c)$. If, there is two non trivial separation open subsets O and U, then $E = O \cup U$ and $O \cap U = \emptyset$, so O^C, U^C are two disjoint closed sets in E and $E = O^C \cup U^C$, contradiction with b). $c) \Longrightarrow d)$. If, there is a nontrivial clopen subset A in E, then: A and A^C are two disjoint separation clopen of E, contradiction with c). $d) \Longrightarrow e)$. If, there is a nontrivial part A of E such that $bd(A) = \emptyset = cl(A) \cap cl(A^C)$, if $x \in cl(A)$, $x \notin cl(A^C)$ then $x \notin A^C$ or $x \in A$, so A is closed and if $x \in cl(A^C)$, $x \notin cl(A)$ then $x \notin A$, or $x \in A^C$ then A^C is closed it follows that A is open, therefore A is clopen, contradiction with d). $e) \Longrightarrow f)$. If there is a continuous function f from E to a discrete space $F = \{a, b\}$, then the nontrivial subset $\{a\}$ in F is clopen therefore, by continuity the nontrivial subset $f^{-1}(\{a\})$ is clopen in E, so $bd(f^{-1}(\{a\})) = f^{-1}(\{a\}) \cap (f^{-1}(\{a\}))^C = \emptyset$, contradiction with e). $f) \Longrightarrow g)$. If, there is a continuous function from E to a discrete space $F = \{a, b\}$, witch is non constant, there are $x, y \in E, x \neq y$ such that $f(x) = a$ and $f(y) = b$ then, a,b∈f(E), so $F = \{a, b\} \subset f(E)$ then f is surjective, contradiction with f). $g) \Longrightarrow a)$ If, E is written as the union of two disjoint nonempty open subsets O and U, then the function $f: E \to F = \{a, b\}$, defined by: $\forall x \in E, f(x) = a$, if $x \in O$, and $f(x) = b$, if $x \in U$, is such that $f^{-1}(\{a\}) = O$ and $f^{-1}(\{b\}) = U$ so, f is continuous, nonconstant, contradiction with g).
Example 9.1.
a) Let $E = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, E\}$, clearly (E, τ) is connected.
b) The indiscrete space is connected. The only clopen sets, are the trivial sets $\emptyset$ and E.
c) The space $\mathbb{R}$, is connected. Since the only clopen sets are $\emptyset$ and $\mathbb{R}$ (see example 2.2, d)).
d) The singleton space is connected, since the only clopen are $E = \{x\}$ and $\emptyset$.
e) The discrete space is disconnected. Since $\forall A \subset E$, $\{A\}$ is clopen.
f) The finite Hausdorff space containing at last two elements, is disconnected, indeed $\forall x \in E, \{x\}$ is clopen.
Definition 9.2. A subspace A of a space E is a connected set, if it satisfies the definition of connected space under induced topology. A is disconnected if, there exist two open O and U in E, such that $A \subset O \cup U$, $A \cap O \cap U = \emptyset$, $A \cap O \neq \emptyset$ and $A \cap U \neq \emptyset$.
Example 9.2. In the space $\mathbb{R}$.
a) The subspace $\mathbb{R}^*$ is disconnected, since $\mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$ and $]-\infty, 0[\cap]0, +\infty[= \emptyset$.
b) The subspace $\mathbb{Q}$ is disconnected, since $\mathbb{Q} = \left(\mathbb{Q} \cap \left]-\infty, \sqrt{2}\right[\right) \cup \left(\mathbb{Q} \cap \left]\sqrt{2}, +\infty\right[\right)$ and $\left(\mathbb{Q} \cap \left]-\infty, \sqrt{2}\right[\right) \cap \left(\mathbb{Q} \cap \left]\sqrt{2}, +\infty\right[\right) = \emptyset$.
c) In (E, τ_{cof}). Any infinite part A is connected, but any finite part B is disconnected . Indeed, if the infinite part A is disconnected, there exist two open O and U in E, such that $A \subset O \cup U$, $A \cap O \cap U = \emptyset$, $A \cap O \neq \emptyset$ and $A \cap U \neq \emptyset$, then $A \subset (O \cap U)^C = O^C \cup U^C$ witch is finite, contradiction. If $B = \{b_1, \dots, b_\alpha, \dots, b_n\}$ is finite, in a T_1-space (E, τ_{cof}), then B is closed and $B = \{b_1\} \cup \{b_2, \dots, b_\alpha, \dots, b_n\}$, with two closed disjoint $\{b_1\}$ and $\{b_2, \dots, b_\alpha, \dots, b_n\}$, then B is disconnected.
Theorem 9.1. In the space $\mathbb{R}$, a part $A \subseteq \mathbb{R}$, is connected$\Longleftrightarrow A$ is an interval.
Proof. If, A is not an interval, then A is disconnected, indeed, as A is not an interval, there are $x, y \in A, x \leq y$, such that $[x, y] \nsubseteq A$, so there is $a \in [x, y]$ and $a \notin A$. The sets $A \cap]-\infty, a[$ and $A \cap]a, +\infty[$ are a separation of A. Conversely, if A is a disconnected interval, there is two open O and U in E, such that $A = (A \cap O) \cup (A \cap U)$ and $A \cap O \cap U = \emptyset$. Let $x, y \in E, x <$

$y, x \in A \cap O$ and $y \in A \cap U$, as $x, y \in A$, witch is an interval, then $[x, y] \subset A$, since the set $B = A \cap O \cap [x, y] = O \cap [x, y] \subset [x, y]$ then B is bounded, there is $b \in [x, y]$ such that $b = supB$, so $b \in A \cap O$ or $b \in A \cap U$. If, $b \in A \cap O$, there exists $\delta > 0$, such that $\left[b, b + \frac{\delta}{2}\right] \subset A$ and $A \cap \left[b, b + \frac{\delta}{2}\right] \subset A \cap O$, since $b \in [x, y]$ and $b \notin U$, then $b < y$, so $x < b + \frac{\delta}{2} < y$, indeed if $b < y \leq b + \frac{\delta}{2}$, then $y \in A \cap O \cap U = \emptyset$, therefore $b + \frac{\delta}{2} \in B$, so $b + \frac{\delta}{2} \leq b$, contradiction. If, $b \in A \cap U$, there exists ρ>0, such that $\left[b - \frac{\rho}{2}, b\right] \subset U$, and $A \cap \left[b - \frac{\rho}{2}, b\right] \subset A \cap U$, since $b \notin O$, then $x < b$ a fortiori, $x < b - \frac{\rho}{2} < y$, indeed if, $b - \frac{\rho}{2} \leq x < b, x \in A \cap O \cap U = \emptyset$, therefore $b - \frac{\rho}{2} \in U \cap [x, y]$, since $O \cap U \cap [x, y] \subset A \cap O \cap U = \emptyset$, then $b - \frac{\rho}{2} \notin O \cap [x, y]$ so, $b \leq b - \frac{\rho}{2}$, contradiction. Thus, $b \in [x, y] \subset E$ and $b \notin E$ witch is an interval, impossible.

Example 9.3.
In the space $\mathbb{R}$, the subspaces $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^C$are disconnected. Because, hey are not intervals of $\mathbb{R}$.

Proposition 9.10. If, A is a connected subset of a space E, and there are two open sets O and U in E, such that: $A \cap O \cap U = \emptyset$, and $A \subset O \cup U$, then $A \subset O$ or $A \subset U$.

Proof. if, $A \not\subseteq O$ and $A \not\subseteq U$, there are $x, y \in A$ such that $x \notin O$ and $y \notin U$, since $A \subset O \cup U$, then $x \in A \cap U$ and $y \in A \cap O$, so A is a partition of the two nonempty open $A \cap U$ and $A \cap O$, then A is disconnected, contradiction.

Proposition 9.11. If, A is a connected subset of a space E, and B a subset of E, such that: $A \subset B \subset cl(A)$, then B is connected, in particular $cl(A)$ is connected.

Proof. If, B is disconnected, there are two nonvoide open sets O and U in E, such that: $B \cap O \neq \emptyset, B \cap U \neq \emptyset, B \cap O \cap U = \emptyset$ and $B \subset O \cup U$. If, $x \in B \cap O$, thus $x \in cl(A)$, since $O \in \mathcal{N}(x)$, then $O \cap A \neq \emptyset$, by the same $U \cap A \neq \emptyset$. Moreover, $A \subset O \cup U$ so A is disconnected, contradiction.

Remark 9.1.
a) If, A is a connected subspace of a space E, and $cl(A) = E$, then E is connected. But, if A is a subset of a space E, such that $cl(A) = E$, then A is not always connected, for example in usual $\mathbb{R}$, $cl(\mathbb{Q})$ is connected, but $\mathbb{Q}$ is disconnected.
b) If, A is a connected subset of a space E, the $int(A)$, the intersection and the union are not always connected. Example in $E = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, E\}$ the parts $A = \{a, b, d\}$ and $B = \{a, b, c\}$ are connected, but $int(A) = \{a, b\} = A \cap B$ is disconnected. In Hausdorff space the singletons are connected but their union is disconnected.

Proposition 9.12. Let E be a topological space. If, A is a connected part of E and B a nonvoide part of E, which satisfy: $A \cap B \neq \emptyset$ and $A \cap B^C \neq \emptyset$. Then $A \cap bd(B) \neq \emptyset$.

Proof. If, $A \cap bd(B) = \emptyset$, as $E = int(B) \cup int(B^C) \cup bd(B)$, then $A = U \cup V$ where $U = A \cap int(B)$ and $V = A \cap int(B\text{^}\{C\})$ are disjoint open sets in a subspace A, then A is disconnected, contradiction.

As noted in the remark 9.1 b), in general, the connectedness is not stable by the union and the intersection. However, under suitable conditions, one can have stability for the union, as shown in the following proposition.

Proposition 9.13. A collection of connected parts, of a topological space, is stable for union, if the intersection of its elements is not empty.

Proof. Let $\{A_\alpha, \alpha \in \Delta\}$ be, a collection of connected parts, in a space E, and $A = \cup_{\alpha \in \Delta} A_\alpha$. Since for every $\alpha \in \Delta$, A_α is connected, there exist two disjoint open U_α, W_α in A_α such that, $A_\alpha \subset U_\alpha$, or $A_\alpha \subset W_\alpha$. Thus, there exist two open O_α and O'_α in E, such that $U_\alpha = A_\alpha \cap O_\alpha$, $W_\alpha = A_\alpha \cap O'_\alpha$. As for every $\alpha \in \Delta$, $A_\alpha \neq \emptyset$ then $\forall \alpha \in \Delta$, ($U_\alpha \neq \emptyset$ if $W_\alpha = \emptyset$) and ($W_\alpha \neq \emptyset$ if $U_\alpha = \emptyset$), then, if $W_\alpha = \emptyset$, $A = \cup_{\alpha \in \Delta} A_\alpha \subset \cup_{\alpha \in \Delta} U_\alpha \subset A \cap (\cup_{\alpha \in \Delta} O_\alpha) = A \cap O = U_A$ where

$O = \cup_{\alpha\in\Delta} O_\alpha$ is a nonempty open in E. Let $V_A = A \cap O'$, where $O' = \cup_{\alpha\in\Delta} O'_\alpha$ is an open in E, then $V_A = A \cap \cup_{\alpha\in\Delta} O'_\alpha = \cup_{\alpha\in\Delta} (A_\alpha \cap O'_\alpha) = \cup_{\alpha\in\Delta} W_\alpha = \emptyset$. Because, $A = U_A \cup V_A$ where $V_A = \emptyset$, then A is connected. The same argument used in the case when $W_\alpha \neq \emptyset$ and $U_\alpha = \emptyset$, yields to the same result.

Proposition 9.14 (Bolzano theorem). Let f be a continuous map, from a connected space E, into the space F. Then, the range $f(E)$ is connected.

Proof. Let f be a continuous map, from a connected space E, into the space F. If, the range $f(E)$ is disconnected, there exists a non trivial clopen part $B \subset f(E)$, so $f^{-1}(B)$ is a non trivial clopen in E, witch implies that E is disconnected, contradiction.

As, a direct consequence, of the proposition 9.14 and theorem 9.1, we have:

Corollary 9.1. The image of any interval in the space $\mathbb{R}$, by a continuous function from $\mathbb{R}$ into $\mathbb{R}$ is an interval.

Corollary 9.2. If, E is a connected space, f is a continuous function from E, into $\mathbb{R}$, and $a, b \in f(E)$. Then, for every every $\lambda \in [a, b]$, there exists $x \in E$, such that $f(x) = \lambda$.

Proof. By proposition 9.14, $f(E)$ is connected in $\mathbb{R}$, therefore, by theorem 9.1 $f(E)$ is an interval, then for $a, b \in f(E)$, $[a, b] \subset f(E)$, thus if $\lambda \in [a, b]$ then $\lambda \in f(E)$,so there exists $x \in E$, such that $f(x) = \lambda$.

Corollary 9.3. Let f be a continuous function, defined from the interval I in $\mathbb{R}$ into $\mathbb{R}$. The following assertions are equivalent:

$a)$ f is an homeomorphism from I into $f(I)$.

$b)$ f is one-to-one.

c) f is strictly monotone.

Proof. $a) \Longrightarrow b)$ clear. $b) \Longrightarrow c)$. If, f is not strictly monotone, there exist $a, b, c, d \in I$, such that $a < b$ and $f(a) > f(b)$; $c < d$ and $f(c) < f(d)$. The function $g: [0,1] \to \mathbb{R}$, defined by: $\forall t \in [0,1]$, $g(t) = f(ta + (1-t)c) - f(tb + (1-t)d)$ is obviously continuous and $g(0) < 0 < g(1)$, by corollary 9.2 there exists $u \in [0,1]$ such that $g(u) = 0$ i.e. $f(ua + (1-u)c) = f(ub + (1-u)d)$, but $ua + (1-u)c < ub + (1-t)d$, then f is not one-to-one, contradiction. $c) \Longrightarrow a)$. As f is continuous and strictly monotone, then f is one to one. It remains to prove that, the inverse function $h = f^{-1}: f(I) \to I$, is continuous or equivalently f is open. Let L be an open interval in I, there exists an open interval J in $\mathbb{R}$ such that $L = I \cap J$, then $f(L) = f(I \cap J) \subset f(I)$, since $f(L)$ is connected then $f(L)$ is an interval. As f is bijective and strictly monotone, $f(L)$ is also open, so h is continuous.

Theorem 9.2. The product topological spaces $E = \Pi_{\alpha\in\Delta} E_\alpha$ is connected iffy, $\forall \alpha \in \Delta$, the space E_α is connected.

Proof. Since E is connected and $\forall \alpha \in \Delta$, the projection $\pi_\alpha: E \to E_\alpha$ is continuous, by proposition 9.14, $\pi_\alpha(E) = E_\alpha$ is connected. Conversely, let $x, y \in E$, if y differs from x by only one component x_α, then $x, y \in Y = \Pi_{\beta<\alpha}\{x_\beta\} \times E_\alpha \times \Pi_{\beta>\alpha}\{x_\beta\}$, since by proposition 8.1 E_α and Y are homeomorphic, then Y is connected. If now, y is arbitrary, using $g)$ in proposition 9.1, we shall demonstrate that, any continuous map $f: E \to F = \{a, b\}$ is constant, i.e. $\forall x, y \in E, f(x) = f(y)$. If $f(x) = a$, because $\{a\}$ is open in the discrete space F and f is continuous then $f^{-1}(\{a\})$ is an open in E, containing a, there exists an elementary open set $O = O_{\alpha_1} \times \ldots \times O_{\alpha_n} \times \Pi_{\alpha\notin\{\alpha_1,\ldots,\alpha_n\}} E_\alpha$ (only a finite number $O_\alpha \neq E_\alpha, O_\alpha \in \tau_\alpha$), such that $x \in O \subset f^{-1}(\{a\})$, so $\forall i \in I = \{\alpha_1, \ldots, \alpha_n\}$ the component $x_i$of x belongs to O_i then $z = ((x_i)_{i\in I}, (y_i)_{i\notin I}) \in O$ and $f(z) = a$. The passage from z to y is given, by modifying only a finite number of components, whose indices are in I. According to the above step f is constant for any modified component, so $f(x) = f(y) = f(z)$ i.e. f is constant.

9.2-Components

Definition 9.3. Let E be a topological space and $x \in E$.

a) A component of E, is the maximal connected part in E.
b) A component of x, is the maximal connected part of E, containing x. It will be noted by $C(x)$.
Remark 9.2.
a) A connected part of a space E, not strictly included, in any other connected, part of this space, is a component of E.
b) C(x) is connected (see, proposition 9.13).
c) $C(x)$ is closed. Indeed, $cl(C(x))$ is connected then $cl(C(x)) \subset C(x) \subset cl(C(x))$.
d) In the connected space, $C(x) = E$.
Example 9.4.
a) Since in $\mathbb{Q}$, the singleton are connected, for $r \in \mathbb{Q}$, $C(r) = \{r\}$.
b) In the subspace $\mathbb{R}^*$, if $x \in]-\infty, 0[$, then $C(x) =]-\infty, 0[$ and if $x \in]0, +\infty[$ then $C(x) =]0, +\infty[$.
c) The subspace $\mathbb{Q}\times\mathbb{R}$ of the space $\mathbb{R}^2$ has for components, the lines $\{r\} \times \mathbb{R}$, where $r \in \mathbb{Q}$.
Proposition 9.15. The components, of a space E, form a separation of E.
Proof. Let $\{C_\alpha, \alpha \in \Delta\}$ be a collection of components of the space E. i). Suppose there exist different $\alpha, \beta \in \Delta$, such that $C_\alpha \cap C_\beta \neq \emptyset$, by the proposition 9.13, $C_\alpha \cup C_\beta$ is a connected containing C_α and C_β, which are the maximal connected part in E, contradiction. ii), let $x \in E$, since the singleton $\{x\}$ is connected, there exists $\alpha_0 \in \Delta$ such that $x \in \{x\} \subset C_{\alpha_0}$, so $x \in \cup_{\alpha\in\Delta} C_\alpha$.
From proposition 9.15, it follows that $C(x)$ is an equivalence class of x, i.e. the relation $\mathcal{R}$ on E, defined by: $\forall x, y \in E$, $x\mathcal{R}y \Leftrightarrow y \in C(x)$ is an equivalence relation.
Corollary 9.4. Any part A of a space E, is a union of a family of two by two disjoint connected parts.
Proof. By the proposition 9.15, the components of a subspace A form a separation of A.

9.3-Localy connected space
Definition 9.4. A topological space, is said to be locally connected, if any element of this space, has a fundamental system of connected neighborhoods.

Example 9.5.
a) In a space E, any element x∈E, is contained in its connected component. Indeed, $\forall x \in E$, $\{x\}$ is connected, since the connected component of x, i.e. $C(x)$, is the maximal connected set in E containing x, then $x \in \{x\} \subset C(x)$.
b) The space $\mathbb{R}$ is locally connected, since $\forall x \in \mathbb{R}$, $\exists \delta > 0$; such that the neighborhood $I(x, \delta)$ of x, is connected.
c) The discrete space E, is locally connected, since $\{x\}$ is a connected neighborhoods of x.
d) The cofinite space E, is locally connected, since the open sets in E is an infinite part, then any neighborhood is connected.
e) The subspace $\mathbb{Q}$, is locally disconnected, since $\forall\varepsilon > 0$; the neighborhoods $\mathbb{Q} \cap I(x, \varepsilon)$ of x in $\mathbb{Q}$, are not connected.
Remark 9.3.
a) A connected space is not necessary locally connected. In fact, the function $f:]0.1] \to \mathbb{R}^2$, defined by: $\forall x \in]0.1], f(x) = \left(x, sin\left(\frac{1}{x}\right)\right)$ is continuous and]0.1] is connected, by Bolzano theorem, $f(]0,1]) = A = \left\{\left(x, sin\left(\frac{1}{x}\right)\right), 0 < x \leq 1\right\}$ is connected in $\mathbb{R}^2$, therefore the subspace $cl(A)$ of the space $\mathbb{R}^2$, is connected. As, $cl(A) = A \cup B$, where $B = \{0\} \times]-1,1[$.

Since, $\forall y \in]-1,1[$, $(0,y)$ has not a connected neighborhood in B, then $cl(A)$ is not locally connected.
b) A locally connected part, in topological space, can not have its closure, locally connected. In fact, in the space $\mathbb{R}$, the subspace $A = \left\{\frac{1}{n}, n \in \mathbb{N}^*\right\}$ is locally connected, but $cl(A) = A \cup \{0\}$ is not locally connected, since $\forall \varepsilon > 0$, the neighborhood $cl(A) \cap]-\varepsilon, \varepsilon[$ of 0 in the subspace $cl(A)$ is an infinite part which is disconnected.

The following theorem gives a characterization, of the locally connected space.

Theorem 9.2. A space E, is locally connected⟺any connected component of any open in E is open.

Proof. Let C be, the maximal connected part of an arbitrary open O in E, and $x \in C \subset O$, since E is locally connected, there exists a connected $N \in \mathcal{N}(x)$. As $N \subset O$, then $N \subset C$, by N_4 in theorem 2.1 and proposition 2.2, C is an open neighborhood. Inversely, let $x \in E$ and $N \in \mathcal{N}(x)$, there exists an open O in E, such that $x \in O \subset N$, since a connected component $C(x)$ of x is an open in O, then $C(x)$ is a connected neighborhood of x, therefore E is locally connected.

Corollary 9.5. In a locally connected space, the connected component is open.

Proof. Let C be, the connected component of E and $x \in C$, as E is locally connected, there exists a connected $N \in \mathcal{N}(x)$, therefore $N \subset C$, then by N_4 in theorem 2.1 and proposition 2.2, C is an open neighborhood of x.

Corollary 9.6. The collection of the components of any nonempty open in the space $\mathbb{R}$, is finite or countable.

Proof. Let $O \in \tau_u$, since the space $\mathbb{R}$ is locally connected, by theorem 9.1; theorem 9.2 and corollary 9.4, the components of O, are two by two disjoint open intervals of $\mathbb{R}$. Since $cl(\mathbb{Q}) = \mathbb{R}$, their intersection with $\mathbb{Q}$ is not empty. Let I be one of these intervals, and $x \in I \cap \mathbb{Q}$, it is clear that $I = C(x)$ in O, as I contains at last one element of $\mathbb{Q}$, then the collection of the components of O, are finite or infinite countable.

9.3 Path and arc connectedness

Path and arc connectedness, related to the existence of certain continuous applications, from the unit interval, into a part of a space.

Definition 9.5. Let E be a topological space and the nonvoide part $A \subseteq E$.
a) Continuous application, from $[0,1]$ into A is said to be path.
b) The one-to-one path is called arc.
c) A is said to be path connected if, for every pair of points α and β in A, there exists a path f such that $f(0) = \alpha$ and $f(1) = \beta$. α (respectively β) is called the origin (respectively the end) of the path.
d) A is said to be locally path (respectively locally arc) connected if, every $x \in A$ has a fundamental system of path connected (respectively arc connected) neighborhoods.
e) A is said to be arc connected if, for every pair of points α and β in A, there exists an arc f such that $f(0) = \alpha$ and $f(1) = \beta$.
f) The maximal subsets with respect to path (respectively arc) connectedness are called path (respectively arc) components.
g) We say that a path crosses a part B of a space E if, there exists x in $[0,1]$ such that $f(x) \in B$

Remark 9.4. Since in the space $\mathbb{R}$, the interval $[a,b]$ is homeomorphic to $[0,1]$, it is equivalent to define a path on $[a,b]$.

Corollary 9.7. Let B be a part of the space E. If, a path crosses both B and B^C, then it crosses $bd(B)$.

Proof. Since, there exist $x, y \in [0,1]$, such that $f(x) \in B, f(y) \in B^C$, $[x,y] \subset [0,1]$, $f([0,1])$ is connected and $f([x,y]) \subset f([0,1])$, by proposition 9.12 $f([0,1]) \cap bd(B) \neq \emptyset$, then there exists $z \in [0,1]$, such that $f(z) \in bd(B)$.

Example 9.6

a) The space $\mathbb{R}$, is arc connected, since for every $a, b \in R$, the continuous function f from $[0,1]$ into $\mathbb{R}$, defined by $\forall x \in [0,1], f(x) = (b-a)x + a$, satisfies $f(0) = a$ and $f(1) = b$.

b) The subspaces $\mathbb{Q}$, and $\mathbb{Q}^C$, in the space $\mathbb{R}$ are not arc connected.

Corollary 9.8. Every arc connected space E is connected.

Proof. Suppose that E is disconnected, then there exist two disjoint open O and U in E, such that $E = O \cup U$. Let $\alpha \in O$ and $\beta \in U$, since E is arc connected, there exists an arc f such that $f(0) = \alpha$ and $f(1) = \beta$. Let $f([0,1]) = F$ witch is connected, as the nonvoide open $F \cap O$ and $F \cap U$ form a partition of F, then F is disconnected, contradiction.

Remark 9.10. The converse in corollary 9.8, is not true, returning to remark 9.3.a) for $\alpha \in A$ and $\beta \in B$ there is no arc connected.

Similar results of connected spaces are obviously valid for arcs connected, let us quote:

Corollary 9.10.

a) The image by a continuous map, of an arc connected space is arc connected.

b) A collection of arc connected parts, of a space, is stable for union, if the intersection of its elements is not empty.

c) The product space $E = \Pi_{\alpha\in\Delta} E_\alpha$ is connected$\Leftrightarrow$ $\forall\alpha \in\Delta,$ the space E_α is connected

It is also easy to verify that:

Corollary 9.10. Both connected and locally arc connected space are arc connected.

10-compacteness, separation and continuity

10.1 Compact space and separation

The closed and bounded interval $[a,b]$ in the space $\mathbb{R}$ satisfies the Borel-Lebesgue property, i.e. every open cover of $[a,b]$ has a finite subcover. Therefore, several important results in the space $\mathbb{R}$ are closely related to this type of interval as: Weierstrass-Bolzano theorem, Hein theorem, Weierstrass theorem, Rolle theorem,...etc. In this chapter, we will introduce a special and important topological spaces called compact spaces, whose closed and bounded intervals in the space $\mathbb{R}$ are a particular case.

Definition 10.1. The space E is called:

a) Compact, if it satisfies the Borel-Lebesgue property, i.e. every open cover of E has a finite subcover.

b) Countably compact, if every countably open cover of E has a finite subcover.

c) Sequentially compact, if every sequence has a convergence subsequence.

d) Lindelôf, if every open cover of E, has a countably subcover.

Definition 10.2. A nonvoide subset A, in a space E, is compact if, the subspace A is compact.

Proposition 10.1. A nonvoide subspace A, in a space E is compact$\Leftrightarrow$every collection of open of E, which cover A, has a subcover.

Proof. Let $\{O_\alpha, \alpha \in\Delta\}$ be a collection of opens in E, such that $A = \cup_{\alpha\in\Delta} O_\alpha$, since $\{A \cap O_\alpha, \alpha \in\Delta\}$ is the collection of open sets in A satisfying $A = \cup_{\alpha\in\Delta} (A \cap O_\alpha)$ and A is compact, there exists a finite open subcover $\{A \cap O_{\alpha_i}, 1 \leq i \leq n\}$. So, $A = \cup_{i=1}^{n} \left(A \cap O_{\alpha_i}\right) = A \cap \left(\cup_{i=1}^{n} O_{\alpha_i}\right)$, hence $A = \cup_{i=1}^{n} O_{\alpha_i}$. Conversely, let $\{U_\alpha, \alpha \in\Delta\}$ be a collection of opens in A, such that $A = \cup_{\alpha\in\Delta} U_\alpha$, there exists a collection $\{O_\alpha, \alpha \in\Delta\}$ of opens in E, such that $A = \cup_{\alpha\in\Delta} (A \cap O_\alpha) = A \cap (\cup_{\alpha\in\Delta} O_\alpha)$, where $U_\alpha = A \cap O_\alpha, \forall\alpha \in\Delta$, then $A = \cup_{\alpha\in\Delta} O_\alpha$. By hypothesis, there exists a finite open collection $\{O_{\alpha_i}, 1 \leq i \leq n\}$ such that $A = \cup_{i=1}^{n} O_{\alpha_i}$, witch implies that $A = A \cap (\cup_{i=1}^{n} O_{\alpha_i}) = \cup_{i=1}^{n} (A \cap O_{\alpha_i})$, then A is compact.

Example 10.1.

$a)$ 2D-space, is Lindelôf (cf, proposition 5.4).

$b)$ The finite space is compact.

$c)$ The subset $A = \left\{\frac{1}{n}, n \in \mathbb{N}^*\right\}$ in the space $\mathbb{R}$ is not compact, since their subsequences converges to $0 \notin A$. But $A \cup \{0\}$ is compact, because in the hausdorff space if the sequence$\{x_n\}$ converges to $x \in E$, the subset $A = \{x_n\} \cup \{x\}$ is compact, in fact if, the collection $\{O_\alpha, \alpha \in \Delta\}$ of open in E, is a cover of A, there is some $\alpha \in \Delta$, such that $x \in O_\alpha$, then there existes $n_0 \in \mathbb{N}$, such that for every $n \in \mathbb{N}, n > n_0, x_n \in O_\alpha$ so $A \subset \left(\cup_{i=1}^{n_0-1} O_{\alpha_i}\right) \cup O_\alpha$ by proposition 10.1, A is compact.

$d)$ A discrete space is compact (respectively Lindelôf)⟺it is a finite (respectively countable) space, indeed if E is finite, then $E = \cup_{\alpha \in \Delta} \{x_\alpha\}$, where the indices set Δ is finite and if E is countable, then $E = \cup_{\alpha \in \mathbb{N}} \{x_\alpha\}$.

$e)$ The space $\mathbb{R}$ is Lindelôf, but it is not compact. In fact, $\mathbb{R} = \cup_{n \in \mathbb{N}^*}]-n, n[$, but there is not a subcovert of these open cover.

$f)$ The discrete $\mathbb{R}$, is not compact. In fact, there is no subcovert of the open cover $\{\{x\};\ x \in \mathbb{R}\}$.

$g)$ Let E be a non countable set $a \in E$ and $F = \{a\}^C$, the family $\tau_a = \{\mathcal{P}(F), E\}$ is a topology on E and (E, τ_a) is Lindelôf.

Lemma 10.1. Let E be a topological space, the following assertions are equivalents:

C_1-E is compact.

C_2-Every family of a closed subsets, whose intersection in empty, has a finite subfamily, whose intersection is empty (the finite intersection property).

Proof. $C_1 \Longrightarrow C_2$. Let $\{F_\alpha, \alpha \in \Delta\}$ be a family of a closed subset, such that $\cap_{\alpha \in \Delta} F_\alpha = \emptyset$, then $(\cap_{\alpha \in \Delta} F_\alpha)^C = \cup_{\alpha \in \Delta} {F_\alpha}^C = E$, since E is compact there exists a finite open subcovert $\left\{{F_{\alpha_i}}^C, 1 \leq i \leq n\right\}$ of the open cover $\{F_\alpha, \alpha \in \Delta\}$, therefore $\cup_{i=1}^{n} {F_{\alpha_i}}^C = E$, so $\left(\cup_{i=1}^{n} {F_{\alpha_i}}^C\right)^C = \cap_{i=1}^{n} F_{\alpha_i} = \emptyset$. $C_2 \Longrightarrow C_1$. Let $\{O_\alpha, \alpha \in \Delta\}$ be an open cover of E, then $(\cup_{\alpha \in \Delta} O_\alpha)^C = \cap_{\alpha \in \Delta} {O_\alpha}^C = \cap_{\alpha \in \Delta} F_\alpha = \emptyset$ where $F_\alpha = {O_\alpha}^C$ witch is closed, by C_2 there exists a finite closed subsets $\{F_{\alpha_i}, 1 \leq i \leq n\}$ such that $\cap_{i=1}^{n} F_{\alpha_i} = \emptyset$, so $\left(\cap_{i=1}^{n} F_{\alpha_i}\right)^C = \cup_{i=1}^{n} {F_{\alpha_i}}^C = \cup_{i=1}^{n} O_{\alpha_i} = E$, then E is compact.

Corollary 10.1. In the compact space, any nonempty closed collection, totally ordered by inclusion, has a nonempty intersection. In particular any intersection of nonvoide decreasing sequence of closed sets has a nonempty intersection.

Proof. If the collection has an empty intersection, as the space is compact, it has a finite subcollection whose empty intersection, contradiction with the finite subcollection has an nonempty minimum which is its intersection.

We will give, in the form of a lemma, an equivalent of the Weierstrass-Bolzano theorem.

Lemma 10.2. Any infinite part of a compact space has at last an accumulation point. Equivalently, any part of a compact space without accumulation points is finite.

Proof. Suppose that there exists an infinite part A of space E, which has no accumulation point, then for every $x \in E$, there exists an open O_x containing x and only one element of A (this element is $x \in A$). Since, the family $\{O_x, x \in E\}$ is an open cover of E, which is compact, it has a finite subcover $\{O_{x_i}, 1 \leq i \leq n\}$, then $A \subset \cup_{i=1}^{n} O_{x_i}$ and A has at most n elements, so it is finite, contradiction.

As the accumulation point is an adherent element, by the proposition 7.5 and the lemma 10.2, we have:

Corollary 10.2. Any sequence in the compact 1D-space has a convergence subsequence.
Proof. Let $A = \{x_n\}$ be a sequence in the space E, as A is a countable part it is an infinite part in the compact space, so by lemma 10.2, A has an accumulation point $x \in cl(A)$. Because E is 1D-space, by proposition 7.5, there is sequence $\{x_m\} \subset A$, which converges to x i.e. there is a convergence subsequence $\{x_{\varphi(n)}\}$ of the sequence $\{x_n\}$
Corollary 10.3. In the compact space, if a sequence has only one limit point, it converges towards this limit point.
Proof. By proposition 7.8, the set of the limit points, of the sequence $\{x_n\}$, in the arbitrary space E, is the closed $A = \cap_{n\geq 0} cl(A_n)$, where $A_n = \{x_k;\ k \geq n\}$, for all $n \in \mathbb{N}$. Let a be the unique limit point of A and $N \in \mathcal{N}(a)$ an open neighborhood, because $\{cl(A_n) \cap N^C, n \in \mathbb{N}\}$ is a decreasing sequence of closed sets whose empty intersection, if not there exists $x \in cl(A_n) \cap N^C$, $\forall n \in \mathbb{N}$, then $x \in A$ and $x \neq a$, contradiction. Therefore, there exists $n_0 \in \mathbb{N}$ such that $cl(A_{n_0}) \cap N^C = \emptyset$, so $cl(A_{n_0}) \subset N$ as, for every $n \geq n_0, cl(A_n) \subset cl(A_{n^0})$), which implies that, for every $n \geq n_0, cl(A_n) \subset N$, hence $x_n \in N$, it follows that $x_n \longrightarrow a$.
Note that, in every space, neither direction of the equivalence holds, betwin the compact space and the sequentially compact space.
Theorem 10.1. The Lindelôf sequentially compact space is compact.
Proof. Suppose that, the space E is not compact, there is some collection $\{O_\alpha, \alpha \in\Delta\}$ of open cover of E, which has no finite subcover, as E is lindelôf, there is some countable open subcover $\{O_n, n \in \mathbb{N}\}$ of E. The sequence $\{x_n\}$ defined by: for every $n \in \mathbb{N}^*, x_n \notin\cup_{i=1}^{n} O_i$, has by assumption a subsequence $\{x_{\varphi(n)}\}$ which converges towards $a \in E$, as $\{O_n, n \in \mathbb{N}\}$ covers E, there is some $p \in \mathbb{N}$ such that $a \in O_p$, because $x_{\varphi(n)} \notin\cup_{i=1}^{\varphi(n)} O_i$ then, for every $\varphi(n) \geq p, x_{\varphi(n)} \notin O_p$, contradiction with the definition of a. Then E is compact.
Remark 10.1. As by proposition 5.4, 2D-space is Lindelôf, then the theorem 10.1 is valid, in 2D-space i.e. If, the 2D-space E is sequentially compact, then E is compact.
With the same arguments used in lemma 10.1, we have:
Corollary 10.4. Let E be a topological space, the following assertions are equivalents:
CC_1-E is countably compact.
CC_2-Every family of a countably closed subsets, whose intersection is. empty, has a finite subfamily, whose intersection is empty (finite countably intersection axiom).
It is obvious that, any compact space is countably compact, but the reverse is not always true.
A condition ensuring that, countable compactness implies compactness is given by:
Lemma 10.3. Any countably compact 2D-space, is compact.
Proof. By the proposition 5.4, any open cover $\{O_\alpha, \alpha \in\Delta\}$, of the 2D-space E, has a countable subcover $\{O_{\alpha_n}, \mathrm{n} \in \mathbb{N}\}$, since E is countably compact there exists a finite $I \subset \mathbb{N}$, such that $E = \cup_{n\in I} O_{\alpha_n}$, then E is compact.
Lemma 10.4. A compact Hausdorff space E, is normal.
Proof. It suffices to demonstrate that E is a T_3-space. By using proposition 5.9, c) it remains to demonstrate that, if $x \in E$ and O is an open containing x, O contains a closed neighborhood of x. Let $x \in E$ and O an open containing x, suppose that every $N \in \mathcal{N}_f(x)$ (the set of a closed neighborhoods of x), $N \subset O^C = F$ and consider a finite family of a closed neighborhoods of x, denoted $\mathcal{M}_f(x)$, as $\cap_{N\in\mathcal{M}_f(x)} (N \cap F) = \left(\cap_{N\in\mathcal{M}_f(x)} N\right) \cap F = M \cap F \neq \emptyset$ where $\cap_{N\in\mathcal{M}_f(x)} N = M \in \mathcal{M}_f(x)$ and E is compact by C_2 in lemma 10.1, $\left(\cap_{N\in\mathcal{N}_f(x)} N\right) \cap F \neq \emptyset$, since by proposition 5.8, $\cap_{N\in\mathcal{N}_f(x)} N = \{\mathrm{x}\}$, then $\{x\} \cap F \neq \emptyset$ i.e. $x \in F$ contradiction.
Compactness is weakly hereditary:

Proposition 10.2. Any closed part in a compact space is compact.
Proof. Let A be a subspace of the space E, and let $\{G_\alpha, \alpha \in\Delta\}$ be a collection of closed set in A such that $\cap_{\alpha\in\Delta} G_\alpha = \emptyset$, as there exists a collection $\{F_\alpha, \alpha \in\Delta\}$ of closed sets in E, such that $\cap_{\alpha\in\Delta} G_\alpha$=$\cap_{\alpha\in\Delta} (A \cap F_\alpha)$=$\emptyset$, where G_α=$A \cap F_\alpha$, $\forall\alpha \in\Delta$, as A is closed in E, by proposition 6.2, the collection $\{A \cap F_\alpha, \forall\alpha \in\Delta\}$ is closed in a compact E, therefore there exists a finite subcollection $\{A \cap F_{\alpha_i}, 1 \le i \le n\}$ with $\cap_{i=1}^{n} \left(A \cap F_{\alpha_i}\right) = A \cap \left(\cap_{i=1}^{n} F_{\alpha_i}\right) = \emptyset$ or $\cap_{i=1}^{n} G_{\alpha_i} = \emptyset$, so the subspace A is compact.
Proposition 10.3. In Hausdorff space, every compact subspace is closed.
Proof. Let A be a subspace of the space E. We will show that A^C is open. Let $x \in A^C$, since E is Hausdorff for every $y \in A$, there exist two disjoint open $O_x \ni x$ and $O_y \ni y$, as $A = \cup_{y\in A} O_y$ and A is compact by proposition 10.1, there exists a finite collection $\{O_{y_i}, 1 \le i \le n\}$ such that $A =\cup_{i=1}^{n} O_{y_i}$. As $O =\cap_{i=1}^{n} O_{x_i}$, is an open containing x, and $O\subset\cap_{i=1}^{n} {O_{y_i}}^C$=$\left(\cup_{i=1}^{n} O_{y_i}\right)^C = A^C$, then A^C is a neighborhood of the arbitrary x, by proposition 2.2, A^C is open and A is closed.
Note that, there exists a space, which is not Hausdorff, but every compact subsets in the space is closed.
Example 10.2. It is shown in example 5.4 b) that, the cocountable space E is not Hausdorff, but any compact subspace in this space is closed. Note that the infinite subsets of E are not compact, indeed, if A is an infinite subspace of E, then $A = B \cup \{a_1, a_2, \dots\}$, where $\{a_1, a_2, \dots\}$ is a countably infinite subspace of A and B its complement in A. Let for avery $n \in \mathbb{N}^*$, $O_n = \{a_{n+1}, a_{n+2}, \dots\}^C$, then the collection $\{O_n, n \in \mathbb{N}^*\}$ of open sets in E, is a cover of A witch has not a subcover, therefore A is not compact. But, the finite or countable subsets of E are compact and closed. Therefore every compact sets in the cocountable space is closed.
Corollary 10.5. If, every compact subspace in a space E is closed, E is T_1.
Proof. Let x an element of a space E, as $\{x\}$ is compact, by the assumption it is closed, using proposition 5.6, E is T_1.
The following example shows that the converse of the corollary 10.5 is false.
Example 10.3. It is shown in example 5.4 a) that the cofinite space$E$ is T_1. But there exists a compact subset of E, which is not closed. Let A be a nonempty subset of E, $\mathcal{U}$ any open cover of A it is clear that any $U \in \mathcal{U}$ is infinite and U^C is finite, hence it contains at most a finite number of points, say n of A. The number of open sets from $\mathcal{U}$ needed to cover these n points does not exceed n, hence the maximum number of sets from $\mathcal{U}$ needed to cover A is $n + 1$. Therefore A is compact subset of E with finite complements are not closed.
Lemma 10.5.
a) Compactness of subspaces in any space is stable by the finite union.
b) Compactness of subspaces in Hausdorff space is stable by intersection
Proof. a) Let I be the finite set of indices, $\{A_i, i \in I\}$ be a finite family of compact subspaces, $A =\cup_{i\in I} A_i$ and let $\{O_\alpha, \alpha \in\Delta\}$ be a collection of open subsets in the space E, witch cover A, since $\forall i \in I$, $A_i =\cup_{\alpha\in\Delta} O_\alpha$, and A_i is compact, by proposition 10.1, there exists a finite subset $\{\alpha_{k,i}, 1 \le k \le n\}$ of Δ, such that $\forall i \in I$, A_i=$\cup_{k=1}^{n} O_{\alpha_{k,i}}$, so $A =\cup_{i\in I} \left(\cup_{k=1}^{n} O_{\alpha_{k,i}}\right)$ witch is a finite open subcover, then A is compact. b) Let $\{A_\alpha, \alpha \in\Delta\}$ be a family of compact subspaces, $A =\cap_{\alpha\in\Delta} A_\alpha$, as $\forall\alpha \in\Delta$, $A \subset A_\alpha$ and by proposition 10.3, A_α is closed, A is obviously closed, proposition 10.2 implies that A is compact.

Being given, the importance of closed and bounded intervals, in real analysis, we will present here, one of the demonstrations of their compactness using the Borel-Lebesgue property.
Theorem 10.2. (Borel-Lebesgue). The bounded and closed interval in the space $\mathbb{R}$ is compact.

Proof. Let a, b are two elements or $\mathbb{R}$ and let $\{O_\alpha, \alpha \in\Delta\}$ be a family of opens in $\mathbb{R}$, which cover $A = [a, b]$, we will prove according to the proposition 10.1 that there exist a subcover of A. For that, consider the subset H of A defined by: $x \in H \Leftrightarrow$there is a finite $I \subset\Delta$, such that $[a, x] = \cup_{i\in I} O_i$. As [a,a]={a} then $a \in H$. We will prove that, H is clopen in the connexe subspace $A = [a, b]$, then by proposition 9.1 d), $A = H$. Let us show that H is an open in the subspace A. Let $x \in H$, by construction there exists a finite $I \subset\Delta$, such that $[a, x] = \cup_{i\in I} O_i$, there is some $i \in I$, such that $x \in O_i$, so $x \in A \cap O_i$ which implies that, $H \subset A \cap O_i$. If now $y \in A \cap O_i$, $y \in A$ and $y \in [a, x]$, because [a,x] is an interval, then $[a, y] \subset [a, x]$, it follows that $y \in H$, hence H is open in A. It remains to prove that. $cl(H) \subset H$. Let $z \in cl(H)$, as $cl(H) \subset cl(A) = A$, then, there is some $\beta \in\Delta$ such that $z \in A \cap O_\beta$, it follows that $(A \cap O_\beta) \cap H = A \cap (O_\beta \cap H) \neq \emptyset$, thus $O_\beta \cap H \neq \emptyset$. Let $s \in O_\beta \cap H$, as $s \in H$, there is a finite $I \subset\Delta$, such that $[a, s] = \cup_{i\in I} O_i$. It is obvious that, if $z \leq s, z \in H$, when $s < z$, $[a, z] = [a, s] \cup [s, z]$. As, $s, z \in O_\beta$ then $[s, z] \subset O_\beta$ Then $[a, z] = (\cup_{i\in I} O_i) \cup O_\beta$, which is a finite subcover, it follows that $z \in H$, so $A = H$, therefore, there exists a subcover of A which implies that A is compact.

Corollary 10.6. The part A in the space $\mathbb{R}$ is compact⟺it is bounded and closed.

Proof. As A is bounded, then A is contained in some interval $[a, b]$ witch is compact, because A is closed, by proposition 10.2, A is compact. For the inverse, as A is compact in a Hausdorff space $\mathbb{R}$, by proposition 10.3 A is closed, also A is bounded, in fact, for every $x\in A$, there is some $n \in \mathbb{N}^*$, such that $x \in]-n, n[$, so $A \subset\cup_{n\in\mathbb{N}^*}]-n, n[$, as A is compact, there is a finite set $N \subset \mathbb{N}^*$, such that $A \subset\cup_{n\in N}]-n, n[=]max_{n\in N}(-n), max_{n\in N}(n)[$, then A is bounded.

10.2 Compact space and continuity

A property is said to be a topological property (or topological invariant) if whenever one space possesses a given property, any homeomorphic to it also possesses the same property. Similarly, property preserved by continuous (respectively open or closed) functions are called continuous (respectively open or closed) property or continuous (respectively open or closed) invariant. Compactness are continuous property.

Proposition 10.4. let E, F are two topological space and let f:E⟶F be a continuous map: If, A is a compact subspace in E, then $f(A)$ is a compact subspace in F.

Proof. Let $\{U_\alpha, \alpha \in\Delta\}$ be a collection of opens in F, which covers $f(A)$, by continuity of f the elements of the collection $\{f^{-1}(U_\alpha), \alpha \in\Delta\}$ are open in E as $f(A) = \cup_{\alpha\in\Delta} U_\alpha$, and $A \subset f^{-1}(f(A)) = f^{-1}(\cup_{\alpha\in\Delta} U_\alpha) = \cup_{\alpha\in\Delta} f^{-1}(U_\alpha)$, then $\{ f^{-1}(U_\alpha), \alpha\in\Delta\}$ covers A, which is compact, there exists a finite set $I \subset\Delta$, such that $A \subset\cup_{\alpha\in I} f^{-1}(U_\alpha)$, then $f(A) \subset f(\cup_{\alpha\in I} f^{-1}(U_\alpha)) \subset\cup_{\alpha\in I} f(f^{-1}(U_\alpha)) \subset\cup_{\alpha\in I} U_\alpha$, by proposition 10.1 $f(A)$ is compact.

As a direct consequence of the proposition 10.4, we have:

Corollary 10.7. If, a map f from a compact space E into a space F is continuous. Then, $f(E)$ is a compact subspace in F.

Proof. It suffices to take $A = E$, in the proposition 10.4.

Corollary 10.8. If, a map f from a compact space E, into a Hausdorff space F is continuous, then f is closed. Moreover, if f is one to one, then f is an homeomorphism.

Proof. It suffices to demonstrate that f is closed. If, C is closed in the compact space E, by proposition 10.2, C is compact, as f is continuous from C into F, by proposition 10.4, $f(C)$ is compact in a Hausdorff space F, by proposition 10.3, $f(C)$ is closed. As, the inverse map f^{-1} from F into E is continuous, f is then an homeomorphism.

Theorem 10.3 (Heine theorem). A continuous function, from a compact space E into the space $\mathbb{R}$, is bounded and reaches these bounds.

Proof. By corollary 10.7, $f(E)$ is compact, the corollary 10.6 implies that $f(E)$ is closed and bounded in $\mathbb{R}$. Let $M = supf(E)$, then $\forall\varepsilon > 0$, there is some $x_\varepsilon \in E$ such that $M - \varepsilon <$

$f(x_\varepsilon) \leq M < M + \varepsilon$, so $\forall \varepsilon > 0, I(M, \varepsilon) \cap f(E) \neq \emptyset$, therefore $M \in cl(f(E)) = f(E)$, then there exits $x_1 \in E$, such that $M = f(x_1)$. By the same, if $m = inf f(E)$, $\forall \varepsilon > 0$, there is some $x_\varepsilon \in E$ such that $m - \varepsilon < m \leq f(x_\varepsilon) < m + \varepsilon$, so $\forall \varepsilon > 0, I(m, \varepsilon) \cap f(E) \neq \emptyset$, therefore $m \in cl(f(E)) = f(E)$, then there is $x_2 \in E$, such that $m = f(x_2)$.
Remark 10.2. In the case when, the space is not compact, the continuous function can be bounded but not reaches these bounds or not bounded. For example, the function $f: \mathbb{R} \longrightarrow \mathbb{R}_+^*$; $x \longmapsto e^x$ is continuous but not bounded above. While the function $f: \mathbb{R} \longrightarrow \mathbb{R}$; $x \longmapsto \frac{x}{1+|x|}$ is continuous and bounded, but not reaches these bounds.

Subsequently, we will give, the proof of the Tyckonoff theorem, concerning the compactness of any product of compact spaces. For that, we need to recall the famous Zorn's lemma, which is used to prove Alexander subbase theorem bellow. Let $(E, \leq)$ be a **partially ordered set** i.e. a binary relation $\leq$ is: reflexive, antisymmetric, transitive, and E may contain elements x, y such that neither $x \leq y$ nor $y \leq x$ holds, such pair of elements is said to be incomparable. One example is $E = \{\{1\}, \{2\}, \{1,2\}\}$, with set inclusion $\subset$ as a partial ordering. It is clear that $\{1\}$ and $\{2\}$ are not comparable. A pair $x, y \in E$ are comparable if $x \leq y$ or $y \leq x$ or both. A partially ordered set E is said to be **totally ordering**, where every pair of its elements is comparable. An **upper bound** (if it exists) of a subset A of a partially ordered set E is an element $u \in E$ such that $x \leq u$ for all $x \in A$. Note that, the upper bound need not be an element of A, but it must be an element of E. A **maximal element** (if it exists) of a partially ordered set E is an element $M \in E$, such that, if $M \leq x$ for some $x \in E$, then $x = M$, in other words, there is no $x \in E$ such that $M \leq x$ but $x \neq M$. If a maximal element exists for a totally ordered set, then it must be unique. Consider the set A=$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$, with set inclusion $\subset$ as a partial ordering. The maximal elements are $\{1,2\}$ and $\{3\}$ and if, we view A as a subset of the set $E = \{1,2,3\}$, then the upper bound of A is the element $\{1,2,3\}$. The partially ordered set E is called **inductive** if, any totally ordered part of E, has an upper bound. We are now ready to formulate Zorn's lemma.
Zorn's lemma. Every nonempty inductive, partially ordered set, has at last one maximal element.
Lemma 10.6. (Alexander subbase theorem). Let (E, τ) be a topological space and $\mathcal{S}$ a subbases of τ. If every open cover of E by the elements of $\mathcal{S}$, has a subcover, then E is compact.
Proof. Suppose that, E is not compact. Then, there is some open cover of E, with no finite subcover. Let $\mathcal{F}$ be, the collection of all open covers of E, with no finite subcover, provided with a partially ordered set inclusion. Let $\mathcal{A} = \{\mathcal{U}_\alpha, \alpha \in \Delta\}$ be the collection of subsets of $\mathcal{F}$ witch is totally ordered, then: its upper bound, $\mathcal{U} = \cup_{\alpha \in \Delta} \mathcal{U}_\alpha$ has no finite open subcover. Indeed, if $\mathcal{U}$ contains a finite open subcover, $\{U_i, 1 \leq i \leq n\}$, then for each i, there exists $\alpha_i \in \Delta$, such that $U_i \in \mathcal{U}_{\alpha_i}$, as $\mathcal{A}$ is totally ordering, there is some $\alpha_0 \in \Delta$, such that $\{U_i, 1 \leq i \leq n\} \subset \mathcal{U}_{\alpha_0}$, therefore this finite subcover cannot cover E . It follows that, $\mathcal{A}$ is nonempty inductive, partially ordered subset of $\mathcal{F}$, by Zorn's lemma, there exists $\mathcal{M} \in \mathcal{F}$ such that for every $\alpha \in \Delta$, $\mathcal{U}_\alpha \subset \mathcal{M}$. As, the set $\mathcal{W} = \mathcal{M} \cap \mathcal{S}$ is an open cover of E. If not, there is some $x \in E$, that is not in any element of $\mathcal{W}$, as $\mathcal{M}$ covers E, there exists $O \in \mathcal{M}$, containing x. As $\mathcal{S}$ is a subbasis of τ, there are a finite open $S_1, \ldots, S_k \in \mathcal{S}$ such that $x \in \cap_{j=1}^{k} S_j \subset O$. Because, for every $j \in \{1, \ldots, k\}$, $S_j \notin \mathcal{M}$, in not, x would be an element of some member of $\mathcal{W}$. By maximality of $\mathcal{M}$, for each j, the open cover $\mathcal{M} \cup \{S_j\}$ of E, contains a finite subcover $\mathcal{M}_j \cup \{S_j\}$ where $\mathcal{M}_j$is a finite union of sets in $\mathcal{M}$, then, for each j $E = \mathcal{M}_j \cup \{S_j\}$, so $E \subset \cap_{j=1}^{k} (\mathcal{M}_j \cup S_j) = (\cap_{j=1}^{k} S_j) \cup (\cap_{j=1}^{k} \mathcal{M}_j) \subset O \cup (\cup_{j=1}^{k} \mathcal{M}_j)$, witch is impossible by construction of $\mathcal{M}$. Because, $\mathcal{W}$ is an open cover of E containing in $\mathcal{S}$, by assumption it has a

finite sucover, this is a contradiction, with the fact that, $\mathcal{W}$ is contained in $\mathcal{M}$. Therefore, the collection $\mathcal{F}$ must be empty, so that E is compact.
Theorem 10.4 (Tychonoff theorem). The product space, $E = \Pi_{\alpha\in\Delta}E_\alpha$ is compact$\Leftrightarrow$ $\forall\alpha \in\Delta$ the space E_α is compact.
Proof. As $\forall\alpha\in\Delta$, the coordinate projection $\pi_\alpha: E \longrightarrow E_\alpha$ is continuous, then if E is compact, by proposition 10.4, $\pi_\alpha(E) = E_\alpha$ is compact. Conversely, we will prove that if, $\forall\alpha \in\Delta, E_\alpha$ is compact, then $E = \Pi_{\alpha\in\Delta}E_\alpha$ is compact, by using Alexander subbase theorem and the following lemma:
Lemma 10.7. Let $\{(E_\alpha,\tau_\alpha),\ \alpha\in\Delta\}$ be a collection of a compact topological spaces and let $E = \Pi_{\alpha\in\Delta}E_\alpha$ be a product space. Then, any open cover $\mathcal{U} = \{\pi_\alpha^{-1}(O), O \in \tau_\alpha\}$ of E, has a finite subcover.
Proof. Let, for every $\alpha \in\Delta, \mathcal{U}_\alpha = \{O \in \tau_\alpha, \pi_\alpha^{-1}(O) \in \mathcal{U}\}$. We claim that, there is some $\alpha_0\in\Delta$, such that $\mathcal{U}_{\alpha_0}$ covers E_{α_0}. If not, for every $\alpha \in\Delta$, there exists $x_\alpha\in E_\alpha$, witch is not containing in any $O \in \mathcal{U}_\alpha$, so $x_\alpha \in O^C$, therefore for every $O \in \mathcal{U}_\alpha$, $\pi_\alpha^{-1}(\{x_\alpha\}) \subset \pi_\alpha^{-1}(O^C) = (\pi_\alpha^{-1}(O))^C$, then $\pi_\alpha^{-1}(\{x_\alpha\})$ is not containing in any $\pi_\alpha^{-1}(O) \in \mathcal{U}$, witch is by assumption a cover of E, contradiction. Choose α such that $\mathcal{U}_\alpha$ is a cover of E_α, by compactness, there are a finite subcover $O_1, \dots, O_n$,, as $E = \pi_\alpha^{-1}(E_\alpha) = \pi_\alpha^{-1}(\cup_{i=1}^n O_i) = \cup_{i=1}^n \pi_\alpha^{-1}(O_i)$, then $\{\pi_\alpha^{-1}(O_1), \dots, \pi_\alpha^{-1}(O_n)\}$ is a finite subcover of E.
To finish the proof of the theorem, take as a subbase in the product topology E, the collection $\mathcal{S} = \{\pi_\alpha^{-1}(O),\ O \in \tau_\alpha\}$, where $\alpha \in\Delta$. Any collection of $\mathcal{S}$ which covers E, by lemma 10.7 has a finite subcover, thus by Alexander subbase theorem, E is compact.
Corollary 10.9. The part A in the space $\mathbb{R}^n$ is compact$\Leftrightarrow$it is bounded and closed.
Proof. As A is compact in the Hausdorff $\mathbb{R}^n$, by proposition 10.3, A is closed and it is also bounded, if not $\forall n \in \mathbb{N}^*$, there is some $x_n \in A$ such that, $\|x_n\| > n$, so $\{x_n\}$ has no convergent subsequences, as by corollary 10.2, A is sequentielly compact, contradiction.
Conversely, as A is bounded, then $A \subset \Pi_{i=1}^n[a_i, b_i]$, where $\forall 1 \le i \le n$, the constants $a_i, b_i \in \mathbb{R}$. Borel-Lebesgue and Tychonoff theorems say that $\Pi_{i=1}^n[a_i, b_i]$ is compact, because A is closed, by proposition 10.2, A is compact.
Example 10.4.
$a)$ The ellipse $E = \left\{(x, y) \in \mathbb{R}^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\right\}$ where, the constants $a, b \in \mathbb{R}^*$, is a compact in $\mathbb{R}^2$. Because, the function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ is continuous, then $A = f^{-1}(\{0\})$ is closed. Moreover $\forall(x, y) \in A, \frac{x^2}{a^2} \le 1$ and $\frac{y^2}{b^2} \le 1$, then $(x, y) \in [-a, a] \times [-b, b]$, so $A \subset [-a, a] \times [-b, b]$. A is closed and bounded in $\mathbb{R}^2$, by corollary 10.9, it is compact.
$b)$ Let $\mathbb{R}$ be a space, the sphere $S_{n-1} = \{x = (x_1, \dots, x_i, \dots, x_n) \in \mathbb{R}^n, \sum_{i=1}^n x_i^2 - 1 = 0\}$ is compact in $\mathbb{R}^n$. Because, the function $f: \mathbb{R}^n \longrightarrow \mathbb{R}, x \longmapsto f(x) = \sum_{i=1}^n x_i^2 - 1$ is continuous, then $S_{n-1} = f^{-1}(\{0\})$ is closed. Moreover, $\forall 1 \le i \le n, x_i^2 \le 1$, then $S_{n-1} \subset \Pi_{i=1}^n[-1,1]$ witch is compact, thus S_{n-1} is compact.
$c)$ As the circle S_1 is compact, then the torus $(S_1)^p, p \in \mathbb{N}^*$ is compact.
Definition10.3. Let $\{(E_\lambda, \tau_\lambda), \lambda \in\wedge\}$ be a family of spaces, such that for all $\lambda \in\wedge$, $(E_\lambda, \tau_\lambda)$ is homeomorphic to $[0,1]$ usual. Then the product space $\Pi_{\lambda\in\wedge}E_\lambda$ is denoted $I^\wedge$ and it is called a cub.
As a consequence of the Tychonoff's theorem, and because the subspace $[0,1]$ is compact and Hausdroff. We have:
Corollary 10.10. For any set $\wedge$, the cube $I^\wedge$ is compact and Hausdroff.

10.3 Locally compact space

Starting from the fact that, in non compact space $\mathbb{R}$, the closure of an open interval is compact, we will define a new topological space:

Definition 10.3. The part of topological space is called relatively compact, if its closure is compact.

It is easy to see that:

a) Any interval in the space $\mathbb{R}$ is relatively compact (since its closure is closed and bounded)

b) Any compact is relatively compact ($E = cl(E)$).

c) Any part of the relatively compact part is relatively compact (see, proposition 10.2).

d) Any part of a compact space is relatively compact (see, proposition 10.2).

e) The finite union of relatively compact parts is relatively compact ($cl(A \cup B) = cl(A) \cup cl(B)$ witch is compact).

f) The intersection of relatively compact parts is relatively compact ($cl(\cap_{\alpha\in\Delta} A_\alpha) \subset \cap_{\alpha\in\Delta} cl(A_\alpha)$ witch is compact)

Definition 10.4. The space E, is said to be locally compact, if every element of E, has a compact neighborhood.

It is clear that, in a space E, if an element has a basis of compact neighborhoods, then it has a compact neighborhood, therefore E is locally compact. For the reverse, we have:

Proposition 10.5. In locally compact, Hausdorff space E, any element has a basis of compact neighborhoods. Therefore E is regular.

Proof. By assumption, every $x \in E$ has a compact $U \in \mathcal{N}(x)$, as E is Hausdorff, by proposition 10.3, U is closed. If now, we take an open $O \in \mathcal{N}(x)$, then $U \cap O$ is an open neighborhood of x in U. Because, from proposition 6.5, c) and lemma 10.4, U is a normal space, by proposition 5.9, b) $U \cap O$ contains a closed neighborhood W of x in the subspace U, so W is compact. As, there is some closed C in E, such that $W = U \cap C$, then by proposition 6.2, W is a closed neighborhood of x in E, containing in O. Therefore W is both closed and compact neighborhood of x in E, so by proposition 5.9, b) E is regular.

Example 10.5.

a) A compact space E, is locally compact. In fact, E is an open neighborhood of each its elements, as it is compact then, it is locally compact.

b) The discret space is locally compact. In fact, any element is open and compact.

c) The space $\mathbb{R}$ is locally compact, since $\forall x \in \mathbb{R}$, there is $n \in \mathbb{N}^*$ such that $x \in [-n, n]$ witch is a compact neighborhood.

d) In the space $\mathbb{R}$, $\forall a, b \in \mathbb{R}$, $]a, b[$ is locally compact. It suffieces to see that for every $x \in]a, b[$, the interval $[x - \delta, x + \delta]$, where $\delta = \frac{1}{2} min(|x - a|, |x - b|)$, is a compact neighborhood of x.

e) In the space $\mathbb{R}$, $\mathbb{Q}$ is not compact nor relatively compact nor locally compact. In deed, $cl(\mathbb{Q}) = \mathbb{R}$, then $\mathbb{Q}$ is not closed, so it is not compact nor relatively compact. If $\mathbb{Q}$ is locally compact, and V is a compact neighborhood of 0 in $\mathbb{Q}$, then V contains the closed neighborhood of 0 of the form $A = \mathbb{Q} \cap [-a, a]$, where $a \in \mathbb{R}_+^*$, A is then compact. Clearly, for any $x \in \mathbb{Q}^C \cap [-a, a]$, the decreasing sequence of closed sets $\left\{A_n = A \cap \left[x - \frac{1}{n}, x +\right.\right.$ *1n, n∈N∗* in A, has an empty intersection, which contradicts the corollary 10.1. So $\mathbb{Q}$ is not locally compact.

Proposition 10.6. The finite locally compact subspaces is stable by intersection.

Proof. Let $x \in A = \cap_{i=1}^n A_i$. Because, $\forall\ 1 \le i \le n, x$ belongs to the locally compact A_i, there is some compact $N_i \in \mathcal{N}_i(x)$, where $\forall\ 1 \le i \le n$, $\mathcal{N}_i(x)$ is the set of neighborhoods in A_i. So $N = \cap_{i=1}^n N_i$ is a compact neighborhood of x then A is locally compact.

Remark 10.3. The union of two locally compact subspaces is not necessarily locally compact. In fact, $A = \{(x, y) \in \mathbb{R}^2, x > 0\}$ and $B = \{(0,0)\}$ are locally compact subspaces in $\mathbb{R}^2$, but $A \cup B$ is not locally compact subspace, because $(0,0)$ has no compact neighborhood in the subspace $A \cup B$.

Proposition 10.7. The closed (respectively open) subspace of the locally compact, Hausdorff space is locally compact. In particular the open in the compact, Hausdroff space is locally compact.

Proof. Let x be an element of the closed subspace C of the locally compact space E, there is a compact neighborhood N of x in E, as by proposition 10.3, N is closed, then $C \cap N$ is a closed neighborhood of x in C containing in the compact N, by proposition 10.2, $C \cap N$ is compact. Let now O an open subspace in E, as O is a neighborhood of any its element x, by proposition 10.5, there is some compact neighborhood of x, containinig in O, then O is locally compact. If O is an open in the compact Hausdorff space E, which is locally compact then O is locally compact.

As a direct consequence of the proposition 10.6, and proposition 10.7, we have:

Corollary 10.11. The finite intersection of closed (respectively open) subspaces, of the locally compact, Hausdorff space is locally compact.

Proposition 10.8. The finite product of locally compact space is locally compact.

Proof. Let $E = \Pi_{i=1}^{n} E_i$ be a finite product of the spaces E_i.and $x = (\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{i}, \dots, \mathrm{x}_\mathrm{n}) \in E$ As, $\forall\ 1 \leq i \leq n, \mathrm{x}_\mathrm{i} \in E_i$ which is locally compact, there exists a compact neighborhood of x_i, say N_i, then $N = \Pi_{i=1}^{n} N_i$ is a compact neighborhood of x in E.

Example 10.6. The spaces $\mathbb{R}^\mathrm{n}$ and $S_1 \times \mathbb{R}$ are locally compact.

At the end of this section, let us give some useful results on the Lindelôf space, which will be used in the following chapter.

Lemma 10.7. The open cover of the closed set in a Lindelôf space has a countable subcover.

Proof. Let F be a closed set in a Lindelôf space E and let $\{O_\alpha;\ \alpha \in \Delta\}$ be an open cover of F. As F^C is an open, and the collection $\{O_\alpha;\ \alpha \in \Delta\} \cup F^C$ is a cover of E, there is a countable subcover $\{O_n;\ n \in \mathbb{N}\}$ of E. Then, the collection $\{O_n;\ n \in \mathbb{N}\} \setminus F^C$ is a cover of F, since if $x \in F$, thus $x \in E\ \setminus F^C$, therfore $x \in (\cup_{n\in\mathbb{N}} O_n)\ \setminus F^C$.

The following lemma will transform a regular lindelôf space into a normal space.

Lemma 10.8. (Normality Lemma). Let A and B be subsets of a space E and let $\{O_n;\ n \in \mathbb{N}*$ and $O n'; n{\in}\mathbb{N}*$ be two sequences of open sets such that

$a)$ $A \subseteq \cup_{n\in\mathbb{N}^*} O_n$;

$b)$ $B \subseteq \cup_{n\in\mathbb{N}^*} O'_n$;

$c)$ For every $n \in \mathbb{N}^*$, $\mathrm{cl}(O_n) \cap B = \emptyset$ and $\mathrm{cl}(O'_n) \cap A = \emptyset$.

Then, there are two disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Proof. Define the two set sequences $\{A_n;\ n \in \mathbb{N}^*\}$ and $\{B_n;\ n \in \mathbb{N}^*\}$ as follows: $A_1 = O_1$, $A_n = O_n \cap \left(\cup_{i<n} \mathrm{cl}(O'_i)\right)^C$ and $B_n = O'_n \cap \left(\cup_{i\leq n} \mathrm{cl}(O_i)\right)^C$ then for every $n \in \mathbb{N}^*$, A_n and B_n are open. So $U = \cup_{n\in\mathbb{N}^*} A_n$ and $V = \cup_{n\in\mathbb{N}^*} B_n$ are open, $A \subset U, B \subset V$, and $U \cap V = \emptyset$, indeed: if $x \in A$ by $a)$ there exists $n_0 \in \mathbb{N}^*$, such that $x \in O_{n_0}$, because for every $n \in \mathbb{N}^*$, $\mathrm{cl}(O'_n) \cap A = \emptyset$ then $x \notin \mathrm{cl}(O'_{n_0})$, so $x \notin \cup_{i<n_0} \mathrm{cl}(O'_i)$, thus $x \in \left(\cup_{i<n_0} \mathrm{cl}(O'_i)\right)^C$, therefore $x \in A_{n_0} \subset U$. By the same argument $B \subset V$. Let now $x \in U$, we will show that $x \notin V$. Because $x \in U$, there is $j \in \mathbb{N}^*$ such that $x \in A_j$, then $x \notin \cup_{i<j} \mathrm{cl}(O'_i)$ hence $x \notin O'_i$ for all $i < j$, so $x \notin B_i$ for all $i < j$. Let for $j \leq m$, $B_m = O'_m \cap \left(\cup_{j\leq m} \mathrm{cl}(O_j)\right)^C$. We know that $x \in A_j$, as $j \leq m$, $x \in \cup_{j\leq m} \mathrm{cl}(O_j)$, so $x \notin \left(\cup_{j\leq m} \mathrm{cl}(O_j)\right)^C$, therefore $x \notin B_m$. Conclusion for all $n \in \mathbb{N}^*$, $x \notin B_n$, then $x \notin V$.

Theorem 10.5. Every regular, Lindelôf space is normal.
Proof. Let A and B be two disjoint closed set a space E. As, E. is regular, for every $x \in A$ there exists two disjoint open sets U containing x and V containing B such that $U \cap V = \emptyset$, by proposition 5.9 c), U contains a closed neighborhood F of x, so there is an open O_x containing x such that $x \in O_x \subset F \subset U$, thus $x \in cl(O_x) \subset U$, therefore $cl(O_x)$ ∩B=∅. Because the family $\{O_x, x \in A\}$ covers the closed set A and the space E is Lindelôf, by lemma 10.7, there is a countable subcover $\{O_n, n \in \mathbb{N}\}$ of A such that for every $n \in \mathbb{N}$, $cl(O_n)$ ∩B=∅. By the same argument there is a countable subcover $\{O'_n, n \in \mathbb{N}\}$ of B such that for every $n \in \mathbb{N}$, $cl(O'_n)$ ∩A=∅. The conditions of the normality lemma 10.8 are therefore satisfied, so there are two disjoint open sets containing respectively A and B, so E is normal.

Since 2D-space is Lindelôf, it is straightforward that.
Corollary 10.12. The regular 2D-space is normal.

11-Nets and filters

11.1 Nets.

We have seen in the previous chapters that: In a 1D-space an adherent point of a part of space is a limit of the sequence containing in this part. In a Hausdorff space, the limit of a sequence when it exists is unique. In a 1D-space, a limit of a function in the neighborhood of a point is the limit of the image of any sequence, which converges to this point. To obtain in some natural way, the above results as well as others. We will introduce the concept of the generalized sequence or the net, which is defined to general the sequence and to overcome the short coming of the sequence. The net allows us to find results obtained by the sequences without additional conditions on the space. Let E be any set and let $(D, \geq)$ be **a directed set** i.e. D is partially ordered and every two elements of D have an upper bound.
Definition 11.1. Let the map $f: d \in D \mapsto f(d) = x_d \in E$ be. The subset $\{x_d, d \in D\}$ of E is called a net and it is denoted $(x_d)_{d\in D}$.
Definition 11.2. Let E be a space and $x \in E$

i) The net $(x_d)_{d\in D}$ is said to be converges to x and we write $x_d \longrightarrow x$, if for every $N \in \mathcal{N}(x)$, there is $d_0 \in D$ such that for every $d \in D$, satisfaying $d \geq d_0$, we have $x_d \in N$

ii) x is said to be an adherent value (or a limit point) of the net $\{x_n\} \subset E$, if $\forall N \in \mathcal{N}(x)$, and $\forall d \in D$, there is $k \in D, k \geq d$ such that $x_k \in N$.

Note that if x is an adherent value for $(x_d)_{d\in D}$. Then $\forall d \in D$, the set A_d=$\{x_k \in E,\ k \geq d\}$ satisfies, for every $e, d \in D$, $cl(A_e) \cap cl(A_d) \neq \emptyset$. Indeed, there is $l \geq e$ and $l \geq d$, so $A_l \subset A_e \cap A_d \subset cl(A_e) \cap cl(A_d)$. Therefore for any finite part $I \subset D$, $\cap_{d\in I}\ cl(A_d) \neq \emptyset$

Before giving in a general space, a characterization of the closure by nets. Note that if $(x_d)_{d\in D}$ is a net in the subset A of the space E which converges to $x \in E$ then $x \in cl(A)$. Indeed if, $N \in \mathcal{N}(x)$ there is $d_0 \in D$, such that for every $d \in D$, satisfaying $d \geq d_0$, we have $x_d \in N$, then $x_d \in N \cap A$, so $x \in cl(A)$. The converse is given without E being 1D-space.
Proposition 11 1. For every $x \in cl(A)$, there is a net in A which converges to x.
Proof. If $x \in cl(A)$, for every $N \in \mathcal{N}(x)$, $N \cap A \neq \emptyset$, $(\mathcal{N}(x), \supseteq)$ being directed, thus there is a net $(x_N)_{N\in\mathcal{N}(x)}$ in $N \cap A$, then for every $V \in \mathcal{N}(x)$, satisfying $V \supseteq N$, we have $x_N \in V$, it follows that $x_N \longrightarrow x$.
Let E and F are two spaces and let x be an element of E. We have the following equivalence without E being 1D-space.
Proposition 11 2. The map $f: E \longrightarrow F$ is continuous in x ⟺fore every net $(x_d)_{d\in D}$ in E converging to x, the net $(f(x_d))_{d\in D}$ converges to $f(x)$ in F.
Proof. Let $V \in \mathcal{N}(f(x))$, because, f is continuous in $x \in E$, there is $N \in \mathcal{N}(x)$ such that $f(N) \subset V$. As the net $(x_d)_{d\in D}$ converges to x, there is $d_0 \in D$ such that for every $d \in D$,

satisfaying $d \geq d_0$, we have $x_d \in N$, so $f(x_d) \in V$, it follows that $f(x_d) \longrightarrow f(x)$. Conversely, if f is not continuous, there is an open U in F such that $f^{-1}(U)$ is not open in E. Then, $(f^{-1}(U))^C$ is not closed in E, so there is x in $cl((f^{-1}(U))^C)$ which is not in $(f^{-1}(U))^C$. By proposition 11.1, there is a net $(x_d)_{d\in D}$ in $(f^{-1}(U))^C$ witch converges to x. As, $x \in f^{-1}(U)$, thus $f(x) \in U$, because for every $d \in D$, $x_d \in (f^{-1}(U))^C$, then x_d is not in $f^{-1}(U)$, so $f(x_d)$ is not in U. Therefore, the net $(f(x_d))_{d\in D}$ not converges to $f(x)$, contradiction.

Recall that (see example 7.1), a convergence sequence can have a unique limit without the Hausdorff property. This is not the case for the nets, if every net have a unique limit, then the space is Hausdorff:

Proposition 11 3. A space E is Hausdorff⟺every net has a unique limit.

Proof. Let $(x_d)_{d\in D}$ be a convergence net in the Hausdorf space E. If the net has two distinct limits x and y in E, there are two disjoint open set $O_x \ni x$ and $O_y \ni y$, so there are d_1 and d_2 in D such that for every $d \in D$ satisfying $d \geq d_1$ and $d \geq d_2$ we have $x_d \in O_x \cap O_y$, contradiction. Conversely, suppose that E is not Hausdorff i.e. there are two different points x and y in E such that every $(N,V) \in \mathcal{N}(x)\mathcal{N}(y)$ satisfies $N \cap V \neq \emptyset$. Let $\left(z_{(N,V)}\right)_{(N,V)\in\mathcal{N}(x)\times\mathcal{N}(y)}$ be a net in $N \cap V$, then for every $U \in \mathcal{N}(x)$ with $(N,V) \geq (U,E)$ i.e. $U \supseteq N$ and $E \supseteq V$, we have $z_{(N,V)} \in U$ it follows that $\left(z_{(N,V)}\right)_{(N,V)\in\mathcal{N}(x)\times\mathcal{N}(y)}$ converges to x. By the same, we have $\left(z_{(N,V)}\right)_{(N,V)\in\mathcal{N}(x)\times\mathcal{N}(y)}$ converges to y as the limit is unique, contradiction.

Definition 11.3. A subnet of the net $(x_d)_{d\in D}$ is a net $\left(x_{\varphi(k)}\right)_{k\in K}$, where $(K,\geq)$ is a directed subset of D and the map $\varphi: k \in K \longmapsto \varphi(k) \in D$ is such that:

  $i)$ If $k \geq l$, then $\varphi(k) \geq \varphi(l)$ (φ is order preserving).

  ii) For every $d \in D$, there is $k \in K$ such that $\varphi(k) \geq d$ ($\varphi(K)$ is cofinal in D).

Before giving, a characterization of compact spaces by the net. Note that, it is obvious to check, that, the finite intersection property⟺all directed and decreasing family of closed sets has a nonempty intersection.

Theorem 11.1. A space is compact⟺every net has a convergence subnet.

Proof. Let $(x_d)_{d\in D}$ be a net in E . As, the family $(\{cl(A_d), d \in D\}, \supseteq)$, where for every $d \in D$, $A_d = \{x_k,\ k \geq d\}$, is directed and decreasing, and E is a compact space, then $\cap_{d\in D}\, cl(A_d) \neq \emptyset$. Let $x \in \cap_{d\in D}\, cl(A_d)$ then, for every $d \in D$ and for every $N \in \mathcal{N}(x)$, $N \cap A_d \neq \emptyset$, so, there is $k \geq d$ such that $x_k \in N$. Consider the set $\mathcal{D} = \{(d,N) \in D \times \mathcal{N}(x)$ such that $x_d \in N\}$, then $(\mathcal{D}, \succcurlyeq)$ where, for every $(d,N),(e,V) \in \mathcal{D}$; $\{(d,N) \succcurlyeq (e,V)\} \Leftrightarrow \{d \geq e$ and $N \supseteq V\}$ is a directed set. It is clear that the binary relation $\succcurlyeq$ is partially ordered and for every $(d,N),(e,V) \in \mathcal{D}$; there is an upper bound p of d and e in D. As $N \cap V \in \mathcal{N}(x)$, then $(N \cap V) \cap A_p \neq \emptyset$, there is, $l \geq p$ such that $x_l \in N \cap V$. Thus $(l, N \cap V)$ is an upper bound of (d,N) and (e,V) in $\mathcal{D}$. In the other hand, the projection $f: (d,N) \in \mathcal{D} \longmapsto f(d,N) = d \in D$ is obviously an order preserving and it is surjective, then $f(\mathcal{D})$ is cofinal in D, thus $\left(x_{f(d,N)}\right)_{(d,N)\in\mathcal{D}}$ is a subnet of the net $(x_d)_{d\in D}$. If now, $N \in \mathcal{N}(x)$, then, for every $d \in D$, $N \cap A_d \neq \emptyset$, so there is $k \geq d$, such that $x_k \in N$, then $(k,N) \in \mathcal{D}$. By definition of $\mathcal{D}$, for every $(e,V) \in \mathcal{D}$, satisfaying $(e,V) \succcurlyeq (k,N)$, we have $x_e = x_{f(e,V)} \in N$, so $x_{f(d,N)} \longrightarrow x$. Conversely, if E is not compact, then there is a family $\mathcal{F}$ of closed sets in E such that, every finite sets of $\mathcal{F}$ has a nonempty intersection, but the intersection of all its elements is empty. Let $\mathfrak{D}$ be the collection of all finite subfamily of $\mathcal{F}$. It is clear that $(\mathfrak{D}, \supseteq)$ is a directed set. We can choose a net $(x_{\mathcal{B}})_{\mathcal{B}\in\mathfrak{D}}$ where $x_{\mathcal{B}} \in \cap_{F\in\mathcal{B}} F$. If, there is a subnet $\left(x_{f(e)}\right)_{e\in\mathcal{D}}$ of $(x_{\mathcal{B}})_{\mathcal{B}\in\mathfrak{D}}$ witch convergence to x in E. By assumption, there is $F \in \mathcal{F}$ such that

$x \notin cl(F) = F$, so there is $N \in \mathcal{N}(x)$, $N \cap F = \emptyset$, then, $x_{\mathcal{B}} \notin N$ for all $\mathcal{B} \supseteq \{F\}$, As $f(\mathcal{D})$ is cofinal in D, there is $e' \in \mathcal{D}$ such that $f(e') \supseteq \{F\}$. Also, it exists $e'' \in \mathcal{D}$, such that, $x_{f(e)} \in N$,. for all e $\succcurlyeq$ e''. Let l an upper bound of e' and e'', then we must have $x_{f(l)} \in N$. As $f(l) \supseteq f(e') \supseteq \{F\}$, also $x_{f(l)} \notin N$, contradiction. So the net does not have a convergence subnet.
Don't believe that a subnet is a subsequence; if not as in the compact space any sequence has a convergence subsequence (see Lemma 10.2) then by the theorem 11.1, the compact space is sequentially compact which is false.
Definition 11.4. A net in E is universal or an ultranet, if for every $A \subset E$, the net is either eventually in A or eventually in A^C.
Proposition 11.4. Every net has a universal subnet.

11.2-Filtres

Along with the net, we introduce filters, general notions of limits and we show that most of the results obtained using nets can equally well be proven using filters. Let E be a set, $\mathcal{P}(E)$ the family of all parts of E and $\mathcal{F}$, the nonempty subfamily of $\mathcal{P}(E)$.
Definition 11.5. $\mathcal{F}$ is said to be a filter if:
F_1-$\emptyset \notin \mathcal{F}$.
F_2-If, $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
F_3- If for $A \in \mathcal{F}$, there is $B \supset A$, then $B \in \mathcal{F}$.
It is obvious that $E \in \mathcal{F}$ and if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.
Definition 11.6. The nonempty subfamily $\mathcal{B}$ of the filter $\mathcal{F}$ is said to be a basis of $\mathcal{F}$. If, for every $A \in \mathcal{F}$, there is B$\in \mathcal{B}$ such that $B \subset A$.
It is clear that, if $\mathcal{B}$ is a basis of the filter $\mathcal{F}$. Then
B_1-$\emptyset \notin \mathcal{B}$.
B_2-For every $A, B \in \mathcal{B}$, there is $C \in \mathcal{B}$ such that $C \subset A \cap B$.
Conversely, any part $\mathcal{B} \subset \mathcal{P}(E)$ satisfying B_1 and B_2 generates a unique filter $\mathcal{F}$ on E such that, for every $A \in \mathcal{F}$, there is B$\in \mathcal{B}$ with $B \subset A$.
Example 11.1.
$a)$ Let $x \in E$ and $\mathcal{F}(x)$.the family of all parts of E containing x. Then $\mathcal{F}(x)$ is a filter in E. Such a filter is said to be **a trivial filter**. In the case where E is a space, $\mathcal{N}(x)$ is a filter on E.
$b)$ $\mathcal{B} = \{B_n, n \in \mathbb{N}\}$, where $B_n = \{p \in \mathbb{N},\ p \geq n\}$, for every $n \in \mathbb{N}$, is a basis of the filter, called **a basis of Fréchet filter** on $\mathbb{N}$.
$c)$ If, A is nonvoide part of the set E, , then $\mathcal{B} = \{X \in \mathcal{P}(E), X \supseteq\ A\}$ is a basis of the filter.
$d)$ $\mathcal{B} = \{[x, +\infty[, x \in \mathbb{R}\}$ is a basis of the filter on $\mathbb{R}$, called a **basis of the filter of the neighborhood of** $+\infty$.
$e)$ If, A is a part of the space E and $x \in cl(A)$, then $\mathcal{B} = \{N\ \cap A, N \in \mathcal{N}(x)\}$ is a basis of a filter, called **a basis of the adherent filter to A on x.**
The family $(\mathfrak{F}, \supseteq)$ of all filters on the set E is a directed set. A filter $\mathcal{U} \in \mathfrak{F}$ is said to be an **ultrafilter** if, it is maximal i.e a filter on E which contains either A or A^C, for all $A \in E$. For example, a trivial filter is an ultrafilter. The importance of ultrafilters lies above all, in the following proposition.
Proposition 11.5. Every filter has an ultrafilter.
Proof. It remains to demonstrate that $(\mathfrak{F}, \supseteq)$ is inductive. If, $\{\mathcal{F}_\alpha, \alpha \in \Delta\}$ is a totally ordering collection of $\mathfrak{F}$, then the filter $\cup_{\alpha \in \Delta} \mathcal{F}_\alpha$ is an upper bound of $\mathfrak{F}$, then $(\mathfrak{F}, \supseteq)$ is inductive. So, by Zorne's lemma it has an maximal element $\mathcal{U} \in \mathfrak{F}$.
Proposition 11.6. Let E and F are two sets and a map $f: E \longrightarrow F$. If, $\mathcal{B}$ is a basis of the filter on E, then $f(\mathcal{B})$ is the basis of the filter on F, called the image filter basis.

Proof. $\emptyset \notin f(\mathcal{B})$, if not there exists $A \in \mathcal{B},\ A \neq \emptyset$, such that $f(A)$=$\emptyset$, as $A \subset f^{-1}(f(A)) = f^{-1}(\emptyset) = \emptyset$, contradiction. Also $f(\mathcal{B}) \neq \emptyset$, since for every $A \in \mathcal{B}$, $f(A) \in f(\mathcal{B})$. If now, $U, W \in f(\mathcal{B})$, there are $A, B \in \mathcal{B}$ such that $f(A) = U$ and $f(B)$=W, therefore there is $C \in \mathcal{B}$, such that $C \subset A \cap B$, then $f(C) \subset f(A \cap B) \subset f(A) \cap f(B)$= $U \cap W$. By B_1 and B_2, $f(\mathcal{B})$ is a basis of a filter on *F*.

Definition 11.7. We say that a point x of the space *E* is a limit of the filter $\mathcal{F}$ on *E*, or $\mathcal{F}$ converges to x and we write, $lim\mathcal{F} = x$, if $\mathcal{F}$ contains any $N \in \mathcal{N}(x)$

Definition 11.8. We say that a point x of the space *E* is a limit of the basis $\mathcal{B}$ of the filter $\mathcal{F}$ on *E*, or $\mathcal{B}$ converges to x and we write $lim\mathcal{B} = x$ if, for any $N \in \mathcal{N}(x)$, there is $B \in \mathcal{B}$, such that $B \subset N$.

It is obvious that if the filter $\mathcal{F}$ is generated by a basis $\mathcal{B}$ then: $lim\mathcal{F} = x \iff lim\mathcal{B} = x$.

Proposition 11.7. The space *E* is Hausdorff ⟺ the limit of the basis of the filter on *E,* when it exists is unique.

Proof. Suppose that the basis $\mathcal{B}$ has two different limits x and y. As, the space *E,* is Hausdorff, there are $N \in \mathcal{N}(x)$ and $N' \in \mathcal{N}(y)$ whose $N \cap N' = \emptyset$. Because $lim\mathcal{B} = x$ and $lim\mathcal{B} = y$, there are A,B∈ $\mathcal{B}$ such that A⊂ N and B⊂ N', therefore $A \cap B = \emptyset \in \mathcal{B}$, contradiction. Conversely, if $x, y \in E$, $x \neq y$ and all $N \in \mathcal{N}(x)$ and $N' \in \mathcal{N}(y)$ satisfy $N \cap N' \neq \emptyset$. Then, the basis $\mathcal{B} = \{N \cap N', N \in \mathcal{N}(x)$ and $N' \in \mathcal{N}(y)\ \}$ is such that for all $N \in \mathcal{N}(x)$ and all $N' \in \mathcal{N}(y), B = N \cap N' \in \mathcal{B}$, $B \subset N$ and $B \subset N'$, then $\mathcal{B}$ converges to x and y, contradiction.

Lemma 11.1. The family of the closed sets has the property of the empty intersection⟺the directed decreasing family of the closed set has a nonempty intersection.

Proof. Let $(\Delta, \geq)$ be a directed set and let $(D, \supseteq)$, where $D = \{F_\alpha, \alpha \in \Delta\}$ be a decreasing family of the closed sets in the space *E*, then $(D, \supseteq)$ is directed. Suppose that $\cap_{\alpha\in\Delta} F_\alpha = \emptyset$, by assumption there is a finite $I \subset \Delta$, such that $\cap_{\alpha\in I} F_\alpha = \emptyset$. Then there is $i \in I$, with $F_i = \emptyset$. As, for every j∈ I, $j \geq i$, $F_i \supseteq F_j \supseteq F_i \cap F_j$, then $F_i \cap F_j = \emptyset$ and, as $(\Delta, \geq)$ is directed, there is $k \in I$ such that $k \geq i$ and $k \geq j$, thus $F_i \supseteq F_k \neq \emptyset$ and $F_j \supseteq F_k$, therefore $F_i \cap F_j \neq \emptyset$, the contradiction. Reciprocally, let $D = \{F_\alpha, \alpha \in \Delta\}$ be a family of the closed sets in the space *E*. Suppose that for any finite $I \subset \Delta$, $\cap_{\alpha\in I} F_\alpha \neq \emptyset$. Let $L = \{G_\beta, \beta \in \nabla\}$ be, where $(\nabla, \geq)$ is directed and for all $\beta \in \nabla$, $G_\beta = \cap_{\alpha\in I} F_{\alpha,\beta}$. Then $(L, \supseteq)$ is a decreasing directed family of the closed sets, which by hypothesis $\cap_{\beta\in\nabla} G_\beta \neq \emptyset$, so $\cap_{\beta\in\nabla} \left(\cap_{\alpha\in I} F_{\alpha,\beta}\right) \neq \emptyset$ or $\cap_{(\beta,\alpha)\in\nabla\times I} F_{\alpha,\beta} \neq \emptyset$, which implies that $\cap_{\alpha\in\Delta} F_\alpha \neq \emptyset$.

Definition 11.9. We say that, the point x of the space *E,* is an adherent point of the filter $\mathcal{F}$ on *E*, if $x \in cl(A)$ for every $A \in \mathcal{F}$. The set $cl(\mathcal{F})$ of all adherent points of $\mathcal{F}$, is called the closure of $\mathcal{F}$, and it is equal to $\cap_{A\in\mathcal{F}} cl(A)$. This definition is obviously valid for the basis of the filter.

Lemma 11.2. A filter on the space *E* has an adherent point ⟺the ultrafilter has a limit.

Proof. Let $\mathcal{U}$ be an ultrafilter, as the filter, by hypothesis $\mathcal{U}$ has an adherent point $x \in E$, then for all $A \in \mathcal{U}$ and all $N \in \mathcal{N}(x)$, $N \cap A \neq \emptyset$. As the family $\mathcal{F} = \{(N \cap A), A \in \mathcal{U}$ and $N \in \mathcal{N}x$ is a filter on *E*, $\mathcal{N}x \supseteq \mathcal{F}$, $\mathcal{U} \supseteq \mathcal{F}$, and $\mathcal{U}$ is maximal, then $\mathcal{U}$=$\mathcal{F}$. As $\mathcal{U} \supseteq \mathcal{N}x$, then for all $N \in \mathcal{N}(x)$, $N \in \mathcal{U}$, so $\mathcal{U}$ converges to x. Inversely, let $\mathcal{F}$ be a filter on *E*, by proposition 11.5, $\mathcal{F}$ has an ultrafilter $\mathcal{U}$, as by hypothesis, $\mathcal{U}$ converges to $x \in E$, then for all $N \in \mathcal{N}(x)$, N∈ $\mathcal{U}$. Because, fore all $N \in \mathcal{N}(x)$ and all $A \in \mathcal{F}$; $N, A \in \mathcal{U}$ by F_2, $N \cap A \in \mathcal{U}$, thus $N \cap A \neq \emptyset$, therefore x is an adherent point of $\mathcal{F}$.

Corollary 11.1. When the filter converges to a unique point it adherent is reduced to this point.

Proof. If, $\mathcal{F}$ converges to $x \in E$, then for all $N \in \mathcal{N}(x)$, N∈ $\mathcal{F}$; so for all $A \in \mathcal{F}$, $N \cap A \in \mathcal{F}$, thus $N \cap A \neq \emptyset$, therefore x is an adherent point of $\mathcal{F}$, as x is unique then $cl(\mathcal{F})$=$\{x\}$. If not there is $y \in cl(\mathcal{F})$ and $y \neq x$, because by proposition 11.7, the space E is Hausdorff, there are $N \in \mathcal{N}(x)$ and $N' \in \mathcal{N}(y)$ such that $N \cap N' = \emptyset$. Because $N \in \mathcal{F}$, then $N \cap N' \neq \emptyset$ contradiction.

Lemma 11.3. The directed decreasing family of the closed sets has a nonempty intersection $\Leftrightarrow$a filter in the space E has an adherent point.

Proof. Let $\mathcal{F}$a filter in the space E. the family $\mathcal{A} = \{cl(A), A \in \mathcal{F}\}$ provided with the relation $\supseteq$ is a directed decreasing family of the closed sets in the space E, indeed if $A, B \in \mathcal{F}$ with $B \supset A$ then $cl(B) \supseteq cl(A)$ as $A \cap B \in \mathcal{F}$ then $cl(A \cap B) \in \mathcal{A}$, and $cl(B) \cap cl(A) \supseteq cl(A \cap B)$, by assumption $\cap_{A \in \mathcal{F}} cl(A) \neq \emptyset$. Then, there is $x \in cl(A)$, for all $A \in \mathcal{F}$ i.e. x is an adherent point of $\mathcal{F}$. Conversely, let $\mathcal{B} = \{F_\alpha, \alpha \in \Delta\}$ be a directed decreasing family of the closed sets in the space E, then $\mathcal{B}$ is a basis of the filter $\mathcal{F}$ on E. If x is an adherent point of $\mathcal{F}$, then for all $A \in \mathcal{F}$, $x \in cl(A)$. As, for all $\alpha \in \Delta$, $F_\alpha \in \mathcal{F}$ and $cl(F_\alpha) = F_\alpha$. Then for all $\alpha \in \Delta$, $x \in F_\alpha$, therefore $\cap_{\alpha \in \Delta} F_\alpha \neq \emptyset$.

Definition 11.10. Let f a map defined from the set E into the space F. The point $l \in F$ is said to be a limit of f according to the basis $\mathcal{B}$, and we write $lim f(\mathcal{B}) = l$ or $lim_{\mathcal{B}} f = l$. If. for all $V \in \mathcal{N}(l)$ there is $B \in \mathcal{B}$, such that $f(B) \subset V$. The closure of f denoted $cl(f)$, is the closure of $f(\mathcal{B})$, is equal to $\cap_{B \in \mathcal{B}} cl(f(B))$.

Theorem 11.2. The map f from the space E into the space F is continuous in $x \in E$ $\Leftrightarrow$for every convergent basis $\mathcal{B}$ to x, $f(\mathcal{B})$ converges according to $\mathcal{B}$ to $f(x)$.

Theorem 11.3. The space E is compact$\Leftrightarrow$ any filter on E, has an adherent point .

Remarque 11.2. The proof of the theorem 11.3, is a straightforward consequence of the lemmas 11.1-11.3. This theorem can also be used, to give an elegant demonstration of Tychonoff's theorem, by comparison with that given by the nets.

Remarque 11.3. If, $(x_d)_{d \in D}$ is a net in the space E, then the family $\mathcal{F} \subset \mathcal{P}(E)$, defined by: $A \in \mathcal{F} \Leftrightarrow$there is d$\in D$ such that $x_e \in A$, for every $e \geq d$ is a filter on E, which eventually contain the net $(x_d)_{d \in D}$. This filter has the same limits as $(x_d)_{d \in D}$. Conversely if, $\mathcal{F}$ is a filter on E , one can consider the directed set $(\mathcal{F}, \supseteq)$. Then $\mathcal{F}$ converges to a point $x \in E$ $\Leftrightarrow$any net $(x_A)_{A \in \mathcal{F}}$, with $x_A \in A$, for all $A \in \mathcal{F}$ converges to x.

As a consequence of the proposition 11.7 and the theorem 11.3, we have.

Corollary 11.2. In the Hausdorff compact space. The filter converges iffy it has a unique adherent point.

Proof. Let $\mathcal{F}$ be a filter on E which convergences to the limit $x \in E$, as E is Hausdorff, x is unique, so by the corollary 11.1,.$cl(\mathcal{F}) = \{x\}$. Conversely, let $x \in E$ be a unique adherence of $\mathcal{F}$. Then, $\mathcal{F}$ convergence to x, if not there is some open $O \in \mathcal{N}(x)$ not contained in $\mathcal{F}$, such that for all $A \in \mathcal{F}$, $x \in cl(A)$ and $O \cap A \neq \emptyset$. As, the family $\{cl(A) \cap O^C,\ A \in \mathcal{F}\}$ is clearly a directed decreasing family of the closed sets of E which is compact by lemma 11.1 and the theorem 11.3, $\bigcap_{A \in \mathcal{F}}(cl(A) \cap O^C) \neq \emptyset$. Therefore there is $y \in \bigcap_{A \in \mathcal{F}}(cl(A) \cap O^C)$, so $y \in cl(A) \cap O^C$, for all $A \in \mathcal{F}$, then y is an adherent point of $\mathcal{F}$ different from x, contradiction.

Example 11.2.

a) If E=ℕ, $\mathcal{B}$ a Fréchet basis, and $f: n \in \mathbb{N} \longmapsto y_n \in F$. $lim f(\mathcal{B}) = $ l $\Leftrightarrow$for all $V \in \mathcal{N}(l)$ there is $B \in \mathcal{B}$, such that $f(B) \subset V \Leftrightarrow$ for all $V \in \mathcal{N}(l)$, there is $n_0 \in \mathbb{N}$, such that $y_n \in V$, for every $n > n_0 \Leftrightarrow y_n \longrightarrow$ l.

b) If, E=ℝ, $\mathcal{B}$ a basis of the neighborhood of $+\infty$., and $f: \mathbb{R} \longrightarrow F$. If, $lim f(\mathcal{B}) =$ l, then for all $V \in \mathcal{N}(l)$ there is $B \in \mathcal{B}$, such that $f(B) \subset V \Longrightarrow$ for all $V \in \mathcal{N}(l)$, there is $a \in \mathbb{R}_+$, such that for every $x \geq a$, $f(x) \in V \Leftrightarrow \lim_{x \longrightarrow +\infty} f(x) = l$.

c) If, $f: E \longrightarrow F$, where E and F are two spaces, $x_0 \in E$ and $\mathcal{B}$ is a basis of the trivial filter $\mathcal{N}(x_0)$, then $\text{Lim}_{x \longrightarrow x_0} f(x) = f(x_0) \Leftrightarrow lim f(\mathcal{B}) = f(x_0)$.

12 Compactification

Let E and F two topological spaces. We say that E **embeds** into a topological space F, if it is homeomorphe to a everywhere dense subset of F. Being given a non compact space, it is natural to ask the following question: Can you add some things to this space to make it compact? Other ways, is there a way to embed this space into another which is compact?. Under suitable conditions, the answer is affirmative. The process to imbed the non compact space into a compact one is called compactification. The main idea of the compactification, comes from the fact that an open space of a compact and hausdorff space is locally compact (see proposition 10.7). This leads us to concentrate primarily on the compactification of locally compact Hausdorff spaces. There are several paths to follow to compact a space. The most "efficient" or the "smallest" one in the sense that the embedding only misses one point is the Alexandroff-compactification or a point-compactification. Let's also mention the two points-compactification, the most general of all compactifications is the Stone-$\hat{C}$ech-compactifcation. In the sequel, we will mainly focus on the first and last compactification paths.

Definition 12.1. The space $\tilde{E}$ is called a compactification of a given space E, if $\tilde{E}$ is a Hausdorff compact space and containing a everywhere dense part which is homeomorphe to E. In other words, if $\tilde{E}$ is a Hausdorff compact space and there is a map $f: E \longrightarrow \tilde{E}$ such that $f: E \longrightarrow f(E)$, is a homeomorphism and $cl(f(E))$=$\tilde{E}$. A compactification of a topological space E, when it exists, is denoted by $(\tilde{E}, f)$.

Two compactifications $(\tilde{E}, f)$ and $\left(\tilde{\tilde{E}}, f\right)$ of the same space E are called equivalent if there is a homeomorphism h: $\tilde{E} \longrightarrow \tilde{\tilde{E}}$ such that $h(x) = x$, for every $x \in E$.

Let (E, τ) be a locally compact Hausdorff space and let $\hat{E} = E \cup \{\omega\}$ be, where $\omega \notin E$ called a point at infinity of $\hat{E}$ and let $\hat{\tau} = \tau \cup \sigma$ be, where σ is the collection of all set Uin $\hat{E}$ containing ω, whose $U^{C_{\hat{E}}}$ is a compact in E

Theorem 12.1. (Alexandroff-Compactification). The space $\left(\hat{E}, \hat{\tau}\right)$ is a unique (up to an equivalence) Alexandroff-compactified of E.

Proof. We will prove that the couple $\left(\hat{E}, \hat{\tau}\right)$ is the Hausdorff compact space, such that E embeds as a everywhere dense part of $\hat{E}$.

a) Let us show that $\left(\hat{E}, \hat{\tau}\right)$ is a topological space.

O_1-Since $\emptyset \in \tau$ and $\tau \subset \hat{\tau}$, then $\emptyset \in \hat{\tau}$. As $\omega \in \hat{E}$ and $\hat{E}^{C_{\hat{E}}} = \emptyset$ which is a compact in E, then $\hat{E} \in \hat{\tau}$.

O_2-let $\{U_\alpha, \alpha \in \Delta\}$ be a collection of elements of $\hat{\tau}$. There are three cases:

Case 1. All U_α are in τ then $U = \bigcup_{\alpha \in \Delta} U_\alpha \in \tau$ and hence $U \in \hat{\tau}$.

Case 2. All U_α are in σ, then, for every $\alpha \in \Delta$, there is a compact K_α in E such that $K_\alpha = {U_\alpha}^{C_{\hat{E}}}$, then $\bigcup_{\alpha \in \Delta} U_\alpha = \bigcup_{\alpha \in \Delta}\left({K_\alpha}^{C_{\hat{E}}}\right) = \left(\bigcap_{\alpha \in \Delta} K_\alpha\right)^{C_{\hat{E}}}$, as $\bigcap_{\alpha \in \Delta} K_\alpha$ is a compact of E, then $\bigcup_{\alpha \in \Delta} U_\alpha \in \hat{\tau}$.

Case 3. There are two collection $\{U_i, i \in I\}$ in τ and $\{U_j, j \in J\}$ in σ such that $\{U_\alpha, \alpha \in \Delta\} = \{U_i, i \in I\} \cup \{U_j, j \in J\}$. Let's pose $U = \cup_{i \in I} U_i$ and $V = \cup_{j \in J} U_j$, then $\bigcup_{\alpha \in \Delta} U_\alpha = U \cup V$, where by the case 2, $V = K^{C_{\hat{E}}}$ with K a compact in E. Therefore,

$U \cup V = U \cup K^{C_{\hat{E}}} = (U^{C_E})^{C_E} \cup K^{C_{\hat{E}}} = (U^{C_E} \cap K)^{C_{\hat{E}}}$, because $U^{C_E} \cap K$ is a closed in the compact K, it is a compact in E, then $U \cup V \in \hat{\tau}$

O_3-Let $\{U_\alpha, 1 \leq \alpha \leq n\}$ a finite collection of the elements of $\hat{\tau}$. As above there are three cases:

Case 1. All U_α are in τ then $U = \bigcap_{\alpha=1}^{n} U_\alpha \in \tau$ and hence $U \in \hat{\tau}$.

Case 2. All U_α are in σ, then, for every $\alpha \in \{1, \dots, n\}$, there is a compact K_α in E such that $K_\alpha = {U_\alpha}^{C_{\hat{E}}}$, then $\bigcap_{\alpha=1}^{n} U_\alpha = \bigcap_{\alpha=1}^{n} \left({K_\alpha}^{C_{\hat{E}}}\right) = \left(\bigcap_{\alpha=1}^{n} K_\alpha\right)^{C_{\hat{E}}}$, as $\bigcap_{\alpha=1}^{n} K_\alpha$ is a compact of E, then $\bigcap_{\alpha=1}^{n} U_\alpha \in \hat{\tau}$.
Case 3. There are two collection $\{U_i, i \in I\}$ in τ and $\{U_j, j \in J\}$ in σ such that $\{U_\alpha, 1 \leq \alpha \leq$ *n=Ui, i∈I⋃Uj, j∈J*. Let's pose *U=⋂α=1nUi* and *V=∩j∈JUj*, then *∩α∈ΔUα=U∩V*, where by the case 2, $V = K^{C_{\hat{E}}}$ with K a compact in E. Therefore, $U \cap V = U \cap K^{C_E}$, since E is Hausdorff, by proposition 10.3, K is closed, then $U \cap V \in \tau$, therefore $U \cap V \in \hat{\tau}$

b) Let us show that $\hat{E}$ is compact.
Let $\{U_\alpha, \alpha \in \Delta\}$ be a collection of elements of $\hat{\tau}$ such that $\hat{E} = \bigcup_{\alpha \in \Delta} U_\alpha$, then there is $\alpha_0 \in \Delta$, such that $\omega \in U_{\alpha_0}$ so $U_{\alpha_0} = K_{\alpha_0}^{C_{\hat{E}}}$ where K_{α_0} is a compact in E, then $\hat{E} = K_{\alpha_0} \cup U_{\alpha_0}$. It follows that $\left(\bigcup_{\alpha \in \Delta} U_\alpha\right) \setminus U_{\alpha_0} = K_{\alpha_0}$, and there exists a finite $I \subset \Delta$ such that $\left(\bigcup_{\alpha \in I} U_\alpha\right) = K_{\alpha_0} \cup U_{\alpha_0} = \hat{E}$, thus $\hat{E}$ is compact.

c) Let us show that $\hat{E}$ is Hausdorff.
Let x be an element of $\hat{E}$ different of ω, then $x \in E$ which is locally compact, therefore x has in E a compact neighborhood K (K is then closed), because $\omega \in K^{C_{\hat{E}}} \in \hat{\tau}$ and $K \cap K^{C_{\hat{E}}} = \emptyset$, then $\hat{E}$ is Hausdorff.

d) Let us show that $cl(E) = \hat{E}$
Let x be an element of $\hat{E}$ and $\emptyset \neq U \in \hat{\tau}$ containing x if, $U \in \tau$ then $U \cap E = U \neq \emptyset$, so $x \in cl(E)$. If now $U \in \sigma$ there is a compact K in E such that $U = K^{C_{\hat{E}}}$ then $U \cap E = K^{C_{\hat{E}}} \cap E = K^{C_E} \cap E = K^{C_E} \neq \emptyset$, so $x \in cl(E)$.

e) Let us show that $\hat{E}$ is unique, in the sense that if $\hat{\hat{E}}$ is another Alexandroff compactified of E, then $\hat{E}$ is homeomorphic to $\hat{\hat{E}}$. Let $\hat{\hat{E}} = E \cup \{\omega'\}$, where $\omega' \neq \omega$ is the point at infinity of $\hat{\hat{E}}$. It is clear that, the map $f: \hat{E} \longrightarrow \hat{\hat{E}}$ defined by for every $x \in \hat{E}$, $f(x) = \begin{cases} x \text{ if } x \in E; \\ \omega' \text{if } x = \omega \end{cases}$
is an homeomorphism. Indeed f is biunivoque, bicontinuous on E, and also, continuous in ω, because if W is an open in $\hat{\hat{E}}$ containing $f(\omega) = \omega'$, there is a compact K in E such that $K = W^{C_{\hat{\hat{E}}}}$, as f^{-1} is continuous from E into E, then $f^{-1}(K) = f^{-1}(W)^{C_{\hat{E}}}$ is a compact in E, so $f^{-1}(W)$ is an open in $\hat{E}$, containing ω, therefore f is continuous in ω.

f) Clearly E is homeomorphe to $E = \omega^{C_{\hat{E}}}$.

Remark 12.1. Two homeomorphe, locally compact Hausdorff spaces, are the same compactification.

Example 12.1.

a) In the space $\mathbb{R}$. The locally compact Hausdroff subspace $]0,1]$, embeds as a subset of a compact Hausdorff space $[0,1]$ via a natural inclusion map $j:]0,1] \longrightarrow [0,1] =]0,1] \cup \{0\}$, defined by: for every $x \in]0,1]$ $j(x) = x$, so $cl\left(j(]0,1])\right) = cl(]0,1]) = [0,1]$. Then $[0,1]$ is a Alexandroff-compactification or one point compactification space of $]0,1]$.

b) In the space $\mathbb{R}$. The locally compact Hausdroff subspace $]0,1[$ embeds as a subset of $[0,1] =]0,1[\cup \{0,1\}$ via a natural inclusion map. This map misses two points 0 and 1, then $[0,1]$ is a two points-compactification space of $]0,1[$.

c) In the space $\mathbb{R}$. The interval $]0,1[$, embeds as a subset of a compact Hausdorff space $S_1 = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$ (the unite circle), via the function $f:]0,1[\longrightarrow \mathbb{R}^2$ defined by: $f(x) = \left(cos(2\pi x), sin(2\pi x)\right)$, for every x in $]0,1[$. This function misses only the point $(1,0)$ in S_1. Then S_1.is a Alexandroff-compactification of $]0,1[$

d) As, the space $\mathbb{R}$ is homeomorphe to $]0,1[$, then if $\widetilde{\mathbb{R}}$ is a compactification of $\mathbb{R}$, $\widetilde{\mathbb{R}}$ is homeomorphe to the circle S_1. So, S_1 is also a compactification of $\mathbb{R}$.

e) The interval $]0,1[$ in the space $\mathbb{R}$, embeds into the compact Hausdorff $[0,1]^{\mathbb{N}}$, via a for example, the function $f:]0,1[\longrightarrow [0,1]^{\mathbb{N}}$ defined by: $f(x) = (x, 1,1, \dots)$, for every x in $]0,1[$.
f) Let $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ be, (in this case the two points at infinity of $\widetilde{\mathbb{R}}$ are ω =+∞ and $\omega' = -\infty$) enjoyed by the topology $\tilde{\tau} = \tau_u \cup \tau$ where, τ is the collection of the subsets U of $\widetilde{\mathbb{R}}$ such that, $\mathrm{U} = [-\mathrm{a}, \mathrm{a}]^{C_{\widetilde{\mathbb{R}}}} = [-\infty, -a[\cup]a, +\infty]$, where $a \in \mathbb{R}_+^*$. Then $\widetilde{\mathbb{R}}$ is the two points-compactified of $\mathbb{R}$. (to chek !).
g) Another two points-compactification of the space $\mathbb{R}$ is as follows: The space $\mathbb{R}$ is locally compact, Hausdroff. $cl(\mathbb{R}) = \widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$. $\widetilde{\mathbb{R}}$ is compact, indeed, if $\{U_\alpha, \alpha \in \Delta\}$ is a open cover of $\widetilde{\mathbb{R}}$, there are two elements β and γ in Δ such that $U_\beta = [-\infty, a[$ and $U_\gamma =]b, +\infty]$, where a and b are two constants in $\mathbb{R}$, it follows that the collection $\{U_\alpha, \alpha \in \Delta \setminus \beta,\gamma$ is a open cover of the compact a,b, then there exists a finite set I in $\Delta \setminus \beta,\gamma$ such that $\{U_\alpha, \alpha \in I\}$ covers $[a, b]$, thus $\{U_\alpha, \alpha \in I\} \cup \{U_\beta \cup U_\gamma\}$ covers $\widetilde{\mathbb{R}}$.
h) In the space $\mathbb{R}$, the compactified of $]0,1[\cup]3,4[$ is homeomorphe to a figure eight, thought of as a subspace of the space $\mathbb{R}^2$. More generally, the compactified of the union of disjoint open intervals is homeomorphe to n circles in $\mathbb{R}^2$, that are disjoint except for a single common point.
i) Let $\mathbb{R}$ be the space, the compactified of any open ball in $\mathbb{R}^2$ is homeomorphe to the two dimensional unit sphere S_2 in $\mathbb{R}^3$ via the usual stereographic projection $\pi: S_2 \setminus \{(0.0.1)\} \longrightarrow \mathbb{R}^2$ defined by: $(x, y, z) \longmapsto \pi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$, which is biunivoque and bicontinuous (to chek!). Therefore the compactified of two disjoints open balls in $\mathbb{R}^2$ is homeomorphe to a subspace of $\mathbb{R}^3$ consisting of two spheres touching at only a single point (kissing spheres).

Before talking about the "largest" path of compactification. Remember that, the locally compact Hausdroff space is regular (see, proposition 10.5). We still have better: a locally compact Hausdroff space is completely regular space or Tychonoff space. Because its one point-compactification is compact and Hausdorff, hence it is normal (see, lemma 10.4), therefore it is completely regular, and every subspace of a completely regular space is completely regular.

Starting from the following basic question which arises, when we want to compact a topological space. If $\tilde{E}$ is the compactification of the space E, under what conditions, can a continuous real-valued function φ defined on E, be continuously extended in $\tilde{E}$? Obviously, φ must be bounded, because φ will carry the compact space $\tilde{E}$ into $\mathbb{R}$, and compact parts of $\mathbb{R}$, must be bounded. But being bounded is not enough. A standard example is $\varphi(x) = sin\left(\frac{1}{x}\right)$ defined in $]0,1]$, $\varphi(x)$ is bounded and continuous in $]0,1]$, but it is has no extended continuous over $[0,1]$. Historically, this problem of continuously extending any bounded, continuous real valued function defined on E motivated the development of the Stone-$\hat{C}$ech-compactifcation, which will be exhibited after the following existence results:

Corollary 12.1. Let E be a topological space and let h be an embedding from E to a compact Hausdorff space F. Then, there is a unique (up to equivalence), compactification $(\tilde{E}, f)$ of E, such that there is an embedding j from $\tilde{E}$ into F, with the property that $j(f(x)) = h(x)$ for every $x \in E$.

Proof. Because the map $h: E \longrightarrow h(E)$ is a homeomorphism, and F is a compact Hausdorff space then $\tilde{E} = cl(h(E)) \subset F$ is a compact Hausdorff space (see, proposition 10 2). Therefore $(\tilde{E}, f)$ is a compactification of E. Clearly $\tilde{E}$ embeds into F via the natural inclusion map, $j: \tilde{E} \longrightarrow F$, which has the required properties. To demonstrate the uniqueness of the compactification $(\tilde{E}, f)$ up to equivalence. Suppose $(\tilde{\tilde{E}}, f)$ is another compactification of E that embeds into F via a map $j: \tilde{\tilde{E}} \longrightarrow F$ such that $j(f(x)) = h(x)$ for every $x \in E$. Then,

$j(f(E)) = h(E)$. We must first, demonstrate that: $j\left(\tilde{\tilde{E}}\right) = cl\big(h(E)\big)$. Because$\tilde{\tilde{E}}$, is compact and the map j is continuous then $j\left(\tilde{\tilde{E}}\right)$ is compact in a Hausdorff F, so it is closed (see, proposition 10.3), therefore $cl\big(h(E)\big) = cl\big(j(f(E))\big) \subset cl\left(j\left(\tilde{\tilde{E}}\right)\right) = j\left(\tilde{\tilde{E}}\right)$. Conversely, as $cl(f(E)) = \tilde{\tilde{E}}$, then $j\left(\tilde{\tilde{E}}\right) = j\big(cl(f(E))\big)$, as, by the continuity of j, we have $j\big(cl(f(E))\big) \subset cl\big(j(f(E))\big) = cl(h(E))$, so $j\left(\tilde{\tilde{E}}\right) \subset cl(h(E))$. Therefore, $j\left(\tilde{\tilde{E}}\right) = \tilde{E}$. Because, the map: $j^{-1} \circ i: \tilde{E} \longrightarrow \tilde{\tilde{E}}$ is obviously a homeomorphism, with the required property, then the compactification $\big(\tilde{E}, f\big)$ is unique up to equivalence.

Remark 12.2. The corollary 10.11 says that, the compactification $\tilde{E}$ acts as an intermediary compact Hausdroff space, between the space E and the compact Hausdroff space F. There is no hope of finding an interesting "largest" compactification, that can always act as an intermediary as in the above result. A space can be embedded into its one-point compactification $\hat{E}$ and so any such $\tilde{E}$ could not be largest than $\hat{E}$.

Definition 12.2. Let E be a Tychonoff space. The Stone-Čech compactification of E, is the unique (up to equivalence) Hausdorff compact space denoted by $\beta(E)$, satisfying the following universal property: If f is a continuous function from E into a compact Hausdorff F, there is a unique continuous function βf from $\beta(E)$ into F such that $\beta f \circ i = f$. Where i is a homeomorphism map, from E into $f(E)$.

Theorem 12.2. The Tychonoff space has a Stone-Čech compactification.

Proof. Let E be a Tychonoff space, following the theorem 8.3 (Urysohn lemma), the set of the continuous function from E into the $[0,1]$ is nonempty. Let $\mathcal{T}$=$\{f_\alpha, \alpha \in \Delta\}$ be the collection of such functions. Consider for every $\alpha \in \Delta$, the space, $F = \prod_{\alpha\in\Delta} I_\alpha$, where for every $\alpha \in \Delta$, I_α=$[inf f_\alpha(E), sup f_\alpha(E)]$, which is homeomorphic to $[0,1]$, therefore by corollary 10.10, the cube $F = I^\Delta$ is compact and Hausdroff. As obviously, the family $\mathcal{T}$ is separates points and closed sets in E witch is Hausdorff (the singletons are closed), then $\mathcal{T}$ separates points as well. By the lemma 8.2 (embidding lemma), the function $f: E \longrightarrow F = \prod_{\alpha\in\Delta} I_\alpha$ defined by: for every $x \in E$, $f(x) = \prod_{\alpha\in\Delta} f_\alpha(x)$ is an embedding. We will check that, the space $\beta(E) = cl(f(E))$ in F is the Stone-$\hat{C}$ech compactification of E. Since $\beta(E)$ is a closed subspace of the cub I^Δ, then it is a compact Hausdroff. From the corollary 12.1, there is a unique (up to equivalence), compactification $(\beta(E), i)$ of E, such that there is an embedding βf from $\beta(E)$ into F, with the property that $\beta f \circ i = f$. It remains to prove that the desired application βf is well defined and unique. As, $\beta(E)$ is a Hausdroff compact space, therefore it is normal. Since $E$is closed in $\beta(E)$ and for every $\alpha \in \Delta$, $f_\alpha: E \longrightarrow I_\alpha$ is continuous and bounded, by the lemma 8.3 and the Tietze-Hurysohn theorem 8.2, there is a unique continuous extension function βf_α of f_α defined from $\beta(E)$ into I_α. The, the function βf from $\beta(E)$ into F defined by: for every $x \in \beta(E)$, $\beta f(x) = \prod_{\alpha\in\Delta} \beta f_\alpha\,(x)$ is a unique extended continuous function of f. Therefore $(\beta(E), i)$ is the Stone-$\hat{C}$ech compactification of E.

separation axioms,
metric compact space, metrizability

13.1-Metric space and separation axioms

Before giving the properties of metric spaces, we will give a very interesting result related to the countability and separation axioms.

Proposition 13.1. A metric space is 1D-space.

Proof. Let x an element in the metric space E and $N \in \mathcal{N}(x)$, there exists $r > 0$, such that $B(x,r) \subset N$, therefore there exists $n \in \mathbb{N}^*$ such that $B\left(x,\frac{1}{n}\right) \subset B(x,r) \subset N$, the sequence of sets $\left\{B\left(x,\frac{1}{n}\right), n \in \mathbb{N}^*\right\}$ constitute a system of countable neighborhoods of x, then E is 1D-space.

Theorem13.1. Let E be a metric space. Then, E is separable iffy it is 2D-space.

Proof. As if O is an open in of the metric space (E,τ_d), containing the element x of E, there exists $r > 0$, such that $B\left(x,\frac{r}{2}\right) \subset B(x,r) \subset O$, therefore there exists $n_0 \in \mathbb{N}^*$, such that $B\left(x,\frac{1}{n_0}\right) \subset B\left(x,\frac{r}{2}\right)$. Since E has an everwhere dense part $D = \{x_m, m \in \mathbb{N}\}$, then $B\left(x,\frac{1}{n_0}\right) \cap D \neq \emptyset$, so, there exists $m_0 \in \mathbb{N}$ such that $x_{\mathrm{m}_0} \in B\left(x,\frac{1}{n_0}\right)$ which implies that $x \in B\left(x_{\mathrm{m}_0},\frac{1}{n_0}\right)$, as $\forall y \in B\left(x_{\mathrm{m}_0},\frac{1}{n_0}\right)$, $d(x,y) \leq d(x,x_{\mathrm{m}_0}) + d(x_{\mathrm{m}_0},y) < \frac{2}{n_0} < r$, then $B\left(x_{\mathrm{m}_0},\frac{1}{n_0}\right) \subset O$. Therefore the collection $\left\{B\left(x_m,\frac{1}{n}\right), (m,n) \in \mathbb{N} \times \mathbb{N}^*\right\}$ is a countable basis of τ_d, then (E,τ_d) is 2D-space. For the reverse, let $\{B_n, n \in \mathbb{N}\}$ be a countable basis of τ_d, then the countable collection $D = \{x_n, \text{where } \forall n \in \mathbb{N}, x_n \in B_n\}$ is everwhere dense in E. Indeed, if $x \in E$ and $N \in \mathcal{N}(x)$ there exists $n_0 \in \mathbb{N}^*$ such that $x \in B_{n_0} \subset N$, as $x_{n_0} \in B_{n_0}$ then $B_{n_0} \cap D \neq \emptyset$, which implies that $N \cap D \neq \emptyset$, so $x \in cl(D)$.

Proposition 13.2. A metric space is Hausdorff space.

Proof. Let $x,y \in E, x \neq y$ then $B\left(x,\frac{r}{2}\right)$ and $B\left(y,\frac{r}{2}\right)$, where $r = d(x,y)$, are two disjoint open sets in E, since if there exists $z \in B\left(x,\frac{r}{2}\right) \cap B\left(y,\frac{r}{2}\right)$, then $r = d(x,y) \leq d(x,z) + d(z,y) < \frac{r}{2} + \frac{r}{2} = r$. So, E is Hausdorff.

Definition 13.1. The sequence $\{x_n\}$ in the metric space (E,d) converges to $x \in E$, iffy for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}^*$ (n_0 depends to ε) such that for all $n > n_0$, [$x_n \in B(x,\varepsilon) \Leftrightarrow$ *dx,xn<ε.* Equivalently, lim*n⟶+∞dx,xn=0.*

Not that, the definition 13.1, remains true when the inequalities $n > n_0$ and $d(x,x_n) < \varepsilon$, are large.

Proposition 13.3. In a metric space, any convergence sequence is bounded. The reverse is false.

Proof. If $\{x_n\}$ converges to x in the metric space (E,d), then for $\varepsilon > 0$, there is $N \in \mathbb{N}^*$ such that for all $n > n_0$, $d(x,x_n) < \varepsilon$. let $\varepsilon' = \max_{n<N} d(x,x_n)$ then, $\{x_n\} \subset B(x,r)$ where $r = max(\varepsilon,\varepsilon')$. The reverse is false, as in the usual metric space $(\mathbb{R}, d_u)$, the sequence $\{(-1)^n\}$

Proposition 13.4. Let A be a nonvoide part of a metric space (E,d), and $a \in E$. Then:

$a)$ $a \in cl(A) \Leftrightarrow d(a,A) = 0 \Leftrightarrow \forall\varepsilon > 0, \exists a_\varepsilon \in A;\ 0 < d(a_\varepsilon,a) < \varepsilon$.

$b)$ For $r > 0$, the set $N_r(A) = \{x \in E, d(x,A) < r\}$ is an open neighborhood of A.

$c)$ $cl(A) = \cap_{n\in\mathbb{N}^*}\left(N_{\frac{1}{n}}(A)\right)$.

Proof. $a)$ It suffices, to demonstrate the first equivalence, the second one comes from the property of the infimum. Let $a \in cl(A)$, since $d(a,A) \leq d(a,x)\ \forall x \in A$, then $0 \leq d(a,A) \leq d(a,a) = 0$, so $d(a,A) = 0$. Inversely, if $d(a,A) = 0$, for any ε>0, there exists $a_\varepsilon \in A$; $d(a_\varepsilon,a) < \varepsilon$, then $a_\varepsilon \in B(a,\varepsilon)$ therefore $B(a,\varepsilon) \cap A \neq \emptyset$, then $a \in cl(A)$. $b)$ Let $x \in A$, since $d(x,A) \leq d(x,y), \forall y \in A$, then $0 \leq d(x,A) \leq d(x,x) = 0 < r$, so $x \in N_r(A)$ and $A \subset N_r(A)$. As, for $x \in N_r(A)$ and $\rho = r - d(x,A)$, the open ball $B(x,\rho) \subset N_r(A)$, then $N_r(A)$ is open. So for r>0, $N_r(A)$ is an open neighborhood of A. $c)$ If, $x \in cl(A)$, then

$d(x,A)=0<\frac{1}{n},\forall n\in\mathbb{N}^*$, so $x\in N_{\frac{1}{n}}(A),\forall n\in\mathbb{N}^*$ then $cl(A)\subset\cap_{n\in\mathbb{N}^*}\left(N_{\frac{1}{n}}(A)\right)$, if now $x\in N_{\frac{1}{n}}(A),\forall n\in\mathbb{N}^*$, then $d(x,A)<\frac{1}{n},\forall n\in\mathbb{N}^*$ so $d(x,A)=0$, from $a)$ $x\in cl(A)$.

Corollary 13.1. Let (E,τ) be a space, $a\in E$, $A\subset E$, and le d be a metric on E. Then τ is a topology induced by d, iffy the following statement holds:
$a\in cl(A)$ in (E,τ) $\Leftrightarrow$for all $\varepsilon>0$, there is $a_\varepsilon\in A$ such that $d(a,a_\varepsilon)<\varepsilon$. $(*)$

Proof. As (E,d) is a metric space, by the proposition 13.4 $a)$, the statement $(*)$ holds. Conversely, let $O\in\tau$ be and $a\in O$ then, there is $r>0$ such that $B(a,\varepsilon)\subset O$. If, not, for all $\varepsilon>0$, $B(a,\varepsilon)\cap O^C\neq\emptyset$. Then, there is $a_\varepsilon\in O^C$, such that $d(a,a_\varepsilon)<\varepsilon$, so $a\in cl(O^C)=O^C$, contradiction.

Theorem 13.2. A metric space is a T_4-space.

Proof. Let F and G are two closed sets in a metric space E, and the two sets $O_F=\{x\in E,d(x,F)<d(x,G)\}$ and $O_G=\{x\in E,d(x,G)<d(x,F)\}$ by proposition 13.4, $b)$ O_F and O_G are open sets containing respectively F and G. In addition $O_F\cap O_G=\emptyset$, indeed, if $z\in O_F\cap O_G$ then $d(z,G)<d(z,F)$ and $d(z,F)<d(z,G)$, so d(z,G)-d(z,F)<0<d(z,G)-d(z,F), impossible. Therefore E is normal because it is T_1, then it is a T_4-space.

As a consequence of the above results, and the relation between the separation axioms, we have:

Corollary 13.2. The metric space is T_0, T_1, T_2, T_3, T_4, regular and normal.

Definition 13.2. Two distances d_1 and d_2, on a non-empty set E, are said to be equivalent, and we write $d_1\backsim d_2$, if there are two, strictly positive real numbers α and β, such that: $\alpha d_1\leq d_2\leq\beta d_1$, i.e. $\alpha d_1(x,y)\leq d_2(x,y)\leq\beta d_1(x,y),\forall x,y\in E$.

Example 13.1. In the n-dimensional Euclidian space $\mathbb{R}^n$, $n\in\mathbb{N}^*$, d_1, d_2 and d_∞ are equivalents. Indeed, $a)$ $d_\infty\leq d_1\leq\sqrt{n}d_\infty$. $b)$ $d_\infty\leq d_2\leq\sqrt{n}d_1\leq nd_\infty$. c) From $a)$ and $b)$ $\frac{1}{\sqrt{n}}d_1\leq d_2\leq\sqrt{n}d_1$. Then $d_\infty\backsim d_1\backsim d_2\backsim d_\infty$.

Proposition 13.5. Let $d_1 and\ d_2$, are two distances on E, if there exists $\gamma\in\mathbb{R}_+^*$ such that: $d_1\leq\gamma\, d_2$. Then, $\tau_{d_1}\subset\tau_{d_2}$.

Proof. If, $x\in O\in\tau_{d_1}$, there exists r>0, such that $B_1(x,r)=\{y\in E,d_1(x,y)<r\}\subset O$, as $d_1\leq\gamma\, d_2$, then $B_2(x,\frac{r}{\gamma})=\left\{y\in E,\ d_2(x,y)<\frac{r}{\gamma}\right\}\subset B_1(x,r)$, so $O\in\tau_{d_2}$.

It is straightforward to check that:

Corollary 13.3. Two equivalent distances define the same topology and exchange the sequences convergence i.e. $\{x_n\}$ converges in (E,d) iffy, $\{x_n\}$ converges in (E,d').

Definition 13.3. Two metrics d and d' over the space E, are said to be, topologically equivalent or t-equivalent. If, the identity map i: $(E,\tau_d)\longrightarrow(E,\tau_{d'})$, is a homeomorphism, i.e. d and d' induce the same topology.

By, the corollary 13.3, the equivalent distances are t-equivalent. But, the converse is not true.

Example 13.2. Let (E,d) be a metric space, then the distances d and $\delta=\frac{d}{1+d}$ are t-equivalent but are not equivalents. It is clear that $\delta\leq d$, but if there exists, a strictly positive real number α, such that $\alpha d\leq\delta$ then $d(x,y)\leq\frac{1}{\alpha},\forall x,y\in E$, with implies that, in $(\mathbb{R},d)$, $d(x,y)=|x-y|\leq\frac{1}{\alpha},\forall x,y\in\mathbb{R}$, so $d(x,0)=|x|\leq\frac{1}{\alpha},\forall x\in\mathbb{R}$, then for $x=\frac{21}{\alpha}$, we have $2\leq1$ impossible.

Definition 13.4. Let A be a nonvoide part of the metric space (E,d). The restriction d_A of d to A i.e.d_A: $A\times A\to\mathbb{R}_+$, is a metric on A called the induced metric from d and (A,d_A) is called metric subspace of the metric space (E,d).

Let τ_{d_A} be, the associated topology to the metric subspace (A, d_A) and τ_A^d the induced topology of τ_d on A. Let $x \in A$, $B_{d_A}(x,r)$ (respectively $B_A(x,r)$) the open ball in (A, d_A) (respectively the open ball in (A, τ_A^d). Note that, $B_A(x,r) = A \cap B(x,r)$, where B(x,r) is the open ball in E, centered in x, with radius $r > 0$ and $B_{d_A}(x,r) = B_A(x,r)$, indeed if $y \in B_{d_A}(x,r)$, $y \in A$ and $d_A(x,y) = d(x,y) < r$, so $y \in A \cap B(x,r) = B_A(x,r)$, if now, $y \in B_A(x,r) = A \cap B(x,r)$, then $y \in A$ and $d(x,y) < r$, so $d_A(x,y) = d(x,y) < r$ implies $y \in B_{d_A}(x,r)$. Then:

Corollary 13.4. $\tau_A^d = \tau_{d_A}$.

Proof. If, $U \in \tau_A^d$ and $x \in U$, there exists $r > 0$, such that $B_A(x,r) = A \cap B(x,r) \subset U$, since the open ball $B_{d_A}(x,r)$ centred in x, with radius r in (A, d_A), is contained in $B_A(x,r)$, then $U \in \tau_{d_A}$, so $\tau_A^d \subset \tau_{d_A}$. Now, if $x \in U \in \tau_{d_A}$, there exists $r > 0$, such that $B_{d_A}(x,r) \subset U$, since $B_A(x,r) \subset B_{d_A}(x,r)$, then $U \in \tau_A^d$, so $\tau_{d_A} \subset \tau_A^d$.

13.2-Compact metric space

Before giving a characterization of a compact metric space, we will demonstrate the following property.

Lemma 13.1. In a metric space the assertions $a)$ and $b)$ are equivalent:

$a)$ Every infinite part has an accumulation point.

$b)$ Every sequence has a convergence subsequence.

Proof. $a) \Longrightarrow b)$. Let $A = \{x_n\}$ be a sequence in the metric space E, as A is a countable part it is an infinite part by $a)$ A has an accumulation point $x \in cl(A)$. Because E is 1D-space, by proposition 7.5, there is sequence $\{x_m\} \subset A$, which converges to x i.e. $\{x_n\}$ has a convergence subsequence. $b) \Longrightarrow a)$. Let A be the infinite part in the metric space E. As A containing a sequence $\{x_n\}$ which has a subsequence $\{x_{\varphi(n)}\}$ that converges towards x, then x is an accumulation point of A.

The study of compact metric spaces is based on the following fundamental lemma.

Lemma 13.2. If, in the metric space (E, d), any sequence of the closed part A in E, has a convergence subsequence in A. Then, for any open cover $\mathcal{U} = \{U_\alpha, \alpha \in \Delta\}$ of A, there is $r > 0$ such that for all $x \in A$, $B(x,r)$ is containing in at last one of the element of $\mathcal{U}$.

Proof. Suppose that, for all $\varepsilon > 0$, there is $x_\varepsilon \in A$ such that $B(x_\varepsilon, \varepsilon) \not\subset O_\alpha$, for all $\alpha \in \Delta$. Then, for all $n \in \mathbb{N}^*$, there is $x_n \in A$, such that $B\left(x_n, \frac{1}{n}\right) \not\subset U_\alpha$ for all $\alpha \in \Delta$. As there is a subsequence $\{x_{\varphi(n)}\}$ of the sequence $\{x_n\}$ which converges to $a \in A = \bigcup_{\alpha\in\Delta} U_\alpha = cl(A)$, then there is $i \in \Delta$, such that $a \in U_i$ thus, there is $\rho > 0$, such that $B(a, \rho) \subset U_i$. Because, for all $x \in B\left(x_{\varphi(n)}, \frac{1}{n}\right)$, $d(a,x) \leq d\left(a, x_{\varphi(n)}\right) + d\left(x_{\varphi(n)}, x\right) < \frac{1}{n} + d\left(a, x_{\varphi(n)}\right)$, and $x_{\varphi(n)} \longrightarrow a$, then for $\frac{\rho}{2} > 0$, there is $n_0 \in \mathbb{N}^*$ such that for all $n > n_0$, $d\left(a, x_{\varphi(n)}\right) < \frac{\rho}{2}$, when $n \longrightarrow +\infty$ in the inequality we obtain $d(a,x) \leq \frac{\rho}{2} < \rho$, then $B\left(x_{\varphi(n)}, \frac{1}{n}\right) \subset B(a,\rho) \subset U_i$, for all $n > n_0$, contradiction.

We have shown in lemma 10.2 that, in a compact space, every infinite part has an accumulation point, and under the supplementary 1D condition see corollary 10.2 every sequence has a convergence subsequence. In a metric space which is 1D-space, we also have the reciprocal.

Theorem 13.3. In a metric space (E, d). If, every infinite part has an accumulation point, then E is compact.

Proof. Let $\mathcal{O} = \{O_\alpha, \alpha \in \Delta\}$ be an open cover of the closed E, by the lemma 13.1, there is $r > 0$ such that for all $x \in E$, $B(x,r)$ is containing in at last one of the element of $\mathcal{O}$. Let

$x_1 \in E$, then $E \subset B(x_1, r) \subset O_1$, where $O_1 \in \mathcal{O}$, so E is compact. If not, there is $x_2 \in E$ such that $d(x_1, x_2) \geq r$ and $B(x_2, r) \subset O_2$, where $O_2 \in \mathcal{O}$, so $E \subset B(x_1, r) \cup B(x_2, r) \subset O_1 \cup O_2$, so E is compact. After a finite number of iterations, we obtain p-balls $B(x_1, r), B(x_2, r), \dots, B(x_p, r)$ which cover E and therefore p-open $O_1, O_2, \dots, O_p$ which cover E, so E is compact. The points $x_1, x_2, \dots, x_p$ satisfy $d(x_i, x_j) \geq r$ for $i \neq j$. If the number of the points $x_1, x_2, \dots, x_n, \dots$ is infinite, by assumption the sequence $\{x_n\}$, has an accumulation point $x \in E$. Then, for $\frac{r}{2} > 0$ there is $i \in \mathbb{N}^*$, such that $0 < d(x, x_i) < \frac{r}{2}$ and for $\frac{r}{3} > 0$ there is $j \in \mathbb{N}^*$ such that $0 < d(x, x_j) < \frac{r}{3}$. As, $r \leq d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \frac{r}{2} + \frac{r}{3} < r$, contradiction. Conclusion, there is only a finite open bulls centered in $x_1, x_2, \dots, x_p$ with radius $r > 0$, such that $E \subset \cup_1^p B(x_i, r) \subset \cup_1^p O_i$, then E is compact.

As a direct consequence of the lemma 13.1 and theorem 13.3, we have.

Corollary 13.5. In a metric space *E,* if any sequence has a convergence subsequence, *E* is compact.

Lemma 13.3. A part A of the metric space (E, d) is relatively compact iffy, any sequence in *A,* has an adherent value in *E*.

Proof. Let $\{x_n\}$ be a sequence in A, as $A \subset cl(A) \subset E$ then $\{x_n\}$ is a sequence $cl(A)$, which is compact, there is a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ converging to $x \in cl(A)$, then x is an adherent value of $\{x_n\}$ in *E*. Conversely, let $\{x_n\}$ be a sequence in $cl(A)$, then for every $n \in \mathbb{N}^*$, there is $a_n \in A$, such that $\lim_{n\to\infty} d(x_n, a_n) = 0$. As, by assumption the sequence $\{a_n\}$ has an adherent value $a \in cl(A)$, there is a subsequence $\{a_{\varphi(n)}\}$ of the sequence $\{a_n\}$ which converges to a. Because, $0 \leq d(a, x_{\varphi(n)}) \leq d(a, a_{\varphi(n)}) + d(a_{\varphi(n)}, x_{\varphi(n)})$, and $\lim_{n\to\infty} d(x_{\varphi(n)}, a_{\varphi(n)}) = 0$, then $d(a, x_{\varphi(n)}) \longrightarrow 0$, therefore $cl(A)$ is compact.

Lemma 13.4. Let f be a map, from a metric space (E, d), into the topological space (F, τ). Then, f is continuous on E iffy, f is continuous on any compact of *E*.

Proof. If, f is continuous in *E*, then it is continuous in any subset of *E*. Therefore, it is continuous in any compact of *E*. Conversely, let $\{x_n\}$ be a converging sequence to x in *E.* As, by corollary 13.5, the set $\{x, x_1, x_2, \dots\}$ is a compact in *E* and f is continuous on this compact, then $f(x_n) \longrightarrow f(x)$, so f is continuous in the arbitrary x in *E.* Thus, f is continuous on *E*.

Lemma 13.5. A compact metric space is separable.

Proof. Let (E, d) a metric space, because for all $n \in \mathbb{N}^*$, the collection $\left\{B\left(x, \frac{1}{n}\right), x \in E\right\}$ is an open cover of E which is compact, there is a finite set of points of E, say $A_n = \{x_1, x_2, \dots, x_{k(n)}\}$ such $E = \cup_1^{k(n)} B\left(x_i, \frac{1}{n}\right)$. It is clear that, the part $A = \cup_{n\geq 1} A_n$ is a countable subset of *E*. It remains, to show that, $cl(A) =$*E*. Let $x \in E$ be, and $x \in E$ then there is $n_0 \in \mathbb{N}^*$ such that $B\left(x, \frac{1}{n_0}\right) \subset B(x, \varepsilon)$. On the other hand, there is $j \in \{1, \dots, k(n)\}$ such that $x \in B\left(x_j, \frac{1}{n}\right)$ for all $n \in \mathbb{N}^*$, then $x_j \in B\left(x, \frac{1}{n_0}\right)$. Therefore, for all $x \in E$ and all $x \in E$, $\emptyset \neq B\left(x, \frac{1}{n_0}\right) \cap A_n \subset B\left(x, \frac{1}{n_0}\right) \cap A \subset B(x, \varepsilon) \cap A$. As for every $N \in \mathcal{N}(x)$, there is $r > 0$, such that $B(x, r) \subset N$, then $N \cap A \neq \emptyset$, hence $x \in cl(A)$.

Remark 13.1. As by the theorem13.1, a metric space is separable iffy, it is 2D-space. Then, a metric compact space *E* is 2D-space.Therefore, *E* has a countable basis constitute of the open balls. More precisely, the collection $\mathcal{B} = \cup_{n\geq 1} \mathcal{B}_n$, where for all $n \in \mathbb{N}^*$, $\mathcal{B}_n = \left\{B\left(x_i, \frac{1}{n}\right), 1 \leq i \leq k(n)\right\}$ constitutes a basis for the induced metric topology of *E*. Indeed, for any open O in *E* and any $x \in O$, there is $r > 0$, such that $B(x, r) \subset O$. Therefore, there is $n_0 \in \mathbb{N}^*$ such that

$B\left(x,\frac{1}{n_0}\right) \subset B(x,r) \subset O$. As $x \in E = \cup_1^{k(n)} B\left(x_i,\frac{1}{n}\right)$, there is $j \in \{1,\dots,k(n)\}$ such that $x \in B\left(x_j,\frac{1}{n}\right)$ for all $n \in \mathbb{N}^*$, thus $x \in B\left(x_j,\frac{1}{n_0}\right)$. It remains to check that, $B\left(x_j,\frac{1}{n}\right) \subset B(x,r)$. Let $y \in B\left(x,\frac{1}{n}\right)$, as $d(x,y) \le d(x,x_j) + d(x_j,y) \le \frac{1}{n_0} + \frac{1}{n}$, when $n \longrightarrow \infty$, $d(x,y) \le \frac{1}{n_0} < r$. So, $y \in B(x,r) \subset O$.

13.3-Product Metric space

Let (E_i,d_i), $1 \le i \le n$ be a finite collection of metric spaces, it is obvious to check that the space $E = \Pi_1^n E_i$ provided with one of the three distance $D_1(x,y) = \sum_1^n d_i(x_i,y_i)$, $D_2(x,y) = \sqrt{\sum_1^n d_i^2(x_i,y_i)}$ or $D_\infty(x,y) = \max_{1\le i\le n} d_i(x_i,y_i)$, for all $x,y \in E$, is a metric space. Furthermore, $D_\infty \le D_2 \le D_1 \le nD_\infty$, i.e. D_1, D_2 and D_∞ are equivalent and if τ is the induced product topology of τ_{d_i}, $1 \le i \le n$, then $\tau = \tau_{D_\infty}$. It suffices to consider $B_\infty(x,r) = \Pi_1^n B(x_i,r)$, where $B(x_i,r) = \{y_i \in E_i, d_i(x_i,y_i) < r\}$, $1 \le i \le n$.

In the case, where the collection $\{(E_n,d_n), n \in \mathbb{N}\}$ of metric spaces is countable. In general we cannot define D1 because the series $\sum_0^\infty d_n(x_n,y_n)$ is not always convergent. On the other hand, by considering the distances on E_n, $(x_n,y_n) \mapsto \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$ which are t-equivalent to the distance d_n, for all $n \in \mathbb{N}^*$. We can define for all $x,y \in E$ $d(x,y) = \sum_1^\infty \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$ which is well defined, whose induced topology τ_d is identical to the product topology τ. Where for $r > 0$ and $x \in E$, the collection $\{B_n(x,r), 1 \le n \le i\}$, with $B_n(x,r) = \left\{y \in E, \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} < r\right\}$, $1 \le n \le i$ constitute a basis of this product topology. To prove that $\tau = \tau_d$. Let, for $1 \le n \le i$, $B_n(x,r)$ then $B(x,\rho)$ of E, where $0 < \rho \le \frac{r}{2^n} \le r$ is containing in $B_n(x,r)$. Because, if $y \in B(x,\rho)$, $d(x,y) = \sum_1^\infty \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} < \rho$ then $\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} < \rho 2^n$, so for $\rho 2^n \le r$, $B(x,\rho) \subseteq B\left(x,\frac{r}{2^n}\right) \subseteq B_n(x,r)$. Conversely, let $B(x,\rho)$ be in E, then for $1 \le n \le i$, $B_n(x,r) \subseteq B(x,\rho)$, where $r = \frac{\rho}{2}$. Because, if $y \in B_n(x,r)$, then $\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$<r for $1 \le n \le i$. As for i big enough, $\sum_{i+1}^\infty \frac{1}{2^n} \longrightarrow 0$, for $r > 0$, there is $i \in \mathbb{N}^*$ such that for all $n > i$, we have $\sum_{i+1}^\infty \frac{1}{2^n} < r$. So, $d(x,y) = \sum_1^\infty \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} = \sum_1^i \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)} + \sum_{i+1}^\infty \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$<r$\left(\sum_1^i \frac{1}{2^n}\right) + r$<r$\left(\sum_1^\infty \frac{1}{2^n}\right) + r$=2r= ρ.

Example 13.3. let $l_2 = \{x = \{x_n\} \subset \mathbb{R}, \sum_1^\infty x_n^2 < +\infty\}$ be, the set of summable square numerical sequences. The function $d: l_2 \times l_2 \longrightarrow \mathbb{R}^+$ defined by $d(x,y) = \sqrt{\sum_1^\infty (x_n - y_n)^2}$, for all $x,y \in l_2$ is a metric. As $(x_n \pm y_n)^2 \le 2(x_n^2 + y_n^2)$, $d(x,y)$ is well defined. The conditions $m_1)$ and $m_2)$ in the metric's definition are obviously checked. Concerning the condition $m_3)$ it suffices to take the limit when $k \longrightarrow \infty$ in the inequality: $\sqrt{\sum_1^k (x_n - y_n)^2} \le \sqrt{\sum_1^k (x_n - z_n)^2} + \sqrt{\sum_1^k (z_n - y_n)^2}$, where $x,y,z \in l_2$.

Definition 13.5. A space (E,τ), by is said to be metrizable if, there is a metric d in E, such that the topology τ_d induced by d is equal to τ.

As we have already seen, a metric offers one of the most important definition of the topology of a space, and that a metric space is 1D-space and it is normal. Therefore, in a space devoid at least one of these two properties, it is impossible to define the topology using a metric. However, we have the following theorem, which is relatively simple in comparison with to other metrization theorems and it is an highly useful result, for determining whether a given space is metrizable.

Theorem13.4. (Urysohn's metrization theorem). The regular, 2D-space is metrizable.

Proof. As E is regular and 2D-space, by, corollary 10.12, E is normal and has a countable basis $\mathcal{B} = \{B_n, n \in \mathbb{N}\}$. By corollary 5.6, we can consider the family of all pairs (B_i, B_j) of $\mathcal{B}$, with $cl(B_i) \subset B_j$, clearly this family is countable, so we can write the pairs as $P_1, P_2, \dots, P_n, \dots$. Since, $B_{n_j}^C$ is closed and $cl(B_{n_i}) \cap B_{n_j}^C = \emptyset$, by theorem 8.1 (Urysohn's lemma), for all pair $P_n = (B_{n_i}, B_{n_j})$, there is a continuous function $f_n: E \longrightarrow [0,1]$ such that $f_n\left(cl(B_{n_i})\right) = 0$ and $f_n\left(B_{n_j}^C\right) = 1$. We will verify that, a function $d: E \times E \longrightarrow \mathbb{R}_+$, defined by, for all $x, y \in E$, $d(x,y) = \sum_1^\infty \frac{1}{2^n}|f_n(x) - f_n(y)|$ which is well defined, is a metric. It follows to check the conditions $m_1), m_2)$and $m_3)$ in the definition 4.1.

$m_1)$ Let $x, y \in E$ be, such that $x \neq y$, i.e. $x \in \{y\}^C$, which is open in T_1-space, by the corollary 5.6, there is $k \in \mathbb{N}$ such that $P_k = (B, B')$, with $x \in B \subset cl(B) \subset B' \subset \{y\}^C$. By Urysohn's lemma, there is a function $f_k: E \longrightarrow [0,1]$ such that $f_k(cl(B)) = 0$ and $f_k({B'}^C) = 1$, as $x \in cl(B)$ and $y \in {B'}^C$, then $f_k(x) = 0$ and $f_k(y) = 1$, then $|f_k(x) - f_k(y)| = 1$, therefore $d(x,y) > 0$. If, now, $x = y$, then $f_n(x) = f_n(y)$ for all $n \in \mathbb{N}$, so $d(x,y) = 0$.

$m_2)$ As, for all $x, y \in E$ and for all $n \in \mathbb{N}$, $|f_n(x) - f_n(y)| = |f_n(y) - f_n(x)|$, then $d(x,y) = d(y,x)$

$m_3)$ As, for all $x, y, z \in E$ and for all $n \in \mathbb{N}$, $|f_n(x) - f_n(y)| \leq |f_n(x) - f_n(z)| + |f_n(z) - f_n(y)|$, then $d(x,y) = \sum_1^\infty \frac{1}{2^n}|f_n(x) - f_n(y)| \leq \sum_1^\infty \frac{1}{2^n}(|f_n(x) - f_n(z)| +$
fnz−fny=1∞ 12nfnx−fnz+1∞ 12nfnz−fny =dx,z+ dz,y.

It remains to prove that τ is induced by the metric d. By the corollary 13 1 it suffices to prove that, if $A \subset E$ and $a \in E$. Then $a \in cl(A)$ in $E \Leftrightarrow \forall\varepsilon > 0, \exists a_\varepsilon \in A;\ d(a_\varepsilon, a) < \varepsilon$

" $\Longrightarrow$ " Let $a \in cl(A)$ be and $\varepsilon > 0$, since the sequence $\frac{1}{2^n} \longrightarrow 0$, there is N$\in \mathbb{N}^*$ such that, for all $n > N$, $\frac{1}{2^n} < \frac{\varepsilon}{2}$, then $\frac{1}{2^{N+1}} < \frac{\varepsilon}{2}$. As, for all $i \in \{1, \dots, N\}$, the function f_i is continuous from E into $[0,1]$ and $\frac{\varepsilon}{4N} > 0$, by proposition 7.11, there is $O_1, \dots, O_i, \dots, O_N \in \tau$ containing a, such that for all $x, y \in O_i$, $|f_i(x) - f_i(y)| < \frac{\varepsilon}{4N}$. Because the set O$=\cap_{i=0}^N O_i$ is an open containing a, then $O \cap A \neq \emptyset$, so there is $a_\varepsilon \in O \cap A$, then

$d(a, a_\varepsilon) =$
$\sum_1^\infty \frac{1}{2^n}|f_n(a) - f_n(a_\varepsilon)| = \sum_1^N \frac{1}{2^i}|f_i(a) - f_i(a_\varepsilon)| + \sum_{N+1}^\infty \frac{1}{2^i}|f_i(a) - f_i(a_\varepsilon)| < \sum_1^N 2\left(\frac{\varepsilon}{4N}\right) +$
$\sum_{N+1}^\infty \left(\frac{2}{2^i}\right) = \frac{\varepsilon}{2} + \frac{2}{2^{N+1}}\sum_0^\infty \left(\frac{1}{2}\right)^i = \frac{\varepsilon}{2} + \frac{4}{2^{N+1}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

" $\Longleftarrow$ " Suppose that $a \in E$ satisfies $\forall\varepsilon > 0, \exists a_\varepsilon \in A;\ d(a_\varepsilon, a) < \varepsilon$, but $a \notin cl(A)$, then there is $O \in \tau$ containing a, such that $O \cap A = \emptyset$, then the closed $\{a\} \subset O$. Once again, by the corollary 5.6 and Urysohn's lemma, there is a pair of basis elements $P_k = (B, B')$, which satisfies $a \in B \subset cl(B) \subset B' \subset O$ and there is a function $f_k: E \longrightarrow [0,1]$ as $cl(B)$ and ${B'}^C$ are two disjoint closed, then $f_k(a) = 0$ and $f_k(b) = 1$, for all $b \in {B'}^C$, so $|f_k(a) - f_k(b)| = 1$. As for $0 < \varepsilon < \frac{1}{2^k}$ we have $d(a,x) \geq \frac{1}{2^k} > \varepsilon$ for all $x \in A$, ($x \notin O$ then $x \notin B'$, so $x \in {B'}^C$), contradiction.

As we noticed in example 4.1 c) and example 5.2 c) iii). The uncountable discrete space is a metric space which is not 2D-space. Then, we don't have the opposite in the theorem 13.4. But under the compact assumption we have.

Corollary 13.6. Let E a compact Hausdorff space. Then, E is metrizable iffy it is 2D-space.

Proof. As, a compact Hausdorff space is normal, if in addition it is 2D-space, by the theorem 13.4 it is metrizable. Conversely, from the theorem 13.1 and the corollary 13.5, the metric compact space is 2D-space.
Remar 13.1.
a) Let $K = \left[0, \frac{1}{n}\right]^{\mathbb{N}^*}$, be the set of all numerical sequences $\{x_n\} \subset \left[0, \frac{1}{n}\right]$, enjoyed by the metric $d(x,y) = \sqrt{\sum_1^{\infty}(x_n - y_n)^2}$, where $x, y \in K$. It is clear that Q is a metric space containing in l_2. The set $Q = [0,1]^{\mathbb{N}}$ under the sup metric $d(x,y) = \sup_{n\in\mathbb{N}}|x_n - y_n|$, where $x, y \in Q$ is said to be a Hilbert cub and it is identified up to homeomorphisms with K. By Tychonoff's theorem Q is compact. Therefore Q is compact metrizable space and hence it is normal 2D-space. So a space is a normal 2D-space⟺it is homeomorphe to Q.
b) We can also prove the theorem 13.4, by using the lemma 8.2 (embidding lemma) to check, with almost the same tools used in the proof of theorem 12.2, that the space E can be embedded in the compact metric Hilbert cub $[0,1]^{\mathbb{N}}$. Therefore E is metrizable.
Corollary 13.7. Let $\{(E_i, d_i), 1 \le i \le n\}$ be a finite collection of metric spaces and $E = \Pi_1^n E_i$. Then, the sequence $\{x_n\}$ converges to x in (E, D_∞) iffy, the sequence $\{x_n^i\}$ converges to x_i, in (E_i, d_i), for all $1 \le i \le n$.
Proof. As $0 \le d_i(x_p^i, x_q^i) \le D_\infty(x_p, x_q)$, for all $p, q \in \mathbb{N}^*$ and for all $1 \le i \le n$. If, $D_\infty(x_p, x_q) \longrightarrow 0$, then $d_i(x_p^i, x_q^i) \longrightarrow 0$, for all $1 \le i \le n$. Reciprocally, let $\varepsilon > 0$, as for all $1 \le i \le n$, and for all $p, q \in \mathbb{N}^*$, $d_i(x_p^i, x_q^i) \longrightarrow 0$, there is $n_0^i \in \mathbb{N}^*$, such that for all $p > n_0^i, q > n_0^i$, we have $d_i(x_p^i, x_q^i) < \varepsilon$, for all $1 \le i \le n$. So, $\sup_{1\le i\le n} d_i(x_p^i, x_q^i) < \varepsilon$, for $n_0 = \sup_{1\le i\le n} n_0^i$ and all $p > n_0, q > n_0$, we have $D_\infty(x_p, x_q) < \varepsilon$.

14-Complete metric space, fixed point theorem

14.1-Cauchy sequence

Definition14.1. A sequence $\{x_n\}$ in a metric space (E, d) is said to be a Cauchy sequence or simply a Cauchy if, for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$ (n_0 depends to ε) such that, for all $p, q > n_0$, we have $d(x_p, x_q) < \varepsilon$. Equivalently, $\lim_{p,q\to+\infty} d(x_p, x_q)=0$.
Note that, the definition 14.1 remains true when the inequalities $p, q > n_0$, and $d(x_p, x_q) < \varepsilon$,. are large.
Proposition 14.1. The Cauchy sequence is bounded.
Proof. If, $\{x_n\}$ is a Cauchy sequence, for $\varepsilon = 1$, there is $n_0 \in \mathbb{N}^*$ such that, for all $p > n_0$, $q > n_0$ we have $d(x_p, x_q) < 1$. As, for all $n, p \in \mathbb{N}^*$, $d(x_{n_0}, x_n) \le d(x_{n_0}, x_p) + d(x_p, x_n)$ then, for all $n > n_0, p > n_0$, $d(x_{n_0}, x_n) < d(x_{n_0}, x_p) + 1$. So for all $n \in \mathbb{N}^*$, $d(x_{n_0}, x_n) \le max_{1\le p\le n_0} d(x_{n_0}, x_p) + 1$=r i.e. $\{x_n\} \subset B(x_{n_0}, r)$, thus $\{x_n\}$ is bounded.
Proposition 14.2. In a metric space, every convergence sequence is a Cauchy.
Proof. Let $\varepsilon > 0$ be, as $\{x_n\}$ converges to x in the metric space (E, d), then when $p, q \in \mathbb{N}^*$ tend towards $+\infty$, $d(x_p, x)$ and $d(x_q, x)$ tend towards 0. As $0 \le d(x_p, x_q) \le d(x_p, x) + d(x, x_q)$ So, when $p, q \in \mathbb{N}^*$ tend towards $+\infty$, $d(x_p, x_q) \longrightarrow 0$.
Remark 14.1.
a) The reciprocal of the proposition 14.1, is not true. Indeed, in the usual metric space $\mathbb{R}$, the sequences $\{sin(n)\}$, $\{(-1)^n\}$ are bounded but there are not a Cauchy.
b) The reciprocal of the proposition 14.2, is not allows true. Indeed, in the metric space $]0,1]$, the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy but it is not convergent in $]0,1]$.
Proposition 14.3. A subsequence of a Cauchy is also a Cauchy.

Proof. As $\{x_n\}$ is a Cauchy in the metric space (E, d), there is $n_0 \in \mathbb{N}^*$ such that, for all $p > n_0, q > n_0$ we have $d(x_p, x_q) < \varepsilon$. So, if $\{x_{\varphi(n)}\}$ is a subsequence of $\{x_n\}$, $\varphi(p) \geq p > n_0$ and $\varphi(q) \geq q > n_0$ then $d(x_{\varphi(p)}, x_{\varphi(q)}) < \varepsilon$, therefore $\{x_{\varphi(n)}\}$ is a Cauchy.

Proposition 14.4. If a subsequence of a Cauchy converges to x, then the Cauchy also converges to x.

Proof. Let $\{x_{\varphi(n)}\}$ be a subsequence of the Cauchy $\{x_n\}$, which converges to x in the metric space (E, d). As $0 \leq d(x_n, x) \leq d(x_n, x_{\varphi(n)}) + d(x_{\varphi(n)}, x)$ for all $n \in \mathbb{N}$ and $d(x_n, x_{\varphi(n)}) \longrightarrow 0, d(x_{\varphi(n)}, x) \longrightarrow 0$, when $n \longrightarrow 0$, then $d(x_n, x) \longrightarrow 0$, when $n \longrightarrow 0$. So $\{x_n\}$ converges to x.

It is clear that:

Corollary 14.1. If, the sequence $A_n = \{x_k, k \geq n\}$ in the metric space (E, d) satisfies $\delta(A_n) \longrightarrow 0$, then the sequence $\{x_n\}$ is a Cauchy.

Definition 14.2. The map $f: (E, d) \longrightarrow (F, d')$ is said to be continuous in $x_0 \in E$. If, for all $\varepsilon > 0$, there is $\delta > 0$ (δ depends to x_0 and ε), such that, for all $x \in E$ satisfies $0 < d(x, x_0) < \delta$, we have $d'(f(x), f(x_0)) < \varepsilon$. f is said to be continuous on *E,* if it is continuous in any element of *E.*

We will now introduce, a property closely related to metric spaces.

Definition 14.3. The map $f: (E, d) \longrightarrow (F, d')$ is said to be uniformly continuous on E. If, for all $\varepsilon > 0$, there is $\delta > 0$ (δ depends to ε), such that, for all $x, y \in E$ satisfies $0 < d(x, y) < \delta$, we have $d'(f(x), f(y)) < \varepsilon$.

Remark 14.2.

a) The definitions 14.2 and 14.3, remains true when, the second and third inequalities are large.

b) The uniform continuity implies the continuity. It suffices to take $y = x_0$ in the definition 14.2.

c) The Continuity dos not implies the uniform continuity.

Example 14.1. The function $x \in \mathbb{R} \mapsto f(x) = x^2 \in \mathbb{R}_+$ is continuous on $(\mathbb{R}, d_u)$. But, for $\varepsilon = 1$ and for any $\delta > 0$, there is $x_\delta = \frac{1}{\delta}$, $y_\delta = \frac{1}{\delta} + \frac{\delta}{2} \in \mathbb{R}_+^*$, such that $d_u(x_\delta, y_\delta) = |x_\delta - y_\delta| = \frac{\delta}{2} < \delta$ but $d_u(f(x_\delta), f(y_\delta)) = |f(x_\delta) - f(y_\delta)| = 1 + \frac{\delta^2}{4} > 1$. Then, *f* is not uniformly continuous on $\mathbb{R}$.

Corollary 14.2. Let $f: (E, d) \longrightarrow (F, d')$ be uniformly continuous on E Then, the image $\{f(x_n)\}$ in the metric (F, d') of the Cauchy $\{x_n\}$ in the metric space (E, d) is a Cauchy.

Proof. Let $\varepsilon > 0$ be, as f is uniformly continuous on E, there is $\delta > 0$ such that, for any $x, y \in E$ satisfying $0 < d(x, y) < \delta$, we have $d'(f(x), f(y)) < \varepsilon$. As $\{x_n\}$ is a Cauchy, in the metric space (E, d), for $\delta > 0$, there is $n_0 \in \mathbb{N}^*$ such that, for all $p > n_0, q > n_0$ we have $d(x_p, y_q) < \delta$, so $d'\left(f(x_p), f(y_q)\right) < \varepsilon$, it follows that $\{f(x_n)\}$ is a Cauchy in *F.*

Remark 14.3. The image by the continuous map, of the Cauchy is not allows a Cauchy, as shown in the following example:

Example 14.2. the function $f: (\mathbb{R}_+^*, d_u) \longrightarrow (\mathbb{R}_+^*, d_u)$ be, defined by: $f(x) = \frac{1}{x}$, for all $x \in \mathbb{R}_+^*$. The sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy in $(\mathbb{R}_+^*, d_u)$, indeed for any $\varepsilon > 0$, there is $n_0 = \left[\frac{2}{\varepsilon}\right] + 1 \in \mathbb{N}^*$, where $\left[\frac{2}{\varepsilon}\right]$ is the integer part of $\frac{2}{\varepsilon}$, such that, for all $p > n_0, q > n_0$ we have $\left|\frac{1}{p} - \frac{1}{q}\right| < \varepsilon$. But the sequence $\left\{f\left(\frac{1}{n}\right)\right\} = \{n\}$ is not a Cauchy, since for $\varepsilon = 1$, and any $n \in \mathbb{N}^*$, there are $p = n + 1$ and $q = 2n + 1$ such that $|2n + 1 - n| = n \geq 1$.

Theorem 14.1. The map $f:(E,d)\longrightarrow(F,d')$ is uniformly continuous on E, iffy for any sequences $\{x_n\}$ and $\{y_n\}$ in E, satisfying $d(x_n,y_n)\longrightarrow 0$, we have $d'\big(f(x_n),f(y_n)\big)\longrightarrow 0$.
Proof. Let $\varepsilon>0$ be, as f is uniformly continuous on E, there is $\delta>0$ such that, for any $x,y\in E$ satisfying $0<d(x,y)<\delta$, we have $d'\big(f(x),f(y)\big)<\varepsilon$. Because $d(x_n,y_n)\longrightarrow 0$, for $\delta>0$, there is $n_0\in\mathbb{N}^*$ such that, for all $n>n_0$ we have $d(x_n,y_n)<\delta$, so $d'\big(f(x_n),f(y_n)\big)<\varepsilon$. Conversely,
Suppose that, f is not uniformly continuous on E, i.e. there is $\varepsilon>0$, such that for all $n\in\mathbb{N}^*$, there are x_n and y_n in E satisfying $d(x_n,y_n)<\frac{1}{n}$ but $d'\big(f(x_n),f(y_n)\big)\geq\varepsilon$. Because $d(x_n,y_n)\longrightarrow 0$ and $\lim_{n\to\infty} d'\big(f(x_n),f(y_n)\big)\geq\varepsilon>0$, contradiction.
Corollary 14.3. The composition of two uniformly continuous maps, is uniformly continuous.
Proof. Let $f:(E,d)\longrightarrow(F,d')$, $g:(F,d')\longrightarrow(G,d'')$ are two maps ant let $\{x_n\}$ and $\{y_n\}$ in E, satisfying $d(x_n,y_n)\longrightarrow 0$, as f is uniformly continuous on E, by the theorem 14.1, $d'\big(f(x_n),f(y_n)\big)\longrightarrow 0$, because g is uniformly continuous on F, always by the theorem 14.1 $d''\big((g\circ f)(x_n),(g\circ f)(y_n)\big)\longrightarrow 0$, i.e. the composition map $g\circ f:(E,d)\longrightarrow(G,d'')$ is uniformly continuous on E.

As given, in the following result. In the compact metric space, the continuity implies the uniform continuity.
Theorem 14.2. Any continuous map f from the compact metric space (E,d), into the metric one (F,d'), is uniformly continuous.
Proof. Suppose that, f is not uniform continuous from (E,d) into (F,d'), by the theorem 14.1, there are sequences $\{x_n\}$ and $\{y_n\}$ in E, satisfying $d(x_n,y_n)\longrightarrow 0$, but $d'\big(f(x_n),f(y_n)\big)\nrightarrow 0$. As E is compact there are subsequences $\{x_{\varphi(n)}\}$ and $\{y_{\varphi(n)}\}$, such that $x_{\varphi(n)}\longrightarrow x\in E$, $y_{\varphi(n)}\longrightarrow y\in E$. Because d is continuous, $d\big(x_{\varphi(n)},y_{\varphi(n)}\big)\longrightarrow 0=d(x,y)$, then $x=y$. By the continuity of d' and f, $d'\left(f\big(x_{\varphi(n)}\big),f\big(y_{\varphi(n)}\big)\right)\longrightarrow d'\big(f(x),f(x)\big)=0$, contradiction. Because $d'\big(f(x_n),f(y_n)\big)\nrightarrow 0$, implies that, there is $\delta>0$, such that for any $n\in\mathbb{N}^*$ there is $N\in\mathbb{N}^*$, when $n>N$, $d'\big(f(x_n),f(y_n)\big)\geq\delta$, as for any $n\in\mathbb{N}^*$, $\varphi(n)\geq n>N$ then $d'\left(f\big(x_{\varphi(n)}\big),f\big(y_{\varphi(n)}\big)\right)\geq\delta$, it follows that $d'\left(f\big(x_{\varphi(n)}\big),f\big(y_{\varphi(n)}\big)\right)\nrightarrow 0$.
Remark 14.4. Any continuous map f from the compact subspace (K,d_K) of the metric space (E,d), into the metric one (F,d'), is uniformly continuous on K.

On a metric spaces, in addition to the definition of a topological isomorphism or the homeomorphism, we also define, the **uniform isomorphism**, i.e. a uniformly continuous, bijective map $f:(E,d)\longrightarrow(F,d')$ where its reciprocal map $f^{-1}:(F,d')\longrightarrow(E,d)$ is uniformly continuous. Clearly, the uniform isomorphism is an homeomorphism; but the reverse is false. The bijective map $f:(E,d)\longrightarrow(F,d')$ is said to be a **uniform isometric** if, for all $x,y\in E$, $d'\big(f(x),f(y)\big)=d(x,y)$. It is clear, from the theorem 14.1 that, the uniform isometric is a uniform isomorphism.
Example 14.3. The identity map $i:(\mathbb{R}_+,d)\longrightarrow(\mathbb{R}_+,d')$ where for all $x,y\in\mathbb{R}_+$, $d(x,y)=|x-y|$ and $d'(x,y)=|x^2-y^2|$ is neither uniform isometric nor uniform isomorphism. Because, for $x=0, y=\frac{1}{2}$, $d\left(0,\frac{1}{2}\right)=\frac{1}{2}\neq d'\left(0,\frac{1}{2}\right)=\frac{1}{4}$ and for $x_n=n, y_n=n+\frac{1}{n}$, $d\Big(n,n+$ 1n=1n⟶0, but d′n,n+1n=2+1n2⟶2.

Definition 14.4. Two metrics d and d' over the space E, are said to be u-equivalent if, both the identity map $i:(E,d)\longrightarrow(E,d')$, and its inverse $i^{-1}:(E,d')\longrightarrow(E,d)$, are uniformly continuous.

Proposition 14.5. Two equivalent metrics are u-equivalent.
Proof. Let d and d' two equivalent metrics over the space E i.e. there are $\alpha,\beta \in \mathbb{R}_+^*$ such that for all $x,y \in E, \alpha d(x,y) \leq d'(x,y) \leq \beta d(x,y)$. $(*)$ Then, for any sequences $\{x_n\}$ and $\{y_n\}$ satisfying $d(x_n,y_n) \longrightarrow 0$, we have $d'(x_n,y_n)$=$d'\big(i(x_n),i(y_n)\big) \longrightarrow 0$, by the theorem 14.1 the identity map $i:(E,d) \longrightarrow (E,d')$ is uniformly continuous. As $(*)$ implies that $d'(x,y) \leq \beta d(x,y) \leq \frac{\beta}{\alpha} d'(x,y)$, for all $x,y \in E$. Then, for any sequences $\{x_n\}$ and $\{y_n\}$ satisfying $d'(x_n,y_n) \longrightarrow 0$, we have $d(x_n,y_n)$=$d\big(i^{-1}(x_n),i^{-1}(y_n)\big) \longrightarrow 0$, by the theorem 14.1 the inverse identity map $i^{-1}:(E,d') \longrightarrow (E,d)$, is uniformly continuous. Clearly the identity map is one to one, then it is u-equivalent.
Remark 14.3.
a) The reverse in the proposition 14.5 is not true, as shown in the following example. Let (E,d) be a metric space, we have seen, in the example 13.2 that, the two metrics d and $d' = \frac{d}{1+d}$ over E are not equivalent. Let us proof that, d and d' are u-equivalent. As $0 \leq d'(x,y) \leq d(x,y)$. Then, for any sequences $\{x_n\}$ and $\{y_n\}$ satisfying $d(x_n,y_n) \longrightarrow 0$, we have $d'(x_n,y_n)$=$d'\big(i(x_n),i(y_n)\big) \longrightarrow 0$, by the theorem 14.1 the identity map $i:(E,d) \longrightarrow (E,d')$ is uniformly continuous. In the other hand, for any $\varepsilon > 0$, there is $\delta \in \left]0,\frac{\varepsilon}{1+\varepsilon}\right]$ such that for all $x,y \in E$, $0 < d'(x,y) < \delta$, we have $d\big(i^{-1}(x),i^{-1}(y)\big) = d(x,y) < \varepsilon$, then the inverse identity map $i^{-1}:(E,d') \longrightarrow (E,d)$, is uniformly continuous. Therefore, d and d' are u-equivalent.
b) Clearly, the uniform isometric map, exchange the Cauchy sequences i.e. If, $f:(E,d) \longrightarrow (F,d')$ is a uniform isometric. Then, $\{x_n\}$ is a Cauchy in E iffy $\{f(x_n)\}$ is a Cauchy in F.
Corollary 14.4. Two u-equivalent metrics are t-equivalent.
Proof. It is deduced from the remark 14.2. a).
From the proposition 14.5, the remark 14.3.and the corollary 14.2 it follows that:
Equivalent metrics⟹u-equivalent metrics ⟹t-equivalent metrics
and, none of the reverse implications is true.
Another interesting type of application, also related to metric space is.
Definition 14.5. The map $f:(E,d) \longrightarrow (F,d')$ is said to be Lipshitz with the ratio k or k-Lipshitz. If, there is $k > 0$, such that, $d'\big(f(x),f(y)\big) \leq kd(x,y)$, for all $x,y \in E$. When $0 < k < 1$, f is said to be a contraction mapping.
Remark 14.5. It is obvious that, any k-Lipshitz map is uniform continuous, and the composition of two k-Lipshitz maps is k-Lipshitz.
Example 14.4.
a) Since for all $x,y \in \mathbb{R}$, $\big||x|-|y|\big| \leq |x-y|$, then the function $f:(\mathbb{R},d_u) \longrightarrow (\mathbb{R}_+,d_u)$, defined by $f(x) = |x|$, for all $x \in \mathbb{R}$ is 1-Lipshitze.
b) The function $f:(\mathbb{R},d_u) \longrightarrow ([-1,1],d_u)$, defined by $f(x) = sinx$, for all $x \in \mathbb{R}$ is 1 Lipshitze, because, $|sinx - siny| = 2\left|sin\left(\frac{x-y}{2}\right)cos\left(\frac{x+y}{2}\right)\right| \leq 2\frac{|x-y|}{2} = |x-y|$, for all $x,y \in \mathbb{R}$.
c) The function $f_y:(E,d) \longrightarrow (\mathbb{R},d_u)$, ($y$ is fixed in E) defined by: $f_y(x) = d(x,y)$, for all $x \in \mathbb{R}$ is 1-Lipshitze, because by the proposition 4.1, $\left|f_y(x) - f_y(x')\right| = |d(x,y) - dx',y\leq dx,x'$, for every $x,x' \in E$. By the same the function $fx:E,d \longrightarrow \mathbb{R},du$, x is fixed in E defined by: $f_x(y) = d(y,x)$, for all $y \in \mathbb{R}$ i is 1-Lipshitze.
d) The function $f:(\mathbb{R},d_u) \longrightarrow ([-1,1],d_u)$ defined by $f(x) = \frac{1}{2}\cos x$ is a contraction, because by the finite increment theorem, for every $x,y \in \mathbb{R}$, there is $c \in]x,y[$ such that $f(y) - f(x) = f'(c)(y-x)$ so $|f(y) - f(x)| = |f'(c)||y-x| \leq \frac{1}{2}|y-x|$.

Definition 14.6. Let (E,d) and (F,d') are metric spaces. We say that the map f from E into F has a modulus of continuity if, there is an increasing function, $\varphi: [0,+\infty] \longrightarrow [0,+\infty]$ verfying: $\lim_{u\to 0}\varphi(u) = \varphi(0) = 0$, such that $d'(f(x), f(y)) \leq \varphi(d(x,y))$, for every $x, y \in E$.
Proposition 14.6. Let (E,d) and (F,d') are metric spaces.. Then, the map f from E into F is uniformly continuous on E, iffy f has a modulus of continuity.
Proof. If, f is uniformly continuous on E, the function $\varphi: [0,+\infty] \longrightarrow [0,+\infty]$, defined by, for every $u \in [0,+\infty]$, $\varphi(u) = \sup_{\{x,y\in E;\, d(x,y)\leq u\}} d'(f(x), f(y))$ is clearly a modulus of continuity of f. Conversely, if there is an increasing function, $\varphi: [0,+\infty] \longrightarrow [0,+\infty]$ verfying: $\lim_{u\to 0}\varphi(u) = \varphi(0) = 0$, such that $d'(f(x), f(y)) \leq \varphi(d(x,y))$, for every $x, y \in E$. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $u \in]0,+\infty[$, satisfying $0 < u < \delta$, we have $\varphi(u) < \varepsilon$, then for every $x, y \in E$, such that $d(x,y) < \eta$, we have $d'(f(x), f(y)) \leq \varphi(d(x,y)) < \varepsilon$. So, f is uniformly continuous on E.
Note that, the most used modulus of continuity, are the functions, $\varphi: [0,+\infty] \longrightarrow [0,+\infty]$ of the form $\varphi(u) = k\, u^{\alpha}$, where $k, \alpha \in \mathbb{R}_+^*$. When $\alpha = 1$, we obtain the definition of the k-Lipschitz maps. Also, if φ is a modulus of continuity of $f: E \longrightarrow F$ and ψ is a modulus of continuity of $g: F \longrightarrow G$, then $\psi \circ \varphi$ is a modulus of continuity of $g \circ f: E \longrightarrow G$.
It is easy to show that:
Corollary 14.5.
a) The uniform isometric exchange the Cauchy sequences.
b) Two equivalents metrics exchange the Cauchy sequences.
Proposition 14.7. Let $\{(E_i, d_i), i \in \{1,\dots,m\}\}$ be a finite family of the metric spaces and $E = \Pi_1^m E_i$. Then, the sequence $\{x_n\}$ is a Cauchy in metric space (E, D_∞) iffy for every $i \in \{1,\dots,m\}$, the sequence $\{x_n^i\}$ is a Cauchy in (E_i, d_i).
Proof. Since, for every $p, q \in \mathbb{N}^*$, $i \in \{1,\dots,m\}$, $0 \leq d_i(x_p^i, x_q^i) \leq D_\infty(x_p, x_q)$ then if $\{x_n\}$ is a Cauchy in metric space (E, D_∞), for every $i \in \{1,\dots,m\}$, the sequence $\{x_n^i\}$ is a Cauchy in (E_i, d_i). Conversely, let $\varepsilon > 0$, since for every $i \in \{1,\dots,m\}$, the sequence $\{x_n^i\}$ is a Cauchy in (E_i, d_i), there is $n_0^i \in \mathbb{N}^*$ such that for all $p, q \in \mathbb{N}^*$, $p > q > n_0^i$ we have $d_i(x_p^i, x_q^i) < \varepsilon$, so for $n_0 = \max_{1\leq i\leq m} n_0^i$ and $p > q > n_0$ we have $D_\infty(x_p, x_q) < \varepsilon$.

14.2-Complete metric space
Definition 14.7. The metric space (E,d), is said to be complete if, any Cauchy sequence in E is convergent. The subset A of E is complete if, the metric space (A, d_A) is complete.
Example 14.5.
a) $(\mathbb{R}, d_u)$ is a complete space. In fact, if $\{x_n\}$ is a Cauchy, from the proposition 14.1, $\{x_n\}$ is bounded, then it is containing in a compact of $\mathbb{R}$. So $\{x_n\}$ has a convergence subsequence $\{x_{\varphi(n)}\}$, therefore by the proposition 14.4, the sequence $\{x_n\}$ is convergent.
b) $(\mathbb{Q}, d_u)$ is not complete, because the sequence $x_n = \sum_0^n \frac{1}{k!}$, for all $n \in \mathbb{N}$ is a Cauchy in $\mathbb{Q}$, As for all $p, q \in \mathbb{N}$, $(p > q)$ $|x_p - x_q| = \sum_{q+1}^{p} \frac{1}{k!} = \frac{1}{(q+1)!}\left[1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots + \frac{1}{(q+2)(q+3)\dots(q+p-q)}\right] < \frac{1}{q+1}\left[1 + \frac{1}{2} + \frac{1}{2.3} + \cdots + \frac{1}{2.3\dots p-q}\right] < \frac{1}{q+1}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p-q}}\right] = \frac{2}{q+1}\left(1 - \left(\frac{1}{2}\right)^p\right) < \frac{2}{q+1}$. It follows that, $|x_p - x_q| \longrightarrow 0$,. when $q, p \longrightarrow +\infty$. Then $\{x_n\}$ is a Cauchy and converges to the Euler's number $e \in \mathbb{R} \setminus \mathbb{Q}$. In fact, if we suppose that $e \in \mathbb{Q}$, then there are $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $p \wedge q = 1$ such that $e = \frac{p}{q}$. As the two sequences $\{x_n\}$ and $\{y_n\}$ where $y_n = x_n + \frac{1}{n!}$, for all $n \in \mathbb{N}$ are adjacent and $x_1 \leq x_n \leq y_n \leq y_1$ $\{x_n\}$ is

increasing, and $\{y_n\}$ is decreasing, then, their common limit satisfies $x_n \leq e \leq y_n$, therefore $2 < e < 3$. It follows that $\left[1+\frac{1}{2!}+\cdots+\frac{1}{q!}\right]q! \leq \frac{p}{q}q! \leq \left[1+\frac{1}{2!}+\cdots+\frac{1}{q!}+\frac{1}{q!}\right]q!$ or $[q!+(q-1)!+\cdots+1] \leq p(q-1)! \leq [q!+(q-1)!+\cdots+1]+1$. By setting $[q!+(q-1)!+\cdots+1]$ =N$\in \mathbb{N}^*$, we have N$\leq (q-1)!\,p \leq N+1$ i.e. between two successive natural numbers there is a third, contradiction.

Lemma 14.1. A complete subspace (A, d_A) of the metric space (E, d) is closed.

Proof. Let $x \in cl(A)$, there is a sequence $\{x_n\}$ in A, which converges to x, then $\{x_n\}$ is a Cauchy in the metric space (E, d), hence $\{x_n\}$ is also Cauchy in the complete metric subspace (A, d_A). Then, $x_n \longrightarrow y \in A$. As (A, d_A) is Hausdorff then $y = x$ and $cl(A) = A$.

Lemma 14.2. Any closed subset in the complete metric space is complete.

Proof. Let $\{x_n\}$ be a Cauchy in (A, d_A), as $\{x_n\}$ is also Cauchy in the complete metric space (E, d), there is $x \in E$ such that $x_n \longrightarrow x$, so, for every $\varepsilon > 0$, $B(x,\varepsilon) \cap \{x_n\} \neq \emptyset$. Therefore for every $\varepsilon > 0$, $B(x,\varepsilon) \cap$A$\neq \emptyset$, which implies that $x \in cl(A) = A$, then (A, d_A) is complete.

Lemma 14.3.

$a)$ Completeness of subspaces in any metric space is stable by the finite union.

$b)$ Completeness of subspaces in any metric space is stable by intersection.

Proof. $a)$. Let (A, d_A) and (B, d_B) are two complete subspaces of the metric space (E, d). If, $\{x_n\}$ is a Cauchy in $A \cup B$, then there is a Cauchy subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ in (A, d_A) or (B, d_B) which is complete, so $x_{\varphi(n)} \longrightarrow x \in A \cup B$, by proposition 14.4, $x_n \longrightarrow x$, thus $A \cup B$ is complete.

$b)$ Let $\{A_\alpha, \alpha \in \Delta\}$ be a family of the complete subspaces (A_α, d_α) of a metric space (E, d) and $A = \cap_{\alpha\in\Delta} A_\alpha$. Because, by lemma 14.1, for every $\alpha \in \Delta$, A_α is closed, then A is closed in the complete subspace (A_α, d_α), for every $\alpha \in \Delta$, by lemma 14.2, A is complete in (A_α, d_α), for every $\alpha \in \Delta$ and hence it is complete in the metric space (E, d).

By the corollary 13.7 and proposition 14.6 we have.

Corollary 14.6. Let $\{(E_i, d_i), i \in \{1,\dots,n\}\}$ be, a finite family of the metric spaces and $E = \Pi_1^n E_i$. Then, the metric space (E, D_∞) is complete iffy for every $i \in \{1,\dots,n\}$, (E_i, d_i) is complete.

Remark 14. 6. The corollary 14.6 remains throw for a countable product metric spaces (E, d) where $E = \Pi_{n\in\mathbb{N}^*} E_n$, $d(x,y) = \sum_1^\infty \frac{1}{2^n} d_n(x_n, y_n)$ for every $x, y \in E$ and $d_n(x_n, y_n)$<1, for every n $\in \mathbb{N}^*$.

The use of, corollaries 4.1 and 14.1 and the proposition 14.4, allows us to demonstrate the following characterization of the completeness of metric spaces:

Theorem 14.3. A metric space (E, d) is complete, iffy any decreasing sequence of closed balls in E, whose radius tend to zero, has for intersection a singleton.

Proof. Let $\{\tilde{B}_n(x_n, r_n), n \in \mathbb{N}\}$ be a decreasing sequence of closed balls in E centered in x_n, whose radius $r_n \longrightarrow 0$. As for any $p, q \in \mathbb{N}^*$, $0 \leq d(x_p, x_q) \leq d(x_p, x_n) + d(x_n, x_q) \leq 2r_n$, then the sequence $\{x_n\}$ is a Cauchy in a complete metric E and as for any $n \in \mathbb{N}$, $\tilde{B}_n(x_n, r_n)$ is closed $x_n \longrightarrow x \in \tilde{B}_n(x_n, r_n)$, for every $n \in \mathbb{N}$, therefore $x \in \cap_{n\in\mathbb{N}} \tilde{B}_n(x_n, r_n)$. Because $0 \leq \delta\left(\cap_{n\in\mathbb{N}} \tilde{B}_n(x_n, r_n)\right) \leq \delta\left(\tilde{B}_n(x_n, r_n)\right)$, and $\delta\left(\tilde{B}_n(x_n, r_n)\right) \longrightarrow 0$, we have $\delta\left(\cap_{n\in\mathbb{N}} \tilde{B}_n(x_n, r_n)\right) = 0$ it follows that $\cap_{n\in\mathbb{N}} \tilde{B}_n(x_n, r_n) = \{x\}$. Conversely, let $\{x_n\}$ be a Cauchy in E. We are going to construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to x and therefore the sequence itself converges to x, so E is complete. As for every $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}^*$ such that, for every $n > n_\varepsilon$ d$(x_n, x_{n_\varepsilon})$$< \varepsilon$, then for $\frac{1}{2}$, there is $n_1 \in \mathbb{N}^*$ such that, for every $n > n_1$, d$(x_n, x_{n_1})$$<\frac{1}{2}$ and for $\frac{1}{2^2}$ there is $n_2 \in \mathbb{N}^*$ such that, for every $n > n_2 > n_1$,

$d(x_n, x_{n_2}) < \frac{1}{2^2}$, by iteration until the order $k+1$, for $\frac{1}{2^{k+1}}$ there is $n_{k+1} \in \mathbb{N}^*$ such that, for every $n > n_{k+1} > n_k$, $d(x_n, x_{n_{k+1}}) < \frac{1}{2^{k+1}}$.
Because, the sequence $\left\{\tilde{B}_{k+1}\left(x_{n_{k+1}}, \frac{1}{2^k}\right), k \in \mathbb{N}\right\}$ of closed balls in E is decreasing and $\frac{1}{2^k} \longrightarrow 0$,
then $\cap_{k\in\mathbb{N}} \tilde{B}_{k+1}\left(x_{n_{k+1}}, \frac{1}{2^k}\right) = \{x\}$. Therefore the constructed subsequence $\{x_{n_k}\}$ converges to x.

Even better, one part of the theorem 14 3, can be generalized as follows:
Theorem 14.4. In the metric complete space, any nonempty decreasing closed sets, whose the diameter tends to zero, has for intersection a singleton.
Proof. Let $\{F_n, n \in \mathbb{N}\}$ be a sequence of nonempty decreasing closed sets whose the diameter tends to zero and $x_0 \in F_0, x_1 \in F_1, \dots, x_n \in F_n, \dots$, as for every $p \geq n$, $x_P \in F_n$ then $A_n = \{x_P, p \geq n\} \subset F_n$, so $\delta(A_n) \leq \delta(F_n)$, therefore $\delta(A_n) \longrightarrow 0$, hence $\{x_n\}$ is a Cauchy in a complete metric E, thus $x_n \longrightarrow x \in E$. As, for every $p, n \in \mathbb{N}$ the subsequent $\{x_{n+P}\}$ of $\{x_n\}$ is containing in the closed F_n and converges to x, then $x \in F_n$ for every $n \in \mathbb{N}$, it follows that $x \in \cap_{n\in\mathbb{N}} F_n$. Because $0 \leq \delta(\cap_{n\in\mathbb{N}} F_n) \leq \delta(F_n)$ for every $n \in \mathbb{N}$, then $\delta(\cap_{n\in\mathbb{N}} F_n) = 0 \Leftrightarrow \cap_{n\in\mathbb{N}} F_n = \{x\}$.
Lemma 14.4. Let (E, d) be a metric space and let (F, d') be a complete metric space. If, the map $f: E \longrightarrow F$ is a uniform isomorphism, then E is complete.
Proof. Let $\{x_n\}$ be a Cauchy in E, since the map f is uniformly continuous on E, by corollary 14.2. the sequence $\{f(x_n)\}$ in the metric (F, d') is a Cauchy, however E is complete, $f(x_n) \longrightarrow y \in F$. As the inverse map $f^{-1}: F \longrightarrow E$ is continuous on F, then $f^{-1}(f(x_n)) = x_n \longrightarrow f^{-1}(y) = x \in E$, Therefore E is complete.

We are now, going to give a theorem for the extension, of a uniformly continuous function on an everywhere dense part, of a metric space.
Theorem 14. 5. Let D be an everywhere dense part, of a metric space (E, d). If, the map f from D into the complete metric space (F, d') is uniformly continuous. Then, there is an uniformly continuous extension map of f to E.
Proof. Let $a \in E = cl(D)$, there is a sequence $x_n \in D$ which converges to a. As $\{x_n\}$ is a Cauchy in E, then it is a Cauchy in the subspace (D, d_D), by corollary 14.2. the sequence $\{f(x_n)\}$ in the complete metric (F, d') is a Cauchy. Therefore, $f(x_n) \longrightarrow y \in F$. The map $\tilde{f}: x \in E \longmapsto \tilde{f}(x) = y \in F$ is unique because F is Hausdorff and $\tilde{f}$ is independent of the sequence $\{x_n\}$. Indeed, if another sequence x'_n of D converges to a, we have $0 \leq d(x_n, x'_n) \leq d(x_n, a) + d(a, x'_n)$ then $d(x_n, x'_n) \longrightarrow 0$, by the uniform continuity of f, d' and the theorem 14.1, $d'(f(x_n), f(x'_n)) \longrightarrow 0 = d'(y, y') \Leftrightarrow y = y' = \tilde{f}(x)$ for every $x \in E$. Then, $\lim_{x\to a} f(x) = \tilde{f}(a)$ for any $a \in E$. Because f is continuous on D, $\lim_{x\to a} f(x) = f(a) = fa$, for each $a \in D$. It remains to show that f is uniformly continuous on E. Let $\varepsilon > 0$ be, we will show that, there is $\delta > 0$ such that, for any $a, b \in E$ satisfying $d(a, b) < \delta$ we have $d'\left(\tilde{f}(a), \tilde{f}(b)\right) < \varepsilon$. Since, for $a \in E$, there are $x_n \in D$ and $n_1 \in \mathbb{N}^*$ such that, for any $n > n_1$, $d(x_n, a) < \frac{\delta}{2}$ and for $b \in E$, there are $x'_n \in D$ and $n_2 \in \mathbb{N}^*$ such that, for any $n > n_2$, $d(x'_n, b) < \frac{\delta}{2}$. So, for any $n > n_0 = max(n_1, n_2)$, $d(x_n, a) + d(x'_n, b) < \delta$, since $d(x_n, x'_n) \leq d(x_n, a) + d(a, b) + d(b, x'_n) < 2\delta$, and f is uniformly continuous on D, then $d'(f(x_n), f(x'_n)) < \frac{\varepsilon}{2}$, as d' is continuous, we have $d'(\tilde{f}(a), \tilde{f}(b))) \leq \frac{\varepsilon}{2} < \varepsilon$.

14.3-Fixed point theorem and Baire's lemma

In several mathematic fields, we are led to find the solution of the equation $f\,(x) = x$. This equation can be a numerical equation, differentiated equation, integral equation, implicit equation,... The bellow theorem, known under the name of, fixed point theorem, ensures the existence and uniqueness of this solution.

Definition 14.8. The point a in the E, is said to be a fixed point of the map f form E to E, if $f(a) = a$.

Theorem 14. 6 (Picard-Banach).If, f is a contraction map from a complete metric space E to E. Then, f has a unique fixed point.

Proof. We will construct a Cauchy sequence $\{x_0 \in E, x_{n+1} = f(x_n);\ n \in \mathbb{N}\}$ in E, whose limit is the desired fixed point of f: Let $x_0 \in E$ be, setting successively $x_1 = f\,(x_0) \in E$, $x_2 = f(x_1) = f\big(f\,(x_0)\big) = f^2\,(x_0) \in E, \dots, x_{n+1} = f(x_n) = f^{n+1}\,(x_0) \in E, \dots$ As f is a contraction map:
$d(x_2, x_1) \le kd(x_1, x_0)$, $d(x_3, x_2) \le k^2 d(x_1, x_0)$, ... By induction, $d(x_n, x_{n+1}) \le k^n d(x_1, x_0)$ for every $n \in \mathbb{N}$. Because, $d(x_n, x_m) \le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \le k^m\big[1 + k^1 + k^2 + k^3 + \cdots + k^{(n-1)-m} + \cdots\big]d(x_1, x_0) \le [k^m d(x_1, x_0)] \sum_0^\infty k^p$, for every $n, m \in \mathbb{N}$, $(n > m)$. As, the geometric series $\sum_0^\infty k^p$ with the general term k^p $(0 < k < 1)$, converges to $\frac{1}{1-k}$. Then, $0 \le d(x_n, x_m) \le k^m \left[\frac{1}{1-k} d(x_1, x_0)\right]$, for every $n, m \in \mathbb{N}$, $(n > m)$. Thus $d(x_n, x_m) \longrightarrow 0$, when $m \longrightarrow \infty$ and $n \longrightarrow \infty$. So, $\{x_n\}$ is a Cauchy in the complete E, $x_n \longrightarrow a \in E$. Because, $x_{n+1} = f(x_n)$, and f is continuous on E, $a = f(a)$. For the uniqueness, if, there is $b \in E$ $(a \neq b)$, such that $b = f(b)$ then,
$0 < d(a, b) = d\big(f(a), f(b)\big) \le kd(a, b)$, so $1 \le k$ contradiction.

Remark 14. 7.

a) The method used in the proof of the theorem 14.6, is known as the **successive iteration** or **successive approximation** method. This method gives us not only the existence and uniqueness of the fixed point but the scheme to finding this point via the sequence $\{x_n\}$.

b) As, $\big|d(x_{n+p}, x_n) - d(x_n, a)\big| \le d(x_{n+p}, a)$ for every $p \in \mathbb{N}$, and from the proof $\lim_{p\to\infty} d(x_{n+p}, x_n) \le \left[\frac{1}{1-k} d(x_1, x_0)\right] k^n$, then $d(x_n, a) \le \left[\frac{1}{1-k} d(x_1, x_0)\right] k^n$, for every $n \in \mathbb{N}^*$. This inequality, gives an estimate of the error in our iteration scheme, i.e. we can estimate, how far we are from the solution at each step, and halt our numerical algorithm accordingly.

Example 14.6.

a) The function $f: [a, b] \longrightarrow [a, b]$, which is derivable on $[a, b]$ and its derivate f' satisfies $|f'\,(t)| \le k$, for every $t \in [a, b]$ with $0 < k < 1$, has a unique fixed point in $[a, b]$. Indeed, by the Lagrange theorem, for every $x, y \in [a, b]$ $(x < y)$, there is $c \in\,]x, y[$, such that $f(y) - f(x) = f'\,(c)(y - x)$, so $|f(y) - f(x)| = |f'\,(c)||y - x| \le k|y - x|$, for every $x, y \in [a, b]$. Then, f is a contraction on the complete $[a, b]$, therefore it has a unique fixed point in $[a, b]$. This idea has been used to resolve the numerical equations.

b) The function $f: [1, +\infty[\, \longrightarrow [1, +\infty[$, defined by $f(x) = \frac{x}{2} + \frac{1}{x}$, for every $x \in [1, +\infty[$ is a contraction. Indeed, for every $x, y \in [1, +\infty[$, $(x < y)$, there is $c \in\,]x, y[$, such that $f(y) - f(x) = f'(c)(y - x) = \left(\frac{1}{2} - \frac{1}{c^2}\right)(y - x) \Longrightarrow |f(y) - f(x)| \le \frac{1}{2}|y - x|$, for every $x, y \in [1, +\infty[$. Hence, it has a unique fixed point in the complete $[1, +\infty[$ i.e. there is $a \in [1, +\infty[$, such that $a = \frac{a}{2} + \frac{1}{a}$ then, $a = \sqrt{2}$.

c) The function $f: \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = \sqrt{x^2 + 1}$ for every $x \in \mathbb{R}$, isn't a contraction. If, not there is $a \in \mathbb{R}$, such $a = f(a)$ then $1 = 0$ impossible. But f is 1-Lipschitz, indeed for every $x, y \in \mathbb{R}$, there is $c \in]x, y[$, such that $f(y) - f(x) = f'(c)(y - x) = \left(\frac{c}{\sqrt{c^2+1}}\right)(y - x) \Longrightarrow |f(y) - f(x)| < |y - x|$, for every $x, y \in \mathbb{R}$.

Corollary 14.7. If, f from a complete metric space E to E is such that, for every $p \in \mathbb{N}$, $f^p = f \circ f \ldots \circ f$ $(p -$ times$)$ is a contraction map. Then, f has a unique fixed point.

Proof. As f^p is a contraction from a complete metric space E to E, by the theorem 14.6 there is a unique point $a \in E$ such that $a = f^p(a) \Longrightarrow f(a) = f^p(f(a)) \Longrightarrow$ $f(a)$ is a fixed point of f^p, therefore $a = f(a)$. If, there is $b \in E$ such that $b = f(b)$ then $b = f^p(b)$, the uniqueness gives $a = b$.

Solving some equations, require the following most adequate, parametric version of the fixed point theorem.

Theorem 14. 7. Let (S, τ) be a topological space, (E, d) a complete metric space and let f be a continuous map from $S \times E$ to E. Suppose that, for $s \in S$, the map $f_s : x \in E \longmapsto f(s, x) \in E$ is a contraction of ratio $k \in]0,1[$, which is independent of s. Then, the map $s \in S \longmapsto a_s = f(s, a_s) \in E$ is continuous on S.

Proof. Let $s_0 \in S$ be, let $B(a_{s_0}, \varepsilon)$ be the open bull of the metric space E, centered in $a_{s_0} = f(s_0, a_{s_0})$ with arbitrary radius $\varepsilon > 0$. Since f is continuous from $S \times E$ to E, there is a neighborhood N of s_0 such that, for any $s \in N$, $d\left(f(s_0, a_{s_0}), f(s, a_{s_0})\right) < \varepsilon$, as $d(a_s, a_{s_0}) = d(f(s_0, a_{s_0}), f(s, a_s)) \leq d\left(f(s_0, a_{s_0}), f(s, a_{s_0})\right) + d(f(s, a_{s_0}), f(s, a_s))$, then $d(a_s, a_{s_0}) < \varepsilon + kd(a_s, a_{s_0})$, for every $s \in N$. Hence, for every $s \in N$, every $\varepsilon > 0$, $0 \leq d(a_s, a_{s_0}) < \frac{1}{1-k}\varepsilon$, when $\varepsilon \longrightarrow 0$, $d(a_s, a_{s_0}) = 0$ for every $s \in N$, which implies that, $\lim_{s \to s_0} d(a_s, a_{s_0}) = d(\lim_{s \to s_0} a_s, a_{s_0})=0$, then $\lim_{s \to s_0} a_s = a_{s_0}$. The map $s \in S \longmapsto a_s = f(s, a_s) \in E$ is continuous in the arbitrary $s_0 \in S$, then it is continuous on S.

Lemma 14.5. The compact metric space is complete.

Proof. If, $\{x_n\}$ is a Cauchy sequence in the compact metric space E, by corollary 10.2, there is a subsequence $\{x_{\varphi(n)}\}$ which converges to $x \in E$ By proposition 14.3 and proposition 14.4, the sequence $\{x_n\}$ converges to x, so E is complete.

Corollary 14. 8. If, in the metric space E, for any $\varepsilon > 0$ and for any $a \in E$, , the closed ball $\tilde{B}(a, \varepsilon)$ is compact. Then E is complete.

Proof. If, $\{x_n\}$ is a Cauchy sequence in metric space E, there is $a_0 \in E$ and $\varepsilon_0 > 0$ such that $\{x_n\} \subset \tilde{B}(a_0, \varepsilon_0)$, as the metric subspace $\tilde{B}(a_0, \varepsilon_0)$ is compact, by lemma 14.5 it is complete. So $x_n \longrightarrow x \in \tilde{B}(a_0, \varepsilon_0) \subset E$. Then, E is complete.

We will give now an interesting result in the complete metric spaces known by Baire's property.

Lemma 14.6. (Baire's property). If $\{F_n\}$ is a sequence of closed subsets of a complete metric space (E, d), satisfies $E = \cup_{n \in \mathbb{N}^*} F_n$. Then, it exists $n_0 \in \mathbb{N}^*$ such that $int(F_{n_0}) \neq \emptyset$. Equivalently, if $\{U_n\}$ is a sequence of open subsets of a complete metric space (E, d) such that $cl(U_n) = E$ for all $n \in \mathbb{N}^*$, then $cl(\cap_{n \in \mathbb{N}^*} U_n) = E$.

Proof. Let $U = \cap_{n \in \mathbb{N}^*} U_n$ be, proof that $cl(U) \supset E$. Let $x \in E$ be and $\varepsilon > 0$, because $cl(U_1) \supset E$ then $B(x, \varepsilon) \cap U_1 \neq \emptyset$ hence, it exists $x_1 \in B(x, \varepsilon) \cap U_1$ which is an open thus, it exists $\delta_1 > 0$ such that $B(x_1, \delta_1) \subset B(x, \varepsilon) \cap U_1$. Thus, it exists $0 < r_1 < \frac{\delta_1}{2} \leq \frac{\varepsilon}{2}$ such that $\tilde{B}(x_1, r_1) \subset B(x, \varepsilon) \cap U_1$ By the same, for $x_1 \in E$ and $r_1 > 0$, it exists $x_2 \in B(x_1, r_1) \cap U_2$ and it exists $0 < r_2 < \frac{r_1}{2} < \frac{\varepsilon}{2^2}$ such that $\tilde{B}(x_2, r_2) \subset B(x_1, r_1) \cap U_2 \subset B(x, \varepsilon) \cap (U_1 \cap U_2)$, and $d(x_2, x_1) < \frac{\varepsilon}{2^2}$. By induction, we construct the sequences $\{x_n\}$ in E and $\{r_n\}$ in $\mathbb{R}_+^*$ where $0 < r_{n+1} < \frac{r_n}{2}$ for all $n \in \mathbb{N}^*$, such that $\tilde{B}(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1}$and

$d(x_{n+1}, x_n) < \frac{r_1}{2^n} < \frac{\varepsilon}{2^{n+1}}$. Because for all $n, p \in \mathbb{N}^*$, $0 \le d(x_n, x_{n+p}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) < \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^{n+2}} + \ldots + \frac{\varepsilon}{2^{n+p}} = \frac{\varepsilon}{2^n}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^p}\right) = \frac{\varepsilon}{2^n}\left(1 - \frac{1}{2^p}\right) < \frac{\varepsilon}{2^n}$, it follows that $d(x_n, x_{n+p}) \longrightarrow 0$, when n and p tend to ∞. Thus, the sequence $\{x_n\}$ is a Cauchy in the complete (E, d), hence it converges to $\bar{x} \in E$. As for all $n, p \in \mathbb{N}^*$, $x_{n+p} \in \tilde{B}(x_n, r_n)$ which is closed, then $\bar{x} \in \tilde{B}(x_n, r_n) \subset B(x, \varepsilon) \cap U$ for all $n \in \mathbb{N}^*$, therefore $B(x, \varepsilon) \cap U\ cl(U) = E$.

Remark 14.7. We can also use the Baire's lemma in its following equivalent form:
If $\{F_n\}$ is a sequence of closed subsets of a complete metric space (E, d), such that $int(F_n) = \emptyset$ for all $n \in \mathbb{N}^*$. Then $int(\cup_{n\in\mathbb{N}^*} F_n) = \emptyset$.

14.4-Completion of a metric space

Starting from the fact that, there are Cauchy sequences, in the metric subspace $\mathbb{Q}$ of the complete $(\mathbb{R}, d_u)$ which converge towards elements containing in $\mathbb{Q}^C$ (see, example 14.5, b)), i.e. $\mathbb{Q}$ is neither complete nor closed in $(\mathbb{R}, d_u)$ and $cl(\mathbb{Q})= \mathbb{R}$. It is natural to wonder, if we can embed a given incomplete metric space, into a complete metric one. The answer is yes. The process of embedding a non-complete metric, in a complete one, is called the completion.

Theorem 14.8. A non complete metric space (E, d), has (up to an equivalence), a unique completion.

Proof. We will prove by steps that there is a unique complete metric space $(\tilde{E}, \tilde{d})$, such that E embeds as a everywhere dense part of $\tilde{E}$.

Step 1. Construction of the metric space $(\tilde{E}, \tilde{d})$. Let $\sim$ be the relation defined on the set $\mathcal{C}(E)$ of all Cauchy sequences $\{x_n\}$ in E, defined by: for any $\{x_n\}, \{x'_n\} \in \mathcal{C}(E)$: $\{x_n\} \sim \{x'_n\}$ iffy $d(x_n, x'_n) \longrightarrow 0$. It is clear that, $\sim$ is an equivalence relation in $\mathcal{C}(E)$. Let $\tilde{E}$ be the set of all equivalence class, of Cauchy sequences in E and let $\hat{x}$ be any element of $\tilde{E}$. As, $0 \le |d(x_n, y_n) - d(x_m, y_m)| \le |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \le d(y_m, y_n) + d(x_n, x_m)$ for any $n, m \in \mathbb{N}$, then $\{d(x_n, y_n)\}$ is a Cauchy in $\mathbb{R}_+$, therefore $\lim_{n\to\infty} d(x_n, y_n)$ exists and it is independent of the choice of the representative. Indeed, if $x_n \sim x'_n$ and $y_n \sim y'_n$ then $0 \le |d(x_n, y_n) - d(x'_n, y'_n)| \le |d(x_n, y_n) - d(x_n, y'_n)| + |d(x_n, y'_n) - d(x'_n, y'_n)| \le d(y_n, y'_n) + d(x_n, x'_n)$, for every $n, m \in \mathbb{N}$, so $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_{n_n})$. The map $\tilde{d}: \tilde{E} \times \tilde{E} \longrightarrow \mathbb{R}_+$, defined by $\tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$, for any $\hat{x}, \hat{y} \in \tilde{E}$ is well defined and satisfies the metric properties. It is clear that, for every $\hat{x}, \hat{y}, \hat{z} \in \tilde{E}$

m_1) If $\tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = 0 \Leftrightarrow x_n \sim y_n \Leftrightarrow \hat{x} = \hat{y}$, (properties of the equivalence classes).

m_2) $\tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \tilde{d}(\hat{y}, \hat{x})$.

m_3) $\tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty}[d(x_n, z_n) + d(z_n, y_n)] = \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \tilde{d}(\hat{x}, \hat{z}) + \tilde{d}(\hat{z}, \hat{y})$.

Step 2. The metric (E, d) embeds as a subspace in $(\tilde{E}, \tilde{d})$ i.e. there is an isometric $j: E \longrightarrow \tilde{E}$ and $cl(E)=\tilde{E}$. Let $j: E \longrightarrow \tilde{E}$ be defined by, for every $x \in E$, $j(x) = \hat{x}$ where, $\hat{x}$ is the class of equivalent Cauchy sequences in E converging to x (it contains the Cauchy $\{x\}$). Because, for ever $x, y \in E$, there are Cauchy sequences $\{x_n\}, \{y_n\}$ in E; $x_n \longrightarrow x$, $y_n \longrightarrow y$, as $0 \le |d(x_n, y_n) - d(x, y)| \le |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \le d(x_n, x) + d(y_n, y)$, then $\tilde{d}(j(x), j(y)) = \tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. Thus, j is an isomorphism from E into $j(E)$, so E is topologically equal to $j(E) \subset \tilde{E}$, therefore, (E, d) is a subspace of $(\tilde{E}, \tilde{d})$. Let us show that, with $cl(E)=\tilde{E}$. If $\hat{x} \in \tilde{E}$, $\varepsilon > 0$, and $\{x_n\}$ in E is a representative of $\hat{x}$. As $\{x_n\}$ is a Cauchy, there is $n_0 \in \mathbb{N}^*$ such that, for any $n, m \ge n_0$ we have $d(x_n, x_m) \leq \varepsilon$. Then, for the class $\hat{y}$ of equivalent constant Cauchy sequences $\{x_{n_0}\}$ in E, $\tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, x_{n_0}) < \varepsilon$, therefore $\hat{y} \in B(\hat{x}, \varepsilon) \cap E$ hence, $\hat{x} \in cl(E)$.

Step 3. The metric space $(\tilde{E}, \tilde{d})$ is complete. Let $\{\hat{x}_n\} = \{\hat{x}_0, \hat{x}_1, \dots, \hat{x}_p, \dots\}$ be a Cauchy in $\tilde{E}$. As, $cl(E) = \tilde{E}$, for a fixed $p \in \mathbb{N}^*$, there is a sequence $\{x_{n,p}\}$ in E, such that $\lim_{n\to\infty} \tilde{d}(x_{n,p}, \hat{x}_p) = 0$. So, there is $N(p) \in \mathbb{N}^*$ such that, for any $n \geq N(p)$ we have $\tilde{d}(x_{n,p}, \hat{x}_p) \leq \frac{1}{p}$. Setting $y_p = x_{N(p),p}$, the sequence $\{y_p\}$ satisfies: $0 \leq d(y_p, y_q) = \tilde{d}(y_p, y_q) \leq \tilde{d}(y_p, \hat{x}_p) + \tilde{d}(\hat{x}_p, \hat{x}_q) + \tilde{d}(\hat{x}_q, y_q) \leq \frac{1}{p} + \tilde{d}(\hat{x}_p, \hat{x}_q) + \frac{1}{q}$, for any $p, q \in \mathbb{N}^*$. Since $\{\hat{x}_n\}$ is a Cauchy in $\tilde{E}$, $\lim_{p,q\to\infty} \tilde{d}(\hat{x}_p, \hat{x}_q) = 0$, thus $\lim_{p,q\to\infty} d(y_p, y_q) = 0$ i.e. $\{y_n\}$ is a Cauchy in E. Let $\hat{y}$ be the equivalence class of $\{y_n\}$, because, $0 \leq \tilde{d}(\hat{x}_p, \hat{y}) \leq \tilde{d}(\hat{x}_p, y_p) + \tilde{d}(y_p, \hat{y}) < \frac{1}{p} + \tilde{d}(y_p, \hat{y})$, $\lim_{p\to\infty} \tilde{d}(y_p, \hat{y}) = \lim_{p,n\to\infty} d(y_p, y_n) = 0$ thus, $\lim_{p\to\infty} \tilde{d}(\hat{x}_p, \hat{y}) = 0$. The uniform continuity of $\tilde{d}$ implies that, $\lim_{p\to\infty} \tilde{d}(\hat{x}_p, \hat{y}) = \tilde{d}(\lim_{p\to\infty} \hat{x}_p, \hat{y}) = 0$ hence, $\lim_{p\to\infty} \hat{x}_p = \hat{y} \in \tilde{E}$ and $(\tilde{E}, \tilde{d})$ is complete.

Step 4. The uniqueness up to an isomorphism, of the completion $(\tilde{E}, \tilde{d})$. Suppose that, there is another completion $(\breve{E}, \check{d})$ of the metric (E, d). Because, $cl(E) = \tilde{E}$, for every $\hat{x} \in \tilde{E}$, there is a sequence $\{x_n\}$ in E, which converges to $\hat{x}$, as $E \subset \breve{E}$, then $\{x_n\}$ is a Cauchy in the complete $\breve{E}$, then $x_n \longrightarrow \dot{x} \in \breve{E}$. The map $h: (\breve{E}, \check{d}) \longrightarrow (\tilde{E}, \tilde{d})$ defined by $h(\dot{x}) = \hat{x}$, for every $\dot{x} \in \breve{E}$ is an isomorphism. In fact, it is a surjection by construction and for any $(\dot{x}, \dot{y}) \in \breve{E} \times \breve{E}$, $\tilde{d}(h(\dot{x}), h(\dot{y})) = \tilde{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$, where $\{x_n\}$ and $\{y_n\}$ are two Cauchy in E and $(x_n, y_n) \longrightarrow (\dot{x}, \dot{y})$ in $\breve{E} \times \breve{E}$. As, $\check{d}: \breve{E} \times \breve{E} \longrightarrow \mathbb{R}_+$ satisfies $|\check{d}(x_n, y_n) - \check{d}(\dot{x}, \dot{y})| \leq |\check{d}(x_n, y_n) - \check{d}(x_n, \dot{y})| + |\check{d}(x_n, \dot{y}) - \check{d}(\dot{x}, \dot{y})| \leq \check{d}(y_n, \dot{y}) + \check{d}(x_n, \dot{x})$ then, $\lim_{n\to\infty} \check{d}(x_n, y_n) = \lim_{n\to\infty} d(x_n, y_n) = \check{d}(\dot{x}, \dot{y})$. The map h, is then an isometric, hence it is an isomorphism.

Using the notion of totally bounded metric space or **precompact metric space** and completeness, we will give another characterization of compacts metric space.

Definition 14.8. A metric space (E, d) is said to be totally bounded if, for all $\varepsilon > 0$ there is a finite parts $\{A_1, \dots, A_N\}$ in E such that, for every $i \in \{1, \dots, N\}$ the diameter $\delta(A_i) < \varepsilon$ and $E = \cup_1^N A_i$. Equivalently, for all $\varepsilon > 0$, there is a finite points $\{x_1, \dots, x_N\}$ in E, such that $E = \cup_1^N B(x_i, \varepsilon)$.

It is obvious that, a totally bounded metric space is bounded and a subset in a totally bounded metric space is totally bounded. Let us summarize some elementary properties related to totally bounded metric space in the following.

Proposition 14.8. A part A in a metric space is totally bounded$\Leftrightarrow$ $cl(A)$ is totally bounded.

Proof. It is obvious that if $cl(A)$ is totally bounded, then A is totally bounded. Conversely, if for $\varepsilon > 0$ there is a finite parts $\{A_1, \dots, A_N\}$ in E such that, for every $i \in \{1, \dots, N\}$ the diameter $\delta(A_i) < \varepsilon$ and $A = \cup_1^N A_i$, then $cl(A) = cl(\cup_1^N A_i) = \cup_1^N cl(A_i)$, as for every $i \in \{1, \dots, N\}$ the diameter $\delta(cl(A_i)) = \delta(A_i) < \varepsilon$, then $cl(A)$ is totally bounded.

With the same argument as in the proof of the lemma 13.5 we have.

Proposition 14.9. The totally bounded metric space is separable.

Lemma 14.7. A compact metric space is totally bounded.

Proof. Suppose that the compact metric space(E, d) in not totally bounded. Then, there is $\varepsilon > 0$ such that, not finite number of bulls with radius ε covers E. Hence for $x_1 \in E$, $B(x_1, \varepsilon) \not\supset E$, so it exists $x_2 \in E$ such that $d(x_1, x_2) > \varepsilon$, because $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \not\supset E$ it exists $x_3 \in E$ such that $d(x_1, x_3) > \varepsilon$ and $d(x_2, x_3) > \varepsilon$,…, by induction, for every $n \in N^*$

there is a sequence $\{x_n\} \subset E$ such that $d(x_i, x_j) > \varepsilon$ for all $i, j \in N^*$ $(i \neq j)$. From the compactness of E, the sequence $\{x_n\}$ has a convergence subsequence hence, it has a Cauchy subsequence $\{x_m\}$ then, for $\frac{\varepsilon}{2} > 0$ there is $n_0 \in N^*$ such that for $p > q > n_0$, $\varepsilon < d(x_p, x_q) < \frac{\varepsilon}{2}$, contradiction.

Lemma 14.8. For any sequence $\{x_n\}$ in a totally bounded metric space E, it exists at last one ball of E containing an infinite subsequence of the sequence.

Proof. Since for all $\varepsilon > 0$, there is a finite points $\{x_1, \dots, x_N\}$ in E, such that $E = \cup_1^N B(x_i, \varepsilon)$. If $B(x_i, \varepsilon)$ for all $i \in \{1, \dots, N\}$ containing a finite elements of the sequence $\{x_n\}$, there is a subsequence $\{x_{\varphi(n)}\}$ of the sequence $\{x_n\}$ such that $\{x_{\varphi(n)}\} \not\subset B(x_i, \varepsilon)$ for all $i \in \{1, \dots, N\}$, it follows that $\{x_{\varphi(n)}\} \not\subset \cup_1^N B(x_i, \varepsilon) = E$ contradiction with $\{x_{\varphi(n)}\} \subset \{x_n\} \subset$E.

The following theorem gives, an important characterization of the totally bounded metric space.

Theorem 14.9. A metric space E is compact$\Leftrightarrow$ E iscomplete and totally bounded.

Proof. By the lemma 14.5 a metric compact space E is complete and by the lemma 14.8 E is totally bounded. Conversely, it remains to prove that every sequence in E has a convergence subsequence, from the corollary 13.5. Let$\{x_s\}$ be a sequence in E which is totally bounded, then for $\varepsilon = 1$, there exists $x_1, \dots, x_{N_1}$ in E such that $E = \cup_1^{N_1} B(x_i, 1)$. From the lemma 14.8, it exists at least $m_1 \in \{1, \dots, N_1\}$ such that the ball $B(x_{m_1}, 1)$ containing an infinite subsequence $\{x_s^1\}$ of the sequence $\{x_s\}$. Take the balls such that $B(x_{m_1}, 1) \cap B\left(x_i, \frac{1}{2}\right) \neq \emptyset$ where $i \in \{1, \dots, N_2\}$, as $E = \cup_1^{N_2} B\left(x_i, \frac{1}{2}\right)$, by the same argument for the sequence $\{x_s^1\}$, it exists at least $m_2 \in \{1, \dots, N_2\}$ such that the ball $B\left(x_{m_2}, \frac{1}{2}\right)$ containing an infinite subsequence $\{x_s^2\}$ of the sequence $\{x_s^1\}$. Because $x_s^2, x_s^1 \in B(x_{m_1}, 1)$ then $d(x_s^1, x_s^2) \leq d(x_s^1, x_{m_1}) + d(x_{m_1}, x_s^2) < 1 + 1 = 2 = \delta\left(B(x_{m_1}, 1)\right)$, where $\delta\left(B(x_{m_1}, 1)\right)$ is the diameter of the ball $B(x_{m_1}, 1)$. By induction, if we take the balls, such that $B\left(x_{m_k}, \frac{1}{k}\right) \cap B\left(x_i, \frac{1}{k+1}\right) \neq \emptyset$ where $i \in \{1, \dots, N_{k+1}\}$, as $E = \cup_1^{N_{k+1}} B\left(x_i, \frac{1}{k+1}\right)$. It exists at least $m_{k+1} \in \{1, \dots, N_{k+1}\}$ and an infinite subsequence $\{x_s^{k+1}\}$ of the sequence $\{x_s^k\}$ containing in the ball $B\left(x_{m_{k+1}}, \frac{1}{k+1}\right)$. As for all $k \in \mathbb{N}^*$, $x_k^k, x_{k+1}^{k+1} \in B\left(x_{m_k}, \frac{1}{k}\right)$ then, $d(x_k^k, x_{k+1}^{k+1}) \leq d(x_k^k, x_{m_k}) + d(x_{m_k}, x_{k+1}^{k+1}) < \frac{2}{k} \leq 1 + 1 = 2 = \delta\left(B\left(x_{m_k}, \frac{1}{k}\right)\right)$, where $\delta\left(B\left(x_{m_k}, \frac{1}{k}\right)\right)$ is the diameter of the ball $B\left(x_{m_k}, \frac{1}{k}\right)$. We will demonstrate that, the sequence $\{x_s^s\}$ is a Cauchy. Since $x_p^p \in \{x_s^q\}$ for all $p, q \in \mathbb{N}^*$ with $p > q$, then x_p^p and x_q^q are in the ball of radius $\frac{1}{q}$, so $d(x_p^p, x_q^q) < \frac{2}{q}$. By Archimedean axiom, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$, hence for $p, q \in \mathbb{N}^*$ with $p > q > n_0$ we have $d(x_p^p, x_q^q) < \frac{2}{q} < \frac{2}{n_0} < \varepsilon$. Therefore, the subsequence $\{x_s^s\}$ of the sequence $\{x_s\}$ is a Cauchy in the complete metric space E. Hence $\{x_s^s\}$ converges in E, it follows that E is compact.

15-Convergences in functional spaces, Ascoli and Stone-Weierstrass theorems

15-1. Simple and uniform convergence

Let E be a nonempty set, F a topological space, $\mathcal{F}(E, F)$ the vector space of all maps, defined from E into F. To make the functional space $\mathcal{F}(E, F)$ important, we are led to introduce on this space, the so called point open topology i.e. the topology whose, the family $\{S(x,U);\ x\in E, U$ is an open in $F\}$ and $S(x,U)=\{f\in\mathcal{F}(E,F);\ f(x)\in U\}$ is a subbasis. We then define in $\mathcal{F}(E,F)$ the simple convergence or pointwise convergence.

Definition 15.1. Let $\{f_n, f;\ n\in\mathbb{N}\}$ be a family of maps in $\mathcal{F}(E,F)$. We say that, the sequence $\{f_n\}$ simply converges to f if, for every $x\in E$ the sequence $\{f_n(x)\}$ converges to $f(x)$ in F. In other words, the map f is a simple limit or pointwise limit of the sequence $\{f_n\}$ if, for every $x\in E$ and for every $V\in\mathcal{N}(f(x))$, there is $n_0\in\mathbb{N}^*$ (n_0 depends to x and V) such that, for $n>n_0$, $f_n(x)\in V$. We write $f_n\overset{s.c}{\rightarrow}f$ to express that, f is a simple limit of the sequence $\{f_n\}$. In the case of the metric (F,d'), $\left[f_n\overset{s.c}{\rightarrow}f\right]\Leftrightarrow[\lim_{n\to\infty}d'(f_n(x),f(x))=0,\text{for every } x\in$ $E\Leftrightarrow\forall x\in E$ and $\forall\varepsilon>0,$ there is $n0\in\mathbb{N}*$ $n0$ depends of x and $\varepsilon,$ such that, for any $n>n0,$ $d'fnx, fx<\varepsilon$

Example 15.1.

a) The sequence $f_n:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, defined by $f_n(x)=e^{-nx}$ simply converges to $f(x)=\begin{cases}0, \text{if } x\in\mathbb{R}_+^*;\\ 1,\quad \text{if } x=0\,.\end{cases}$

b) The sequence $f_n:[0,1]\longrightarrow\mathbb{R}_+$, defined by $f_n(x)=\frac{nx}{1+nx}$ simply converges to $f(x)=\begin{cases}1, \text{if } x\in\,]0,1];\\ 0,\quad \text{if } x=0\,.\end{cases}$

c) The sequence $f_n:\mathbb{R}\longrightarrow\mathbb{R}$, defined by $f_n(x)=\frac{1+nx}{1+n^2x^2}$ simply converges to $f(x)=\begin{cases}0, \text{if } x\in\,]0,1];\\ 1,\quad \text{if } x=0\,.\end{cases}$

d) The sequence $f_n:\mathbb{R}\longrightarrow\mathbb{R}_+^*$, defined by a) For any $n\in\mathbb{N}^*$, the sequence $f_n:[0,1]\longrightarrow\mathbb{R}$, defined by $=\frac{1}{1+(x-n)^2}$ simply converges to $f(x)=0$, for every $x\in\mathbb{R}$.

Remark 15.1. We can also define, the simple convergence of a net $(f_\alpha)_{\alpha\in D}$, following a basis $\mathcal{B}$ of the filter $\mathcal{F}$ on D to a function f in $\mathcal{F}(E,F)$ i.e. for every $x\in E$, $f_\alpha(x)\longrightarrow f(x)$ following $\mathcal{B}$.

In the sequel, we assume that (F,d') is a metric space and for all $f\in\mathcal{F}(E,F)$ the diameter of $f(E)$ is finished i.e. $\delta(f(E))=\sup_{y,y'\in f(E)}d'(y,y')<+\infty$. Then, the map $d_\infty:\mathcal{F}(E,F)\times\mathcal{F}(E,F)\longrightarrow\mathbb{R}_+$, defined by: $d_\infty(f,g)=\sup_{x\in E}d'(f(x),g(x)$, for every $f,g\in\mathcal{F}(E,F)$ is a metric. The topology on $\mathcal{F}(E,F)$, induced by d_∞, is said to be the **uniform topology.** We will define in the metric $(\mathcal{F}(E,F),d_\infty)$, another important type of convergence, which is called the uniform convergence and we will give, the relationship between the pointwise convergence and uniform convergence and the properties of their limit when it exist.

Definition 15.2. Let $\{f_n, f;\ n\in\mathbb{N}\}$ be a family of maps in $\mathcal{F}(E,F)$. We say that, the sequence $\{f_n\}$ uniformly converges to f or, the map f is a uniform limit of the sequence $\{f_n\}$ iffy, $\lim_{n\to\infty}d_\infty(f_n,f)=0$ In other words, iffy, for every $\varepsilon>0$, there is $n_0\in\mathbb{N}^*$ (n_0 depends en ε), such that, for any $n>n_0$, $[d_\infty(f_n,f)<\varepsilon\Leftrightarrow\sup_{x\in E}d'(f_n(x),f(x)<\varepsilon]\Leftrightarrow$ $\left[d'(f_n(x),f(x))<\varepsilon,\forall x\in E\right]$. We write $f_n\overset{u.c}{\rightarrow}f$, to express that, f is a uniform limit, of the sequence $\{f_n\}$.

Example 15.2.

a) For any $n\in\mathbb{N}^*$, the sequence $f_n:[0,1]\longrightarrow\mathbb{R}$, defined by $f_n(x)=x^n$ simply converges to $f(x)=\begin{cases}0, \text{if } x\in[0,1[;\\ 1,\quad \text{if } x=1\,.\end{cases}$

But, f_n is not uniformly convergent to f. In fact, $d_\infty(f_n, f) = \sup_{x\in[0,1]}|f_n(x) - f(x)| = \sup_{x\in[0,1]}\frac{x}{n} = 1$. Then, $\lim_{n\to\infty} d_\infty(f_n, f) \neq 0$.

b) For any $n \in \mathbb{N}^*$, the sequence $f_n: [0,1] \longrightarrow \mathbb{R}_+$, defined by: $f_n(x) = 1 + \frac{x}{n}$, simply converges to $f(x) = 1$ ($\max_{x\in[0,1]}|f_n(x) - f(x)| = \max_{x\in[0,1]}\frac{x}{n} = \frac{1}{n} \longrightarrow 0$).

c) For any $n \in \mathbb{N}^*$, the sequence $f_n: [0,1] \longrightarrow \mathbb{R}_+$, defined by:

$$f_n(x) = \begin{cases} n^2x(1 - nx), \text{ if } x \in \left[0, \frac{1}{n}\right]; \\ 0, \text{ if } x \in \left[\frac{1}{n}, 1\right]. \end{cases}$$

simply converges to $f(x) = 0$, for every $x \in [0,1]$. But, $f_n(x)$ is not uniformly convergent to 0. Because, $\max_{x\in[0,1]}|f_n(x)| = \max_{x\in\left[0,\frac{1}{n}\right]}\left(n^2x(1 - nx)\right) = \frac{n}{4} \longrightarrow +\infty$.

d) For any $n \in \mathbb{N}^*$, the sequence $f_n: [0,1] \longrightarrow \mathbb{R}_+$, defined by:

$$f_n(x) = \begin{cases} \frac{x}{n}, \text{ if } x \in \left[0, \frac{1}{2}\right]; \\ 1 + \frac{1}{e^n}, \text{ if } x \in \left]\frac{1}{2}, 1\right]. \end{cases}$$

converges uniformly to $f(x) = \begin{cases} 0, \text{ if } x \in \left[0, \frac{1}{2}\right]; \\ 1, \quad x \in \left]\frac{1}{2}, 1\right]. \end{cases}$

($\max_{x\in[0,1]}|f_n(x) - f(x)| = \frac{1}{e^n} + \max_{x\in\left[0,\frac{1}{2}\right]}\left(\frac{x}{n}\right) = \frac{1}{e^n} + \frac{1}{2n} \longrightarrow 0$).

As seen in the example 15.2, a) and b), the uniform convergence implies the simply convergence, and the converse is not true.

Theorem 15.1. If F is complete, then $(\mathcal{F}(E, F), d_\infty)$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy in $(\mathcal{F}(E, F), d_\infty)$, then for $\varepsilon > 0$ be, there is $n_0 \in \mathbb{N}^*$ such that for $p, q > n_0$, $d'\left(f_p(x), f_q(x)\right) < \varepsilon$ for any $x \in E$. Then, for every $x \in E$ the sequence $\{f_n(x)\}$ is Cauchy in the complete E, so for every $x \in E$ $f_n(x) \longrightarrow f(x) \in E$. As, for $p > n_0$ and $q \longrightarrow \infty$ $d'\left(f_p(x), f(x)\right) < \varepsilon$, for any $x \in E$, then $d_\infty\left(f_p, f\right) \longrightarrow 0$. So $(\mathcal{F}(E, F), d_\infty)$ is complete.

The fact that, we are going to introduce the continuity of the elements of $\mathcal{F}(E, F)$, we must assume that E is a topological space.

Proposition 15.1. Let $\{f_n, f;\ n \in \mathbb{N}\}$ be a family of maps in $\mathcal{F}(E, F)$. If, the sequence $\{f_n\}$ uniformly converges to f and for every $n \in \mathbb{N}$ f_n is continuous on E. Then, f is continuous on E.

Proof. Since $f_n \overset{u.c}{\longrightarrow} f$, for $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$such that $d'\left(f(x), f_{n_0}(x)\right) \leq \frac{\varepsilon}{3}$ for every $x \in E$. As f_{n_0} is continuous in x_0, there is a neighborhood $N \in \mathcal{N}(x_0)$, such that $d'\left(f_{n_0}(x_0), f_{n_0}(x)\right) \leq \frac{\varepsilon}{3}, \forall x \in N$. Then, for any $\varepsilon > 0$ there is $N \in \mathcal{N}(x_0)$ such that, $d'\left(f(x_0), f(x)\right) \leq d'\left(f(x_0), f_{n_0}(x_0)\right) + d'\left(f_{n_0}(x_0), f_{n_0}(x)\right) + d'\left(f_{n_0}(x), f(x)\right) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, for any $x \in N$. Hence, f is continuous in the arbitrary element x_0 of E, thus f is continuous on E.

Corollary 15.1. Let $\{f_n\}$ be a sequence, of the continuous maps from the metric (E, d) into F. If, the restriction of $\{f_n\}$ to any compact $K \subset E$ uniformly converges to f. Then, f is continuous on E.

Proof. Let f be the uniform limit of the restriction of $\{f_n\}$ into the compact K. From the proposition 15.1, f is continuous on the arbitrary K, so by the lemma 13.4 f is continuous on E.

Let us denote by $\mathcal{C}(E,F)$, the subspace of $\mathcal{F}(E,F)$, formed by the continuous maps from E into F. We then have.

Corollary 15.2. $\mathcal{C}(E,F)$ is closed.

Proof. If, $f \in cl(\mathcal{C}(E,F))$, there is a sequence $\{f_n\}$ in $\mathcal{C}(E,F)$ uniformly converging to f. As by proposition 15.1, f is continuous on E, then $f \in \mathcal{C}(E,F)$. Thus $\mathcal{C}(E,F)$ is closed.

In order to give a Dini's theorem, regarding the passage from simple convergence to uniform convergence, we need the following lemma.

Theorem 15.2. Let E be a compact space. If, the family $\{f_n, f;\ n \in \mathbb{N}\}$ in $\mathcal{C}(E,F)$ satisfies: for every $x \in E$, the sequence $d'(f_n(x), f(x))$ is decreasing and converges to 0. Then, $d_\infty(f_n, f) \longrightarrow 0$.

Proof. As $f \in \mathcal{C}(E,F)$ and $d'(f_n(x), f(x)) \longrightarrow 0$. For $\varepsilon > 0$, there is $n_1 \in \mathbb{N}^*$ such that, for each $n > n_1$, $f(x) \in B(f_n(x), \varepsilon)$. So for all $n > n_1$ $A_n = \{x \in E, d(f_n(x), f(x)) \geq \varepsilon\}$ is closed. Because $d'(f_{n+1}(x), f(x)) \leq d'(f_n(x), f(x))$, for every $n \in \mathbb{N}$, then the closed sequence $\{A_n, n > n_1\}$ is decreasing. As the space E is compact, by corollary 10.1, $\cap_{n>n_1} A_n = \emptyset$ or $E = \cup_{n>n_1} (A_n)^C$, it follows that, for $x \in E$ there is $n_0 > n_1$, such that $x \in (A_{n_0})^C$ then $d'\left(f_{n_0}(x), f(x)\right) < \varepsilon$, so for $n > n_0$,
$d'(f_n(x), f(x)) < d'\left(f_{n_0}(x), f(x)\right) < \varepsilon$. Thus $\lim_{n\to\infty} d_\infty(f_n, f) = 0$.

Corollary 15.3. (Dini′s theorem). If the family $\{f_n, f;\ n \in \mathbb{N}\}$ in $\mathcal{C}(E,\mathbb{R})$, where E is a compact space satisfies: the sequence $\{f_n\}$ is monotone and $f_n \overset{s.c}{\rightarrow} f$. Then $f_n \overset{u.c}{\rightarrow} f$.

Proof. As $d(f_n(x), f(x)) = |f_n(x) - f(x)|$ for every $x \in E$, if $\{f_n\}$ is decreasing $f(x) \leq f_{n+1}(x) \leq f_n(x)$ for every $x \in E$ and if, $\{f_n\}$ is increasing $f_n(x) \leq f_{n+1}(x) \leq f(x)$, for every $x \in E$. Then, for every $x \in E$, $d'(f_n(x), f(x))$ is decreasing and $0 \leq d'(f_n(x), f(x)) \leq 0$, therefore $d'(f_n(x), f(x)) \longrightarrow 0$. By the theorem 15.2 $f_n \overset{u.c}{\rightarrow} f$.

15.2-Ascoli and Stone-Weierstrass theorems

In Section 15.1, we have seen that, the space $\mathcal{C}(E,F)$ is closed in $(\mathcal{F}(E,F), d_\infty)$ and, if F is complete, $(\mathcal{F}(E,F), d_\infty)$ is complete, therefore $\mathcal{C}(E,F)$ is complete. It is also important, to find compacts spaces in $(\mathcal{F}(E,F), d_\infty)$. Such a space is closely linked to the concept of equicontinuity.

Definition 15.3. let $\mathcal{F}(E,F)$ be, where E is a topological space. We say that, the subset $\mathcal{H}$ of $\mathcal{F}(E,F)$ is equicontinuous in $x_0 \in E$, if for every $\varepsilon > 0$ there is a neighborhood $N \in \mathcal{N}(x_0)$, such that the diameter $\delta(f(N)) < \varepsilon$, for every $f \in \mathcal{H}$. $\mathcal{H}$ is said to be equicontinuous on E, if $\mathcal{H}$ is equicontinuous in any point of E.

Definition 15.4. let (E,d) and (F,d') are metric spaces. We say that the subset $\mathcal{H}$ of $\mathcal{F}(E,F)$ is uniformly equicontinuous on E, if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every $x, y \in E$, satisfying $d(x,y) < \eta$, $d'(f(x), f(y)) < \varepsilon$ for every $f \in H$.

It is clear that:

$a)$ The subset $\mathcal{H}$ of $\mathcal{F}(E,F)$ is uniformly equicontinuous on E, iffy all the elements of $\mathcal{H}$ has the same modulus of continuity.

$b)$ In the definition 15.3, the neighborhood N depends on $\mathcal{H}$, x_0 and ε but not on f.

$c)$ The equicontinuous implies the uniform continuity.

$d)$ If $\mathcal{H}$ is finite, then $\mathcal{H}$ is equicontinuous.

$e)$ If $\mathcal{H} = \{f_n\}$ and $f_n \overset{u.c}{\rightarrow} f$, then $\mathcal{H}$ is equicontinuous.

Proposition 15.1. Let E be a space. If, the sequence $\{f_n\}$ in $\mathcal{F}(E,F)$, is equicontinuous in $x_0 \in E$, and $f_n \overset{u.s}{\rightarrow} f \in \mathcal{F}(E,F)$. Then, f is continuous in x_0.

Proof. Let $\varepsilon > 0$ be, since for all $x \in E$ $f_n(x) \longrightarrow f(x)$, there is $n_0 \in \mathbb{N}^*$ such that $d'\left(f_{n_0}(x), f(x)\right) \leq \frac{\varepsilon}{3}$. $\{f_n\}$ being equicontinuous in $x_0 \in E$, there is a neighborhood $N \in \mathcal{N}(x_0)$, such that $d'\left(f_{n_0}(x), f_{n_0}(x_0)\right) \leq \frac{\varepsilon}{3}$, for all $x \in N$, therefore $d'\big(f(x), f(x_0)\big) \leq d'\left(f(x), f_{n_0}(x)\right) + d'\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d'\left(f_{n_0}(x_0), f(x_0)\right) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, for all $x \in N$.

Proposition 15.2. Let E be a space. If, the subset $\mathcal{H}$ in $\mathcal{F}(E,F)$ is equicontinuous, then $cl(\mathcal{H})$ is equicontinuous.

Proof. Because for $f \in cl(\mathcal{H})$, there is a sequence $\{f_n\}$ in $\mathcal{H}$ converging to f in $(\mathcal{F}(E,F), d_\infty)$. Then, for $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$ such that, $d'\left(f_{n_0}(x), f(x)\right) \leq \frac{\varepsilon}{3}$ for every $x \in E$. Let $x_0 \in E$, because $\mathcal{H}$ is equicontinuous in x_0, there is a neighborhood $N \in \mathcal{N}(x_0)$, such that, $d'\left(f_{n_0}(x), f_{n_0}(x_0)\right) \leq \frac{\varepsilon}{3}$, for every $x \in N$, thus
$d'\big(f(x), f(x_0)\big) \leq d'\left(f(x), f_{n_0}(x)\right) + d'\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d'\left(f_{n_0}(x_0), f(x_0)\right) \leq \varepsilon$, for every $x \in N$, and all $f \in cl(\mathcal{H})$. Thus $cl(\mathcal{H})$ is equicontinuous.

Proposition 15.3. If, (E,d) is a metric compact space. Any equicontinuous subset $\mathcal{H}$ in $\mathcal{F}(E,F)$ is uniformly equicontinuous.

Proof. Let $\varepsilon > 0$ be, by the equicontinuous of $\mathcal{H}$, for every $x \in E$ there is an open O_x in E such that $x \in O_x$ and $\delta\big(f(O_x)\big) < \varepsilon$, for every $f \in H$. Because $E = \cup_{x\in E} O_x$, and E is a compact metric space all the requirements of the fundamental lemma 13.2 are checked. Then, there is $r > 0$ such that for all $y \in E$, $B(y,r)$ is containing in at last one of the O_x, therefore $\delta\left(f\big(B(y,r)\big)\right) < \varepsilon$, it follows that, as soon as $d(x,y) < r$, where $x, y \in E$, we have $d'(f(x), f(y)) < \varepsilon$, for every $f \in \mathcal{H}$. Thus, $\mathcal{H}$ is uniformly equicontinuous.

Theorem 15.3. (First Ascoli's theorem). Let E be a space, (F,d') a compact metric space, $\{f_n\}$ an equicontinuous sequence in $\mathcal{F}(E,F)$ and a part $D \subset E$ whith $cl(D) = E$. If, $f_n \overset{s.c}{\rightarrow} f$ on D, then

$i)$ it exists a continuous map f from E into F, such that $f_n \overset{s.c}{\rightarrow} f$ on E.

$ii)$ $f_n \overset{u.c}{\rightarrow} f$ on any compact K of E.

Proof. $i)$ Since $\{f_n\}$ is equicontinuous in $\mathcal{F}(E,F)$, for $x \in E$ and $\varepsilon > 0$, there is a neighborhood $N \in \mathcal{N}(x)$, such that for $x' \in N$, $d'\big(f_n(x), f_n(x')\big) < \frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$. Because $cl(D) = E$, then $N \cap D \neq \emptyset$ and $f_n(y) \longrightarrow f(y)$ for $y \in N \cap D$, so $\{f_n(y)\}$ is a Cauchy in E, there is $n_0 \in \mathbb{N}^*$, such that for $n, m > n_0$, $d'\big(f_n(y), f_m(y)\big) < \frac{\varepsilon}{3}$, therefore $d'\big(f_n(x), f_m(x)\big) \leq d'\big(f_n(x), f_n(y)\big) + d'\big(f_n(y), f_m(y)\big) + d'\big(f_m(y), f_m(x)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Hence, for all $x \in$ the sequence $\{f_n(x)\}$ is a Cauchy in the complete F, by the proposition 15.1, it exists a continuous map $f: E \longrightarrow F$ which is the simply limit of $\{f_n(x)\}$. $ii)$ Let $a \in K$, since the maps f_n, f are continuous in a, for $\varepsilon > 0$ there is an open U_a in K containing a such that $d'\big(f_n(x), f_n(a)\big) < \frac{\varepsilon}{3}$ $\forall x \in U_a$, $\forall n \in \mathbb{N}$, and there is an open W_a in K containing a such that $d'\big(f(x), f(a)\big) < \frac{\varepsilon}{3}$, $\forall x \in W_a$. Then for $\varepsilon > 0$, there is an open $O_a = U_a \cap W_a$ in K containing a such that $\forall x \in O_a$, $d'\big(f_n(x), f_n(a)\big) + d'\big(f(x), f(a)\big) < \frac{2\varepsilon}{3}$. Because $K = \cup_{a\in K} O_a$, there is a finite elements $\{a_1, \dots, a_m\}$ in K such that $K = \cup_{1\leq i\leq m} O_{a_i}$. Since for

$i \in \{1, \dots, m\}$, $f_n(a_i)$ simply converges to $f(a_i)$, there is $n_i \in \mathbb{N}^*$ such that for $n > n_i$, $d'\big(f_n(a_i), f(a_i)\big) < \frac{\varepsilon}{3}$, hence for $n > n_0 = \max_{1\le i\le m} n_i$, $d'\big(f_n(a_i), f(a_i)\big) < \frac{\varepsilon}{3}$, therefore for all $x \in K$ i.e. x is containing in some open O_{a_j} we have

$$d'\big(f_n(x), f(x)\big) \le d'\left(f_n(x), f_n(a_j)\right) + d'\left(f_n(a_j), f(a_j)\right) + d'\left(f(a_j), f(x)\right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $f_n \overset{u.c}{\rightarrow} f$ on K.

Corollary 15.4. If, (E, d) is a compact space, (F, d') is a metric one, $\{f_n\}$ an equicontinuous sequence in $\mathcal{F}(E, F)$ and $f_n \overset{s.c}{\rightarrow} f$ on E. Then $f_n \overset{u.c}{\rightarrow} f$ on E.

Proof. It suffices to take, $K = E$ and to repeat, the same proof as that of $f_n \overset{u.c}{\rightarrow} f$ on K in the theorem 15.3.

We now give, the largely used and very important form of Ascoli's theorem.

Theorem 15. 4. (Second Ascoli's theorem). Let E be a compact space, (F, d') a complete metric space, H a part of $(\mathcal{C}(E, F), d_\infty)$ and, $\mathcal{H}(x) = \{f(x);\ f \in \mathcal{H}\}$, where $x \in E$. Then,

$$\mathcal{H} \text{ is relatively compact} \Leftrightarrow \begin{cases} i) & \mathcal{H} \text{ is equicontinuous;} \\ ii) & \mathcal{H}(x) \text{ is relatively compact, for all } x \in E. \end{cases}$$

Proof. $i)$ As $cl(\mathcal{H})$ is compact, from the lemma 14.8 it is totally bounded. Then, for $\varepsilon > 0$, there is a finite elements $\{f_1, \dots, f_m\}$ in $\mathcal{H}$ such that, $\mathcal{H} \subset cl(\mathcal{H}) = \cup_{1\le i\le m} B\left(f_i, \frac{\varepsilon}{3}\right)$, because for $i \in \{1, \dots, m\}$, f_i is continuous in any $x \in E$, there is an open O_x^i in E containing x, such that $d'\big(f_i(x), f_i(x')\big) < \frac{\varepsilon}{3}$ for $x' \in O_x^i$. Since for $f \in \mathcal{H}$ there is $j \in \{1, \dots, m\}$ such that $f \in B\left(f_j, \frac{\varepsilon}{3}\right)$ then, $\sup_{x\in E} d'\left(f(x), f_j(x)\right) < \frac{\varepsilon}{3}$ hence for $x' \in O = \cap_{1\le i\le m} O_x^i$ which is an open in E containing x $d'\big(f(x), f(x')\big) \le d'\left(f(x), f_j(x)\right) + d'\left(f_j(x), f_j(x')\right) + d'\left(f_j(x'), f(x')\right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Thus $\mathcal{H}$ is equicontinuous. $ii)$ Define, the map $\varphi: \mathcal{C}(E, F) \longrightarrow F$ by, $\varphi(f) = f(x)$ for a fixed x in E. As, $d'\big(\varphi(f), \varphi(g)\big) = d'\big(f(x), g(x)\big) \le d_\infty(f, g)$ for all $f, g \in \mathcal{C}(E, F)$, then φ is 1-Lipschitz, hence φ is continuous on $\mathcal{C}(E, F)$. By proposition 10.4 $\varphi\big(cl(\mathcal{H})\big)$ is compact and by the theorem 7.2, 4) $\varphi(\mathcal{H}) \subset \varphi\big(cl(\mathcal{H})\big) \subset cl\big(\varphi(\mathcal{H})\big)$. As $\varphi\big(cl(\mathcal{H})\big)$ is closed by the proposition 10.3. Thus $cl\big(\varphi(\mathcal{H})\big) = \varphi\big(cl(\mathcal{H})\big)$, hence $\varphi(\mathcal{H})$ is relatively compact in F, as for $x \in E$ $\mathcal{H}(x) \subset \varphi(\mathcal{H})$, then $cl\big(\mathcal{H}(x)\big) \subset cl\big(\varphi(\mathcal{H})\big)$, as a closed in the compact $cl\big(\mathcal{H}(x)\big)$ is compact. Conversely, by the proposition 14.8 and the theorem 14.9, it suffices to prove that $cl(\mathcal{H})$ is complete and totally bounded. As F is complete, by the theorem 15.1 $\mathcal{F}(E, F)$ is complete and by the corollary 15.2, $\mathcal{C}(E, F)$ is closed in $\mathcal{F}(E, F)$, hence $\mathcal{C}(E, F)$ is complete. It is clear that $cl(\mathcal{H})$ is complete. To prove that $\mathcal{H}$ is totally bounded. Let $\varepsilon > 0$ and $x \in E$, by the equicontinouity of $\mathcal{H}$, there is an open O_x in E which contains x such that for all $x' \in O_x$, $d'\big(f(x), f(x')\big) < \frac{\varepsilon}{4}$ for all $f \in \mathcal{H}$. Because $E = \cup_{x\in E} O_x$ and E is compact, there is a finite elements $\{x_1, \dots, x_m\}$ in E such that $E = \cup_{1\le i\le m} O_{x_i}$. Since for all $i \in I = \{1, \dots, m\}$, $\mathcal{H}(x_i) = \{f(x_i), f \in \mathcal{H}\}$ is relatively compact, then $\mathcal{L} = \cup_1^m \mathcal{H}(x_i)$ is relatively compact, there is a finite elements $\{y_1, \dots, y_k\}$ in F such that $\mathcal{L} \subset \cup_1^k B\left(y_j, \frac{\varepsilon}{4}\right)$, then there is $\varphi(i) \in J = \{1, \dots, k\}$ such that $f(x_i) \in B\left(y_{\varphi(i)}, \frac{\varepsilon}{4}\right)$. Denote Φ the finite collection of all maps $\varphi: i \in I \mapsto \varphi(i) \in J$. Let $\mathcal{E}_\varphi = \left\{f \in \mathcal{H},\ f(x_i) \in B\left(y_{\varphi(i)}, \frac{\varepsilon}{4}\right)\right\}$, then $\mathcal{H} = \cup_{\varphi\in\Phi} \mathcal{E}_\varphi$. So, for any $\varepsilon > 0$ for any $x \in E$ which belongs to some O_{x_i} and for any $f, g \in \mathcal{E}_\varphi$, $d'\big(f(x), g(x)\big) \le d'\big(f(x), f(x_i)\big) + d'\big(f(x_i), y_{\varphi(i)}\big) + d'\left(y_{\varphi(i)}, g(x_i)\right) + d'\big(g(x_i), g(x)\big) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$ for all $x \in E$. It follows that for every $\varphi \in \Phi$ the diameter $\delta(\mathcal{E}_\varphi) < \varepsilon$, therefore $\mathcal{H}$ is totally bounded, by proposition 15.8 $cl(\mathcal{H})$ is totally bounded. Since $cl(\mathcal{H})$ is totally bounded and complete it is compact.

Remark 15.2. In the theorem 15.4

a) If (F, d') is compact, then it is complete. The condition $\mathcal{H}(x)$ is relatively compact for all $x \in E$ is obviously verified. Then, $\mathcal{H}$ is equicontinuous iffy $\mathcal{H}$ is relatively compact.

b) If E is a metric compact space, we us the separability of E and a modulus of continuity to proof the implication" $\Longrightarrow$ "(see G.Choquet, theorem 23.5, p.97).

It is clear that.

c). It (F, d') is a metric, we obtain the same resultat by utilization of the Tychonoff theorem.

Corollary 15.5. Under the conditions of the theorem 15.4. If, the sequence $\{f_n\}$ in $(\mathcal{C}(E,F), d_\infty)$ is equicontinuous, and the sequence $\{f_n(x)\}$ for all $x \in E$ is relatively compact in F. Then $\{f_n\}$ has a subsequence which uniformly converges.

Remark 15.3.

a) The theorem 15.4, is not valid if E is locally compact. In fact, the sequence $\{f_n\}$ in $\mathcal{C}(\mathbb{R}, [0,1])$ defined by $f_n(x) = \frac{1}{1+(x-n)^2}$ is equicontinuous, but it is not relatively compact in $\mathcal{C}(\mathbb{R}, [0,1])$ (E is not compact).

b) Let $id: [0,1] \longrightarrow [0,1]$ be the identity function. For all $n \in \mathbb{N}$, the sequence$\{f_n = id + n$ is equicontinuous in $\mathcal{C}0,1,\mathbb{R},$ since for all $n \in \mathbb{N}$ and for all $x,y \in 0,1,\ fnx-fny=x-y$ but it is not relatively compact in $\mathcal{C}([0,1], \mathbb{R})$ ($\mathcal{H}(0) = \mathbb{N} = cl(\mathbb{N})$ which is not compact).

c) The sequence $\{f_n\}$ in $\mathcal{C}([0,1],[0,1])$ where $f_n(x) = sin(nx)$, is not relatively compact in $\mathcal{C}([0,1],[0,1])$ ($\mathcal{H} = \{f_n\}$ is not equicontinuous).

d) Let $a \in [2,3]$ and let for all $x \in [0,1]$ $f_a(x) = x + a$. Then, the family $\mathcal{H} = \{f_a, a \in [2,3]\}$ is relatively compact.

In the end of this section, we will present one of the versions of the Stone-Weierstrass theorem, whose the Weierstrass theorem concerning the uniform approximation, of a continuous function on a compact space by a polynomial, becomes a special case. Before the proof of this theorem, let us introduce and prove some concepts and elementary results related at it demonstration. In the sequel, E is a compact space, $\mathcal{A}$ is the nonempty part of $(\mathcal{C}(E,\mathbb{K}), d_\infty)$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and for $f, g \in \mathcal{C}(E,\mathbb{K})$,
$d_\infty(f,g) = \max_{x\in E} d'(f(x), g(x))$ with, $d'(f(x), g(x)) = |f(x) - g(x)|$, for all $x \in E$, (the max and min exists by the Hein's theorem 10.3).

Definition 15.4. The part $\mathcal{A}$ is said to be:

i) A $\mathbb{K}$-subalgebra, if for $f, g \in \mathcal{A}$, $f + g \in \mathcal{A}$, the product $fg \in \mathcal{A}$ and for $\lambda \in \mathbb{K}$, $\lambda f \in \mathcal{A}$.

ii) Separates points if, for all $x, y \in E$ with $x \neq y$ there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

iii) A lattice, if for $f, g \in \mathcal{A}$ we have $f \vee g, f \wedge g \in \mathcal{A}$, where
$(f \vee g)(x) = max(f(x), g(x))$ and $(f \wedge g)(x) = min(f(x), g(x))$, for all $x \in E$.

When, the elements of the subalgebra $\mathcal{A}$ are the complex valued functions, the conjugate $\overline{f}$ of $f \in \mathcal{A}$, is defined by, for all $x \in E$, $\overline{f}(x) = \overline{f(x)}$, and

iv) $\mathcal{A}$ is said to be selfadjoint if, for all $f \in \mathcal{A}$, $\overline{f} \in \mathcal{A}$.

Example 15.3. The space of real coefficients polynomials $\mathcal{P}[E]$, where $E = [a,b] \subset \mathbb{R}$, is a subalgebra and separates points. It is clear that $\mathcal{P}[E]$ is a subalgebra, and for all $s, t \in [a,b]$, $s \neq t$, the polynomial $P \in \mathcal{P}[E]$, defined by $P(x) = x$ for all $x \in E$, satisfies $P(t) \neq P(s)$.

Lemma 15.1. If, the part $\mathcal{A}$ is a separated lattice $\mathbb{R}$-subalgebra, which containing all constant functions. Then

$a)$ $cl(\mathcal{A})$ is a $\mathbb{R}$-subalgebra.
b) If $f \in cl(\mathcal{A})$, then $|f| \in cl(\mathcal{A})$.
$c)$ $cl(\mathcal{A})$ is a lattice.
$d)$ $cl(\mathcal{A})$ separates points strongly i.e. if $x_0, y_0 \in E$, $x_0 \neq y_0$ and $\alpha, \beta \in \mathbb{R}$, there exists $f \in cl(\mathcal{A})$ such that $f(x_0) = \alpha$ and $f(y_0) = \beta$.

Proof. $a)$ If $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{R}$, there are sequences $\{f_n\}$, $\{g_n\}$ in $\mathcal{A}$, such that $f_n \overset{u.c}{\rightarrow} f$ and $g_n \overset{u.c}{\rightarrow} g$, then $\lambda f_n + g_n \overset{u.c}{\rightarrow} \lambda f$+g, $f_n g_n \overset{u.c}{\rightarrow} fg$. Since, the sequences $\{\lambda f_n + g_n\}$ and $\{f_n g_n\}$ are containing in $\mathcal{A}$, then λf+g, $fg \in cl(\mathcal{A})$. It is clear, that all constant functions are in $cl(\mathcal{A})$, hence $cl(\mathcal{A})$ is a subalgebra. b) Setting for no zero f $\in cl(\mathcal{A})$, $a = \sup_{x\in E}|f|(x)$, then $0 < |f| \leq a$. We want to prove, the uniform convergence towards |f| of the following sequence:

$$\begin{cases} f_0 = 0, \\ f_n = f_{n-1} + \dfrac{1}{2a}(f^2 - f_{n-1}^2), \text{for all n} \in \mathbb{N}^*. \end{cases}$$

It is obvious that, $\{f_n\} \subset$ cl($\mathcal{A}$), $0 \leq f_n \leq$ |f| and $f_n \leq f_{n+1}$, for all $n \in \mathbb{N}$. Also, $f_{n+1} \leq$ |f|, for all $n \in \mathbb{N}$, in fact, $f_{n+1} -$ |f|=($f_n -$ |f|)+$\frac{1}{2a}(f^2 - f_n^2) = \frac{1}{2a}[-2a(|f| - f_n) +$ f$-fn$f$+fn=12a$f$-fn$f$+fn-2a$, because f$-fn\geq 0$ and f$+fn-2a\leq 2$f$-a<0$, therefore $f_{n+1} \leq$ |f|, for all $n \in \mathbb{N}$. Furthermore, for all $x \in E$, the real sequence $\{f_n(x)\}$ is increasing and bounded above by $|f(x)| = |f|(x)$, then $\{f_n(x)\}$ converges simply towards a function g: $x \in E \longmapsto g(x) \in \mathbb{R}_+$. By the definition of the sequence $\{f_n\}$, we have $0 = |f|^2 - g^2 = (|f| - g)(|f| + g)$, because $|f| + g > 0$, then $|f| - g = 0$ hence $|f| = g$. All the condition of the corollary 15.3. (Dini′s theorem) are satisfied, hence $f_n \overset{u.c}{\rightarrow} |f| \in cl(\mathcal{A})$. $c)$ Just notice that: $min(f,g) = \frac{|f+g|-|f-g|}{2}$ and $max(f,g) = \frac{|f+g|+|f-g|}{2}$. $d)$ If $x_0, y_0 \in E$, $x_0 \neq y_0$ there is $h \in \mathcal{A}$ such that $h(x_0) \neq h(y_0)$. It is obvious that, the function $f = \alpha + \frac{\beta-\alpha}{h(y_0)-h(x_0)}\big(h -$ $hx0\in\mathcal{A}\subset cl\mathcal{A}$ and satisfies $fx0=\alpha$ and $fy0=\beta$.

Remark 15.4.
$a)$ In the unital $\mathbb{K}$-subalgebra i.e. $1 \in \mathcal{A}$, all constant functions are elements of $\mathcal{A}$.
$b)$ The lemma 15.1, $d)$ is true is $\mathbb{C}$-subalgebra with the supplementary condition $\mathcal{A}$ is fanishes at no point. Indeed, there exist $g, h, k \in \mathcal{A}$ such that $g(x_0) \neq g(y_0)$, $h(x_0) \neq 0$ and $k(y_0) \neq 0$. It is obvious that, the function $f = \alpha \frac{(g-g(y_0))h}{(g(x_0)-g(y_0))h(x_0)} + \beta \frac{(g-g(x_0))k}{(g(y_0)-g(x_0))k(y_0)} \in \mathcal{A} \subset cl(\mathcal{A})$ and satisfies $f(x_0) = \alpha$ and $f(y_0) = \beta$.

Theorem 15.5 ($\mathbb{R}$-Stone-weierstrass theorem). If $\mathcal{A}$ is a separates points $\mathbb{R}$-subalgebra, which containing all constant elements. Then $cl(\mathcal{A}) = \mathcal{C}(E, \mathbb{R})$.

Proof. Let $s, t \in E$ be with $s \neq t$, by the lemma 15.2 d), for all $f \in \mathcal{C}(E, \mathbb{R})$ there exists $h_{s,t} \in cl(\mathcal{A})$ such that $h_{s,t}(s) = f(s)$ and $h_{s,t}(t) = f(t) < f(t) + \varepsilon$, for all $\varepsilon > 0$. As $h_{s,t}$ and f are continuous, then $\left(h_{s,t} - f\right)^{-1}(]-\infty, \varepsilon[) = \{u \in E, h_{s,t}(u) < f(u) + \varepsilon\} = O_t$ is an open in E. Because the collection $\{O_t, t \in E\}$ is an open cover of the compact E, it exists a finite points $\{t_1, \dots, t_n\}$ in E such that $E = \cup_{1\leq i\leq n} O_{t_i}$. So for all $x \in E$ there is $i \in \{1, \dots, n\}$ such that $x \in O_{t_i}$, thus $h_{s,t_i}(x) < f(x) + \varepsilon$. The function $g_s = \min_{1\leq i\leq n} h_{s,t_i}$, which by the lemma 15.2, $c)$ is an element of $cl(\mathcal{A})$ satisfies $g_s(s) = f(s)$, $g_s(x) < f(x) + \varepsilon$, for all $x \in E$. By the continuity of g_s and f, $(g_s - f)^{-1}(]-\varepsilon, +\infty[) = \{v \in E, f(v) - \varepsilon < g_s(v)\} = U_s$ is an open in E and the collection $\{U_s$, s $\in E\}$ is an open cover of the compact E. It exists a finite points $\{s_1, \dots, s_m\}$ in E such that $E = \cup_{1\leq j\leq m} U_{s_j}$. Thus, for all $x \in E$ there is $j \in \{1, \dots, m\}$ such that $x \in U_{s_j}$ and $f(x) - \varepsilon < g_{s_j}(x)$. By the lemma 15.2 c), the function

$g = \max_{1 \le i \le m} g_{s_j}$ is an element of $cl(\mathcal{A})$ and satisfies $f(x) - \varepsilon < g(x) < f(x) + \varepsilon$, for all $x \in E$. Therefore, for all $f \in \mathcal{C}(E, \mathbb{R})$ there is $g \in cl(\mathcal{A})$, such that $d_\infty(f, g) < \varepsilon$ for all $\varepsilon > 0$. It follows that, when $\varepsilon \longrightarrow 0$, $f = g \in cl(\mathcal{A})$, hence $cl(\mathcal{A}) = \mathcal{C}(E, \mathbb{R})$.
Example 15.4. The set $\mathcal{A}$ of the functions defined from $\mathbb{R}$ into $\mathbb{R}$ by $f(x) = \sum_1^n c_k e^{kx}$ where $c_i, \dots, c_n \in \mathbb{R}$, is everywhere dense in $\mathcal{C}([a,b], \mathbb{R})$. Clearly, if $\lambda \in \mathbb{R}$, $\lambda f + g \in \mathcal{A}$ and $fg \in \mathcal{A}$ by the identity e^{s+t}=$e^s e^t$, for $s, t \in \mathbb{R}$. As the function e^t is one to one and strictly positive then, $\mathcal{A}$ separates points. By Stone-Weierstrass theorem $cl(\mathcal{A}) = \mathcal{C}([a,b], \mathbb{R})$.
Let E be a compact of $\mathbb{R}^n$ and $\mathcal{P}[E]$ the unital $\mathbb{R}$-subalgebra, of all polynomials from E into $\mathbb{R}$, in the coordinate $x_1, \dots, x_n$. As a direct consequence of the theorem 15.5, we have.
Corollary 15.6. $cl(\mathcal{P}[E]) = \mathcal{C}(E, \mathbb{R})$.
The Weierstrass approximation theorem, is obtained from the corollary 15.6 by taking $n = 1$. So,
Corollary 15.7 ($\mathbb{R}$-Weierstrass theorem)**.** $cl(\mathcal{P}[[a,b]])$=$\mathcal{C}([a,b], \mathbb{R})$.

Let us give some simple versions of Weierstrass approximation theorem.
Corollary 15.8. The metric space $(\mathcal{C}([a,b], \mathbb{R}), d_\infty)$ is separable.
Proof. It remains, to use corollary 15.7 and $cl(\mathbb{Q}) = \mathbb{R}$.
Corollary 15.9. For every $x \in [-a, a] = E$, there is a real sequence $\{Q_n\}$ in $\mathcal{P}[E]$, uniformly converging towards $|x|$ and $Q_n(0) = 0$.
Proof. As the function $f(x) = |x|$ is an element of $(\mathcal{C}([-a,a], \mathbb{R}), d_\infty)$, by the corollary 15.8 there is a real sequence $\{P_n\}$ in $\mathcal{P}[E]$ which satisfies, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$ such that, for $n > n_0$, $\left|P_n(x) - |x|\right| < \frac{\varepsilon}{2}$ for all $x \in [-a, a]$. Let $Q_n(x) = P_n(x) - P_n(0)$, obviously $Q_n(0) = 0$ and for all $x \in [-a, a]$, $\left|Q_n(x) - |x|\right| = \left|P_n(x) - P_n(0) - |x|\right| \le \left|P_n(x) - |x|\right| + |P_n(0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, as soon as $n > n_0$. Thus $\{Q_n\}$ is the desired sequence.

Denote by $\mathcal{A}_\mathbb{C}$ the complex subalgebra i.e. $\mathcal{A}_\mathbb{C} \subset \mathcal{C}(E, \mathbb{C})$.
Theorem 15.6 ($\mathbb{C}$-Stone-weierstrass theorem). If $\mathcal{A}_\mathbb{C}$ is a selfadjoint separates points $\mathbb{C}$-subalgebra of $\mathcal{C}(E, \mathbb{C})$, which containing all constant elements. Then, $cl(\mathcal{A}_\mathbb{C}) = \mathcal{C}(E, \mathbb{C})$.
Proof. Let $\mathcal{A}$ be the unital subalgebra of $\mathcal{A}_\mathbb{C}$, containing all real valued functions. If $f \in \mathcal{A}_\mathbb{C}$, then $Re(f) = \frac{1}{2}(f + \overline{f}), Im(f) = \frac{1}{2i}(f - \overline{f}) \in \mathcal{A}$. As, for $x, y \in E$ such that $x \ne y$, there is $f \in \mathcal{A}_\mathbb{C}$ such that $f(x) \ne f(y)$ or $Re(f)(x) + iIm(f)(x) \ne Re(f)(y) + iIm(f)(y)$, then either $Re(f)(x) \ne Re(f)(y)$ or $Im(f)(x) \ne Im(f)(y)$, so $\mathcal{A}$ is separates points. By the $\mathbb{R}$-Stone-Weierstrass theorem, $cl(\mathcal{A})$= $\mathcal{C}(E, \mathbb{R})$. As $\mathcal{C}(E, \mathbb{C}) = \mathcal{C}(E, \mathbb{R})$+i $\mathcal{C}(E, \mathbb{R})$ and as $cl(\mathcal{A}_\mathbb{C}) = cl(\mathcal{A}) + icl(\mathcal{A})$= $\mathcal{C}(E, \mathbb{C})$.

Another version of Weierstrass theorem, regarding the approximation of the periodic continuous function, by the trigonometric polynomials is still established. Recall that for any $n \in \mathbb{N}$, the **complex trigonometric polynomial** of the order $\le n$ is a continuous function f from $\mathbb{R}$ into $\mathbb{C}$ defined by: for all $x \in \mathbb{R}$, $f(x) = \sum_{-n}^{n} a_k e^{ikx}$ where $i^2 = -1$, $a_k \in \mathbb{C}$ and $e^{ikx} = cos(kx) + isin(kx)$. Denote $T\mathcal{P}[\mathbb{R}]$ the set of all trigonometric polynomial and $\mathcal{C}_{2\pi per}(\mathbb{R}, \mathbb{C})$, the unital $\mathbb{C}$-subalgebra of 2π-periodic continuous functions from $\mathbb{R}$ to $\mathbb{C}$, i. e. $f \in \mathcal{C}_{2\pi per}(\mathbb{R}, \mathbb{C})$ iff $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ and $f(x + 2k\pi)$=f(x), for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.
Corollary 15.10. $cl(T\mathcal{P}[\mathbb{R}])$=$\mathcal{C}_{2\pi per}(\mathbb{R}, \mathbb{C})$.
Proof. $T\mathcal{P}[\mathbb{R}]$ is a unit subalgebra. Indeed, if $f, g \in T\mathcal{P}[\mathbb{R}]$ and $\lambda \in \mathbb{C}$ then $\lambda f + g \in T\mathcal{P}[\mathbb{R}]$, and for $a_0 = 1$, $a_k = b_k = 0$, for every $k \in \{1, \dots, n\}$ the unit polynomial $1 \in T\mathcal{P}[\mathbb{R}]$, $fg \in T\mathcal{P}[\mathbb{R}]$. by trigonometric identity $e^{i(k+l)x} = e^{ikx}e^{ilx}$, separates points because the function $t \in \mathbb{R} \longmapsto \mathrm{e}^{\mathrm{i}t} \in \mathbb{C}$ satisfies $\mathrm{e}^{\mathrm{i}t} \ne \mathrm{e}^{\mathrm{i}k}$ for all $t \ne k$. Let $S^1(0,1) = S = \{(x_1, x_2) \in \mathbb{R}^2$, $x_1^2 + x_2^2 = 1\}$ the unit compact sphere in $\mathbb{R}^2$; define the surjection map p from $\mathbb{R}$ into S by, $p(x) = (sinx, cosx)$ for every $x \in \mathbb{R}$. As the map $\Phi: f \in \mathcal{C}(S, \mathbb{R}) \mapsto \Phi(f) = f \circ p \in$

$\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{R})$, satisfies for all $f,g\in\ \mathcal{C}(S,\mathbb{R})$ $d_\infty(\Phi(f),\Phi(g))=sup_{x\in\mathbb{R}}|(f\circ p)(x)-(g\circ p)(x)|=sup_{x\in\mathbb{R}}|f(p(x))-g(p(x))|$=$sup_{t\in S}|f(t)-g(t)|$=$d_\infty(\mathrm{f,g})$, then Φ is an isometric. As, for any $h\in\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{R})$, there is $f\colon \mathrm{S}\longrightarrow\mathbb{R}$ such that $h=f\circ p=\Phi(f)$, then Φ is onto. Therefore $\mathcal{C}(S,\mathbb{R})$ and $\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{R})$ are homeomorphic then $\mathcal{C}(S,\mathbb{C})=\mathcal{C}(S,\mathbb{R})$+i $\mathcal{C}(S,\mathbb{R})$ and $\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{C})=\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{R})+i\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{R})$ are homeomorphic. As by $\mathbb{C}$-stone-Weierstrass theorem $\mathcal{C}(S,\mathbb{C})=cl(T\mathcal{P}[\mathbb{R}])$ then $cl(T\mathcal{P}[\mathbb{R}])=\mathcal{C}_{2\pi per}(\mathbb{R},\mathbb{C})$.

Example 15 5. By, the previous trigonometric identity. The set of real trigonometric polynomials f, defined by: $f(x)=a_0$+$\sum_{k=1}^{n}a_k\cos(kx)$, for all $x\in E$, is an subalgebra, which does not separates point in $[-a,a]$, because for all $x\in E$, $f(x)=f(-x)$, for all f. But, it separates point in $[0,\pi]$, as $\cos(t)$ is one to one in this interval.

Corollary 15.11. Let E and F two compact Hausdorff spaces, and let $\mathcal{A}$ be the collection of all continuous functions $\Phi\colon E\times F\longrightarrow\mathbb{K}$, defined by for any $(\mathrm{x,y})\in E\times F$, $\Phi(\mathrm{x,y})=\sum_1^n f_i(x)g_i(y)$ where $n\in\mathbb{N}^*$, $f_i\in\mathcal{C}(E,\mathbb{K})$ and $g_i\in\mathcal{C}(F,\mathbb{K})$ are continuous. Then $cl(\mathcal{A})$= $\mathcal{C}(E\times F,\mathbb{K})$.

Proof. It is clear that $\mathcal{A}$ is a unital selfadjoint $\mathbb{K}$-subalgebra of $\mathcal{C}(E\times F,\mathbb{K})$. Let $(\mathrm{x,y}),(\mathrm{x',y'})$ are two elements of $E\times F$ such that $(\mathrm{x,y})\neq(\mathrm{x',y'})$ suppose that $\mathrm{x}\neq\mathrm{x'}$ by the lemma 10.4 E is normal, as the singletons $\{x\}$ and $\{\mathrm{x'}\}$ are disjoint closed sets in the normal space, by the theorem 8.1 (Urysohn Lemma) there is a continuous function f defined from E into [0,1] such that $f(x)=0$ and $f(\mathrm{x'})=1$. For any $(\mathrm{s,t})\in E$, the continuous function $\psi(\mathrm{s,t})=f(s)$ is such that $0=\psi(\mathrm{x,y})\neq\psi(\mathrm{x',y'})$=1, then $\mathcal{A}$ separates points, all the requirements of the Stone-Weierstrass theorem are satisfied then $cl(\mathcal{A})$= $\mathcal{C}(E\times F,\mathbb{K})$.

16-Normed Vector Spaces

16.1-Definitions and properties

Normed vector spaces are a very important class of metric spaces. They are introduced after Hilbert spaces and much studied by Banach. They constitute a powerful tool in mathematical analysis whose study is relatively simple. In the sequel E is a $\mathbb{K}$-vector space.

Definition 16.1. The function $\|\quad\|\colon E\longrightarrow\mathbb{R}_+$ is said to be a norm on E. If, for all $x,y\in E$ and all $\lambda\in\mathbb{K}$

n_1-$\|x\|=0\Leftrightarrow x=0$ (separation property).

n_2-$\|\lambda x\|=|\lambda|\|x\|$ (homogeneity property).

n_3-$\|x+y\|\leq\|x\|+\|y\|$ (triangle inequality).

The couple $(E,\|\quad\|)$ is called the $\mathbb{K}$-normed vector space, we write $\mathbb{K}$-nvs E for a such space.

Example 16.1.

a) The function $|\quad|\colon z\in\mathbb{C}\longmapsto|\mathrm{z}|\in\mathbb{R}_+$ is a norm on $\mathbb{R}$-vs $\mathbb{C}$.

b).In the Euclidian space $\mathbb{R}^n$, for every $x=(x_1,\dots,x_i,\dots,x_n)\in\mathbb{R}^n$ the functions $\|x\|_1=\sum_1^n|x_i|$, $\|x\|_2=(\sum_1^n|x_i|^2)^{\frac{1}{2}}$ (Euclidean norm) and $\|x\|_\infty=\max_{1\leq i\leq n}|x_i|$ (infinite norm) define a norms on $\mathbb{R}^n$.

c) In the space $\mathbb{R}_\mathrm{n}[\mathrm{x}]$ of the polynomials of degree $n\in\mathbb{N}$, $p(x)=\sum_1^n a_i x^i$, for every $p\in\mathbb{R}_\mathrm{n}[\mathrm{x}]$, the functions $\|p\|_1=\sum_1^n|a_i|$, $\|p\|_2=(\sum_1^n|a_i|^2)^{\frac{1}{2}}$ and $\|p\|_\infty=\max_{1\leq i\leq n}|p_i|$ define a norms on $\mathbb{R}$-vs $\mathbb{R}_\mathrm{n}[\mathrm{x}]$.

d) In the space $\mathcal{M}_\mathrm{n}(\mathbb{R})$ of the square matrices $A=(a_{ij})_{1\leq i,j\leq n}$ with coefficients in $\mathbb{R}$. For every $A\in\mathcal{M}_\mathrm{n}(\mathbb{R})$, the functions $\|A\|_1=\sum_1^n\sum_1^n|a_{ij}|$, $\|A\|_2=\left(\sum_1^n\sum_1^n|a_{ij}|^2\right)^{\frac{1}{2}}$ and $\|A\|_\infty=\max_{1\leq i,j\leq n}|a_{ij}|$ define a norms on $\mathbb{R}$-vs $\mathcal{M}_\mathrm{n}(\mathbb{R})$.

e) In the space $l^1(\mathbb{N},\mathbb{R}) = \{x = \{x_n\}, \sum_{n\geq 0}|x_n| < \infty\}$. For every $x \in l^1(\mathbb{N},\mathbb{R})$, the function $\|x\| = \sum_{n\geq 0}|x_n|$ define a norm on $\mathbb{R}$-vs $l^1(\mathbb{N},\mathbb{R})$.

f) In the space $\mathcal{C}([a,b],\mathbb{R})$, for all $f \in \mathcal{C}([a,b],\mathbb{R})$ the functions $\|f\|_1 = \int_a^b |f(x)|\,dx$, $\|f\|_2 = \left(\int_a^b |f(x)|^2\,dx\right)^{\frac{1}{2}}$ (quadratic norm) and $\|f\|_\infty$=$\max_{x\in[a,b]}|f(x)|$ (infinite norm), define the norms on $\mathbb{R}$-vs $\mathcal{C}([a,b],\mathbb{R})$.

Proposition 16.1. The $\mathbb{K}$-nvs $(E,\|\ \ \|)$ is a metrizable space, where the metric d associated to the norm $\|\ \ \|$ is defined by: $d(x,y) = \|x-y\|$ for all$x,y \in E$.

Proof. It is clear that for all $x,y,z \in E$, $d(x,y) \in \mathbb{R}_+$; $d(x,y) = 0 \Leftrightarrow x = y$ and $d(x,y) = d(y,x)$, while $d(x,y) = \|(x-z)+(z-y)\| \leq \|x-z\| + \|z-y\|$= $d(x,z)+d(z,y)$. Then d is a metric on E.

Remark 16.1.

a). From the proposition 16.1, it follows that a $\mathbb{K}$-nvs $(E,\|\ \ \|)$ is a topological space where the topology is induced by the metric d associated to the norm $\|\ \ \|$.

b) All the properties obtained in the metric space remain true in the $\mathbb{K}$-nvs E whith modification in the form. For example: $B(a,r) = \{x \in E, \|a-x\| < r\}$; $\tilde{B}(a,r) = \{x \in E, \|a-x\| \leq r\}$ and $S(a,r) = \{x \in E, \|a-x\| = r\}$.

c) As $|d(x,0)-d(y,0)| \leq$d(x,y) for all $x,y \in E$, then $|\|x\|-\|y\|| \leq \|x-y\|$ for all $x,y \in E$, it follows that the norm is 1-Lipschitz, therefore it is uniformly continuous and hence it is continuous on E.

d) The metric enjoyed by the norm satisfies: d(λx, λy)= $|\lambda|$d(x,y) and d(x+z,y+z)= d(x,y) for all $x,y,z \in E$ and all $\lambda \in \mathbb{K}$.

Proposition 16.2. In the $\mathbb{K}$-nvs $(E,\|\ \ \|)$ we have.

a). $cl(B(a,r)) = \tilde{B}(a,r)$.

b).int($\tilde{B}(a,r)) = B(a,r)$.

c) $S(a,r) = bd(B(a,r)) = bd\left(\tilde{B}(a,r)\right)$.

Proof. a). As $B(a,r) \subset \tilde{B}(a,r)$ and $\tilde{B}(a,r)$ is closed, then $cl(B(a,r)) \subset \tilde{B}(a,r)$. To demonstrate the reverse inclusion, let $x \in \tilde{B}(a,r)$ and let $\varepsilon > 0$ be, show that $B(x,\varepsilon) \cap B(a,r) \neq \emptyset$. If, $r < \varepsilon$ then $\|a-x\| \leq r < \varepsilon$ so $a \in B(x,\varepsilon)$ and $B(x,\varepsilon) \cap B(a,r) \neq \emptyset$. If $0 < \varepsilon \leq r$ the element $y = x - \frac{\varepsilon}{2r}(x-a)$ is such that $a - y = a - x + \frac{\varepsilon}{2r}(x-a)$ then $\|a-y\|$=$\left|1-\frac{\varepsilon}{2r}\right| \|a-x\| \leq \left(1-\frac{\varepsilon}{2r}\right)r = r - \frac{\varepsilon}{2} < r$ so $y \in B(a,r)$ and $y - x$= $-\frac{\varepsilon}{2r}(x - a$ then $y-x$=$\varepsilon 2ra-x \leq \varepsilon 2 < \varepsilon$, so $y \in Bx,\varepsilon$, hence $Bx,\varepsilon \cap Ba,r \neq \emptyset$. b) As $Ba,r \subset Ba,r$ and $B(a,r) = in(B(a,r))$ then $B(a,r) \subset in\left(\tilde{B}(a,r)\right)$. If now $x \in in\left(\tilde{B}(a,r)\right)$ which is an open neighborhood of x, it exists $\rho > 0$ such that $B(x,\rho) \subset in\left(\tilde{B}(a,r)\right) \subset \tilde{B}(a,r)$, if $x = a$ then $\|a-x\| = 0 < r$ so $x \in B(a,r)$. If $x \neq a$, the element $y = x + \frac{\rho}{\|a-x\|}(x-a)$ is such that $\|y-x\| = \rho$, then $y \in \tilde{B}(x,\rho) \subset \tilde{B}(a,r)$. As $x - a = \frac{1}{1+\frac{\rho}{\|a-x\|}}(y-a)$ then $\|a-x\| < \|a-y\| \leq$r hence $x \in B(a,r)$.

c) $bd(B(a,r))$= $cl(B(a,r)) \cap cl(B(a,r)^c)$=$\tilde{B}(a,r) \cap \left(int(B(a,r))\right)^C$=$\tilde{B}(a,r) \cap B(a,r)^c = S(a,r)$ and

$bd\left(\tilde{B}(a,r)\right)$= $cl\left(\tilde{B}(a,r)\right) \cap cl(\tilde{B}(a,r)^c)$=$\tilde{B}(a,r) \cap \left(int\left(\tilde{B}(a,r)\right)\right)^C$=$\tilde{B}(a,r) \cap B(a,r)^C$= $S(a,r)$.

Remark 16.2. The proposition 16.2 is not valid in any metric space. For example in the discrete metric space E: $\tilde{B}(a,1) = E$ and $cl\big(B(a,1)\big) = B(a,r) = \{a\}$.
Proposition 16.3. In the $\mathbb{K}$-nvs E, two norms $\| \quad \|_1$ and $\| \quad \|_2$ are said to be equivalent and we write $\| \quad \|_1 \sim \| \quad \|_2$, if there are $\alpha, \beta \in \mathbb{R}_+^*$ such that $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$, for all $x \in E$.
Example 16.2. The norms in the example 16.1 b) are equivalent. For example, we have $\|x\|_\infty \leq \|x\|_2 \leq n\|x\|_\infty$; $\frac{1}{\sqrt{n}}\|x\|_2 \leq \|x\|_1 \leq n\|x\|_2$ and $\frac{1}{\sqrt{n}}\|x\|_2 \leq \|x\|_\infty \leq \|x\|_2$ for every $x \in E$.

We have seen in the corollary 13.3 that, the equivalent distances are t-equivalent. But the converse is not true by the example 13.2. We will check that in a $\mathbb{K}$-nvs E the t-equivalent property implies the equivalent norms.
Proposition 16.4. Let τ_1 and τ_2 are tow topologies, enjoyed by tow norms $\| \quad \|_1$ and $\| \quad \|_2$ on $\mathbb{K}$-nvs E. If τ_1=τ_2 then $\| \quad \|_1 \sim \| \quad \|_2$.
Proof. As τ_1=τ_2, then the identity map $i: (E, \tau_1) \longrightarrow (E, \tau_2)$ is a homeomorphism, the continuity of i and i^{-1} in 0 leads to the result. Indeed, for $\varepsilon = 1$, it exists $\alpha > 0$ such for $0 < \|x\|_1 \leq \alpha$ we have $\|x\|_2 \leq 1$, as $\left\| \frac{x}{\|x\|_1}\alpha \right\|_1 = \alpha$ then $\left\| \frac{x}{\|x\|_1}\alpha \right\|_2 \leq 1$ equivalently $\alpha\|x\|_2 \leq \|x\|_1$. In the other hand it exists $\delta > 0$, such that $0 < \|x\|_2 \leq \delta$ implies $\|x\|_1 \leq 1$, as $\left\| \frac{x}{\|x\|_2}\delta \right\|_2 = \delta$ then $\left\| \frac{x}{\|x\|_2}\delta \right\|_1 \leq 1$ equivalently $\|x\|_1 \leq \frac{1}{\delta}\|x\|_2$=$\beta\|x\|_2$. Therefore $\| \quad \|_1 \sim \| \quad \|_2$.
Proposition 16.5. Let $\{x_n\}$ and $\{y_n\}$ are two sequences in the $\mathbb{K}$-nvs E and let $\{\lambda_n\}$ be a sequence in $\mathbb{K}$. If

a) $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then $\|x_n\| \longrightarrow \|x\|$ and $x_n + y_n \longrightarrow x + y$.
b). $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$, then $\lambda_n x_n \longrightarrow \lambda x$.

Proof. a). From, $0 \leq |\|x_n\| - \|x\|| \leq \|x_n - x\|$ and $\|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$, we have the results. b). From $0 \leq \|\lambda_n x_n - \lambda x\| \leq |\lambda_n|\|x_n - x\| + \|x\|\|\lambda_n \longrightarrow \lambda\|$ we have the result.
Corollary 16.1. If, H is a $\mathbb{K}$-subvector space of $\mathbb{K}$-nvs E. The $cl(H)$ is a $\mathbb{K}$-subvector space of E.
Proof. Let $x, y \in cl(H)$ and $\lambda \in \mathbb{K}$, there are sequences $\{x_n\}$ and $\{y_n\}$ in H such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$, as the sequence $\{\lambda x_n + y_n\}$ is containing in H and $\lambda x_n + y_n \longrightarrow \lambda x$+y thus λx+y$\in cl(H)$.

Let $\{(E_i, \| \quad \|_i), 1 \leq i \leq n\}$ be a finite collection of $\mathbb{K}$-nvs E_i and $E = \prod_1^n E_i$ then for all $x = (x_1, \dots, x_i, \dots, x_n) \in E$, the functions $\|x\|_1 = \sum_1^n \|x_i\|_i$, $\|x\|_2 = \left(\sum_1^n \|x_i\|_i{}^2\right)^{\frac{1}{2}}$ and $\|x\|_\infty = \max_{1\leq i\leq n}\|x_i\|_i$ define a norms on the E and E is called a finite product $\mathbb{K}$-nvs.
Corollary 16.2. Let a $\mathbb{K}$-nvs E. For every $x, y \in E$ and every $\lambda \in \mathbb{K}$, the map $f: E \times E \longrightarrow E$ defined by $f(x, y) = x + y$ and the map $g: \mathbb{K} \times E \longrightarrow E$ defined by $g(\lambda, x) = \lambda x$ are continuous.
Proof. Let $\{(x_n, y_n)\}$ be a sequence in $E \times E$ which converges to $(x, y) \in E \times E$, then $f(x_n, y_n) = x_n + y_n \longrightarrow x + y = f(x, y)$ so f is continuous in arbitrary $(x, y) \in E \times E$, thus it continuous on $\in E \times E$. By the same, for (λ_n, x_n) converging to (λ, x) in $\mathbb{K} \times E$, $g(\lambda_n, x_n) = \lambda_n x_n \longrightarrow \lambda x = g(\lambda, x)$ so g is continuous in arbitrary $(\lambda, x) \in \mathbb{K} \times E$, thus it is continuous on $\mathbb{K} \times E$.
Definition 16.2. The $\mathbb{K}$-nvs $(E, \| \quad \|)$ is said to be a Banach space, if it is complete for the metric associated to the norm $\| \quad \|$.
Example 16.3.

a) $(\mathbb{R}, | \quad |)$, $(\mathbb{C}, | \quad |)$ and $(\mathbb{R}^n, \|x\|_1)$ are Banach spaces.

b) The space $(\mathcal{C}([a,b],\mathbb{R}),\|f\|_\infty)$ is a Banach space.

Remark 16.3. The continuity of the maps defined in the corollary 16.2.immediately gives:

a) If O is an open in the $\mathbb{K}$-nvs E, then $\forall\lambda\in\mathbb{K}^*$, λO is an open in E.

b) If O and U are two open in the $\mathbb{K}$-nvs E, then $O+U$ is an open in E.

c) If K is the nonvoide compact in the $\mathbb{K}$-nvs E, then $\forall\lambda\in\mathbb{K}$, λK is a compact in E.

d) The sum of two compacts is a compact.

e) If F is a closed in the $\mathbb{K}$-nvs E, then $\forall\lambda\in\mathbb{K}$, λF is a closed in E.

f) The sum of two closed in the $\mathbb{K}$-nvs E is not always closed. Indeed, The sets $F=\{(x,y)\in\mathbb{R}^2$such that $xy=1\}$ and $G=\{(x,0)$ such that $x\in\mathbb{R}\}$ are two closed in $\mathbb{R}^2$. But $F+G=G^C_{\mathbb{R}^2}$ is an open in $\mathbb{R}^2$.

Lemma 16.1. Let A and B are two subsets of E. If, A is closed and B is compact then for all $\lambda\in\mathbb{R}$, $A+\lambda B$ is closed.

Proof. Let $x\in cl(A+\lambda B)$ be, it exists a sequence $\{a_n\}$ in A which converges to $a\in A$ by the closure of A, and it exists a sequence $\{b_n\}$ in B, which converges to b$\in B$ by the compactness of B. Hence then sequence $\{a_n+\lambda b_n\}$ of $A+\lambda B$ converges to $a+\lambda b=x\in A+\lambda B$, so $A+\lambda B$ is closed.

16.2-Finite dimensional normed vector space

Proposition 16.6. Any norm N on $\mathbb{K}^n$ is k-Lipschitz.

Proof. As in the canonical basis $(e_1,\dots,e_i,\dots,e_n)$ of $\mathbb{K}^n$ any x $\in\mathbb{K}^n$ has a components $(x_1,\dots,x_i,\dots,x_n)\in\mathbb{K}^n$ and x $=\sum_1^n x_i\,e_i$, then $N(x)=N(\sum_1^n x_i\,e_i)\le\sum_1^n N(x_ie_i)=\sum_1^n|x_i|N(e_i)$. Let $k=\max_{1\le i\le n}N(e_i)$ then $N(x)\le k\sum_1^n|x_i|=k\|x\|_1$. Hence for all x, y $\in$ $\mathbb{K}n$, $Nx-y\le kx-y1$ so N is k-Lipschitz.

Proposition 16.7. All the norms in $\mathbb{K}^n$ are equivalent.

Proof. Let N be any norm in $\mathbb{K}^n$. It suffices to proof that N and $\|\ \ \|_1$ are equivalent. Because it exists $k>0$ such that $N(x)\le k\|x\|_1$ for all $x\in\mathbb{K}^n$ by the proof of proposition 16.6 and $S(0,1)=\{x\in\mathbb{K}^n,\|x\|_1=1\}$ is bounded and closed in $\mathbb{K}^n$, then $S(0,1)$ is compact in $\mathbb{K}^n$. Because N is continuous on $S(0,1)$ thus N is bounded on $S(0,1)$. Let $\alpha=\min_{x\in S(0,1)}N(x)$ be, so $\alpha\le N(x)$ for all $x\in S(0,1)$, because $\frac{x}{\|x\|_1}\in S(0,1)$ thus $\alpha\le N(\frac{x}{\|x\|_1})$ for all $x\in\mathbb{K}^n$ $(x\neq 0)$ hence $\alpha\|x\|_1\le N(x)\le k\|x\|_1$ for all $x\in\mathbb{K}^n$. Therefore N and $\|\ \ \|_1$ are equivalent.

Let us now give the fundamental result, which makes it possible to preserve the topological properties of $\mathbb{K}^n$ on any finite dimension $\mathbb{K}$-nvs, i.e. we will establish a (algebraic and topological) homeomorphism between $\mathbb{K}^n$ and a n-dimension $\mathbb{K}$-nvs (E,N), where N is a norm on E.

Theorem 16.1. Any n-dimension $\mathbb{K}$-nvs (E,N) is uniformly homeomorphic to $\mathbb{K}^n$.

Proof. Let $\{\alpha_1,\dots,\alpha_i,\dots,\alpha_n\}$ be a canonical basis of E and let $(\lambda_1,\dots,\lambda_i,\dots,\lambda_n)\in\mathbb{K}^n$ be the components of $x\in E$, then $x=\sum_1^n\lambda_i\,\alpha_i$. We will demonstrate that the map $\psi\colon E\longrightarrow\mathbb{K}^n$ defined
by $\psi(x)=(\lambda_1,\dots,\lambda_i,\dots,\lambda_n)$ is an uniform homeomorphism. By induction:

Step 1. Suppose that $dim\ E=1$, then $\psi\colon E\longrightarrow\mathbb{K}$, is such that for $x\in E$, $\psi(x)=\psi(\lambda\alpha)=\lambda$ where $\{\alpha\}$ is a basis of E and $\lambda\in\mathbb{K}$ the component of x. It is clear that ψ is linear, bijective and for all $x,y\in E$, $N(x-y)=N\big((\lambda-\lambda')\alpha\big)$=$|\lambda-\lambda'|N(\alpha)$ as $N(\alpha)\neq 0$, then $|\psi(x)-$ $\psi y=\lambda-\lambda'=1N(\alpha)Nx-y$, thus ψ is k-Lipschitz with $k=1N(\alpha)$, so it is continuous on E. The inverse $\psi^{-1}\colon\mathbb{K}\longrightarrow$ E defined by $\psi^{-1}(\lambda)=x=\lambda\alpha$ satisfies for every $\lambda,\lambda'\in\mathbb{K}$, $N\big(\psi^{-1}(\lambda)-$ $\psi-1\lambda'=N\lambda-\lambda'\alpha=\lambda-\lambda'N(\alpha)$ then $\psi-1$ is k-Lipschitz with $k=N(\alpha)$, hence it is continuous.

Step 2. Suppose that $\mathbb{K}^{n-1}$ is homeomorphic to the $n-1$-dimension $\mathbb{K}$-nvs (E,N). We will proof that $\mathbb{K}^{n}$ is homeomorphic to the n-dimension $\mathbb{K}$-nvs (E,N). We need the following $a), b), c)$ assertions:

$a)$ For all $1 \leq i \leq n$, H_i=$g_i^{-1}(\{0\})$ is a closed in E, where $g_i: E \longrightarrow \mathbb{K}$ is defined by $g_i(\sum_1^{\mathrm{n}} \lambda_i\, \alpha_i)$=$\lambda_i$. As $H_i = \{x = \sum_1^{\mathrm{n}} \lambda_i\, \alpha_i,\ \lambda_i = 0\}$=$\{y \in E, y = \sum_{1\leq \mathrm{i}\neq \mathrm{j}\leq \mathrm{n}}^{\mathrm{n}} \lambda_i\, \alpha_i\}$ then H_i is a $n-1$-dimension $\mathbb{K}$-nvs by the assumption it is homeomorphic to $\mathbb{K}^{n-1}$ which is a Banach space by the corollary 14.6, hence H_i is also a Banach space in the $\mathbb{K}$-nvs (E,N), then H_i is closed by lemma 14.1.

$b)$ Show that, it exists $b \notin H_i$ such that g_i(b)=1. As $dimH_i = n-1$, then H_i is strictly containing in E, so it exists $a \in E$ and $a \notin H_i$ thus g_i(a) $\neq 0$. It is obvious that $b = \frac{a}{g_i(a)}$ satisfies $g_i(b) = g_i\left(\frac{a}{g_i(a)}\right)$=1, then $b \notin H_i$.

$c)$ Show that $b + H_i$ is closed and it exists $r > 0$ such that $B(0,r) \cap\ b + H_i = \emptyset$. Obviously the map $h: H_i \longrightarrow b + H_i$, defined by $h(x) = b + x$ is a homeomorphism, because H_i is a closed then $h(H_i)$= $b + H_i$ is a closed. In the other hand $-b \notin H_i$ implies that $0 \notin b + H_i = cl(b + H_i)$ which implies that it exists $r > 0$ such that $B(0,r) \cap\ b + H_i = \emptyset$ (by definition of the closure).

$d)$ We will proof that $\forall x \in B(0,r)$, $|g_i(x)|$<1. Let $x \in B(0,r)$ as $B(0,r) \cap\ b + H_i = \emptyset$ then $x \notin\ b + H_i$ or $x - b \notin H_i$ then $g_i(x-b) \neq 0$ or $g_i(x) \neq g_i(b)$=1, if we assume that $g_i(x) > 1$ then $N\left(\frac{x}{g_i(x)}\right) = \frac{1}{g_i(x)} N(x) < N(x) < r$, hence $\frac{x}{g_i(x)} \in B(0,r)$ so $g_i\left(\frac{x}{g_i(x)}\right) = 1$ contradiction. Thus $\forall x \in B(0,r)$, $|g_i(x)|$<1.

$e)$ We will prove that g_i is uniformly continuous. Let $\varepsilon > 0$ be, we search $\delta > 0$ such that if $0 < N(x-y) < \delta$ for all $x, y \in E$, then $|g_i(x) - g_i(y)| = |g_i(x-y)| < \varepsilon$ (g_i is linear). It suffices to take $0 < \delta \leq \varepsilon r$, indeed $N(x-y) < \delta \leq \varepsilon r$ implies that $\frac{x-y}{\varepsilon} \in B(0,r)$ then $\left|g_i\left(\frac{x-y}{\varepsilon}\right)\right| < 1$ or $|g_i(x-y)| < \varepsilon$.

Step 3. In this last step, we return to the proof of the uniform continuity of ψ and ψ^{-1} with for all $\in E$, $x = \sum_1^{\mathrm{n}} \lambda_i\, \alpha_i$, $\psi(x) = (\lambda_1, \dots, \lambda_i, \dots, \lambda_n) = (g_1(x), \dots, g_i(x), \dots, g_n(x))$. It is clear that ψ is a linear isomorphism. Let us proof that ψ is uniformly continuous. Let $\varepsilon > 0$ be, because for all $1 \leq i \leq n$, g_i is uniformly continuous, it exists δ_i>0 such that, if $0 < N(x-y) < \delta_i$ for all $x, y \in E$, then $|g_i(x) - g_i(y)| = |g_i(x-y)| < \frac{\varepsilon}{n}$. Thus for $\delta = \max_{1\leq i\leq n} \delta_i$ we have $0 < N(x-y) < \delta$ for all $x, y \in E$ which implies that $\|\psi(x) -$ $\psi y1$=1n$gix-giy<\varepsilon$, therfore ψ is uniformly continuous. let us show at the end that $\psi^{-1}: \mathbb{K}^{\mathrm{n}} \longrightarrow \mathrm{E}, \lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_n) \mapsto \psi^{-1}(\lambda) = x = \sum_1^{\mathrm{n}} \lambda_i\, \alpha_i$ is k-Lipschitz. Let $\lambda, \lambda' \in \mathbb{K}^{\mathrm{n}}$, we have for every $\lambda, \lambda' \in \mathbb{K}$,
$N(\psi^{-1}(\lambda) - \psi^{-1}(\lambda')) = N(\sum_1^{\mathrm{n}} \lambda_i\, \alpha_i - \sum_1^{\mathrm{n}} \lambda_i'\, \alpha_i)$=$N(\sum_1^{\mathrm{n}} \alpha_i(\lambda_i - \lambda_i')) \leq \sum_1^{\mathrm{n}} \mathrm{N}(\alpha_i(\lambda_i -$ $\lambda i'$=1nN$\alpha i\lambda i-\lambda i'\leq$max$1\leq i\leq nN\alpha i$1n$\lambda i-\lambda i'=k\lambda-\lambda'1$then $\psi-1$ is k-Lipschitz with $k = \max_{1\leq i\leq n} \mathrm{N}(\alpha_i)$, hence it is uniformly continuous.

Lemma 16.2. Any n-dimension subspace in the $\mathbb{K}$-nvs E is closed.

Proof. Let H be a n-dimension subspace in the $\mathbb{K}$-nvs E. By the theorem 16.1, H is uniformly homeomorphic to $\mathbb{K}^n$, then H is complete in the metric E, hence it is closed.

Remark 16.4. From the theorem 16.1, all the properties obtained in $\mathbb{K}^n$ remain valid in the n-dimension pace $\mathbb{K}$-nvs E. In particular:

$a)$ The closed unit bull $\tilde{B}(0,1)$ is compact.

$b)$ The open unit bull $B(0,1)$ is locally compact.

As the two applications in the corollary 16.2 are homeomorphism, it follows that:

Lemma 16.3.

$i)$ A $\mathbb{K}$-nvs E is locally compact$\Leftrightarrow$ $\tilde{B}(0,1)$ is compact.

$ii)$ A $\mathbb{K}$-nvs E is locally compact$\Leftrightarrow$ $B(0,1)$ is relatively compact.

Proof. $i)$ Since E is locally compact, then 0 has a compact neighborhood K, therefore there is $r > 0$ such that $B(0,r) \subset K$, so $\tilde{B}(0,1) \subset \frac{1}{r}cl(K)$=$\frac{1}{r}$K, which is compact, it follows that the closure unit ball $\tilde{B}(0,1)$ is compact. Reciprocally, since for any $x \in E$, there is r>0 such that $B(x,r) \subset \tilde{B}(x,r) = x + r\tilde{B}(0,1)$ which is a compact neighborhood of x by the fact that, the singleton $\{x\}$ and $\tilde{B}(0,1)$ are compact and the remark 16.3 $c)$ and $d)$, then E is locally compact. $ii)$ is a direct consequence of $i)$.and the proposition 16.2 $a)$.

Remark 16.5. As $\tilde{B}(a,r) = a + r\tilde{B}(0,1)$ and $B(a,r) = a + rB(0,1)$, the lemma 16.3 remains valid for $\tilde{B}(a,r)$ and $B(a,r)$.

Theorem 16.2 (Riez-Frédiric). A locally compact $\mathbb{K}$-nvs E is finite dimensional.

Proof. Because E is locally compact, then $\tilde{B}(0,1)$ is compact. It exists a finite number $a_1, \dots, a_i, \dots, a_n \in \tilde{B}(0,1)$ such that $\tilde{B}(0,1) = \cup_1^n B\left(a_i, \frac{1}{2}\right)$. As the n-dimension subspace $\mathrm{H} = [a_1, \dots, a_i, \dots, a_n]$ (H is enjoyed by $a_1, \dots, a_i, \dots, a_n$) is closed in E, by the lemma 16.2. Then, $E \subset H$ therefore $dimE = n.$ If not, if it exists $b \in E$ and $b \notin H$, then $d(b,H) = a > 0$, taking $\varepsilon = \frac{a}{2}$ and using the infimum property, there is $x_\varepsilon \in H$ such that $a \leq d(b,x_\varepsilon) = \|b - x_\varepsilon\| < a + \frac{a}{2} = \frac{3}{2}a$. Because $\frac{b-x_\varepsilon}{\|b-x_\varepsilon\|} \in \tilde{B}(0,1)$, it exists $j \in \{1, \dots, n\}$ such that $\frac{b-x_\varepsilon}{\|b-x_\varepsilon\|} \in B\left(a_j, \frac{1}{2}\right)$ i.e. $\left\|a_j - \frac{b-x_\varepsilon}{\|b-x_\varepsilon\|}\right\| < \frac{1}{2}$. But, $\left\|a_j - \frac{b-x_\varepsilon}{\|b-x_\varepsilon\|}\right\| = \left\|\frac{a_j\|b-x_\varepsilon\|-b+x_\varepsilon}{\|b-x_\varepsilon\|}\right\| = \frac{1}{\|b-x_\varepsilon\|}\|b - x\varepsilon + ajb - x\varepsilon$.Because $x\varepsilon$ and aj are two elements of the subspace H of E and $b - x\varepsilon \in \mathbb{R}+*$, it follows that $x_\varepsilon + a_j\|b - x_\varepsilon\| \in H$, hence $\frac{2}{3} \leq \frac{1}{\|b-x_\varepsilon\|}a \leq \frac{1}{\|b-x_\varepsilon\|}\left\|b - \left(x_\varepsilon + a_j\|b - x_\varepsilon\|\right)\right\| < \frac{1}{2}$. Contradiction.

16.3-Linear maps on $\mathbb{K}$-nvs

Linear maps have some particular and interesting properties. In the sequel, f is a linear map from the $\mathbb{K}$-nvs $(E, \|\ \ \|_E)$ into the $\mathbb{K}$-nvs $(F, \|\ \ \|_F)$. Starting with

Corollary 16.3. If the $\mathbb{K}$-nvs E is n-dimension. Then f is k-Lipschitz.

Proof. Let $(e_i)_{i=1,\dots,n}$ be a basis E and let $(x_i)_{i=1,\dots,n}$ be the components of $x \in E$, then $\|f(x)\|_F = \|f(\sum_1^n x_i\, e_i)\|_F = \|\sum_1^n x_i\, \mathrm{f}(e_i)\|_F \leq \sum_1^n |x_i|\, \|\mathrm{f}(e_i)\|_F \leq (\max_{1\leq i\leq n}\|\mathrm{f}(e_i)\|_F)\|x\|_E$. We conclude as in the proof of the proposition 16.6 that f is k-Lipschitz, where $k = \max_{1\leq i\leq n}\|\mathrm{f}(e_i)\|_F$.

Definition 16.3. The map f is said to be bounded, if there is $k > 0$ such that $\|f(x)\|_F \leq$k$\|x\|_E$, for all $x \in E$.

Theorem 16.2. The following properties are equivalent:

$a)$ f is continuous on E.

$b)$ f is continuous en 0.

$c)$ f is k-Lipschitz.

$d)$ f is bounded on E.

$e)$ f is bounded on $\tilde{B}(0,1)$.

$f)$ f is bounded on $S(0,1)$.

Proof. $a) \Longrightarrow b)$. As f is continuous on E, then it is continuous in 0. $b) \Longrightarrow c)$ Because f is continuous in 0, for all $\varepsilon >$ it exists $\delta > 0$ such that for all $x \in E$, satisfying $0 < \|x\|_E \leq \delta$ we have $\|f(x)\|_F \leq \varepsilon$. Thus for all $x, y \in E$, $\left\|f\left(\frac{x-y}{\|x-y\|_E}\delta\right)\right\|_F \leq \varepsilon$, witch implies that $\|f(x) - f(y)\|_F \leq k\|x - y\|_E$ for all $x, y \in E$, where $k = \frac{\varepsilon}{\delta}$>0. Hence f is k-Lipschitz.

$c) \Longrightarrow d)$ As it exists k>0 such that $\|f(x-y)\|_F \leq k\|x-y\|_E$ for all $x, y \in E$ and $f(0)=0$, then it exists k>0 such that $\|f(x)\|_F \leq k\|x\|_E$ for all $x \in E$, so f is bounded. $d) \Longrightarrow e)$ Because it exists k>0 such that $\|f(x)\|_F \leq k\|x\|_E$ for all $x \in E$, then it exists k>0 such that $\|f(x)\|_F \leq k\|x\|_E$ for all $x \in \tilde{B}(0,1)$. $e) \Longrightarrow f)$ Because it exists k>0 such that $\|f(x)\|_F \leq k\|x\|_E$ for all $x \in \tilde{B}(0,1)$, and $S(0,1) \subset \tilde{B}(0,1)$. Then, it exists k>0 such that $\|f(x)\|_F \leq k\|x\|_E$ for all $x \in S(0,1)$. $f) \Longrightarrow a)$ Since it exists k>0, such that $\|f(x)\|_F \leq k\|x\|_E$ for all $x \in S(0,1)$ then $\|f(x)\|_F \leq k$ for all $x \in S(0,1)$. So for all $x, y \in E$ $(x \neq y)$ $\left\|f\left(\frac{x-y}{\|x-y\|_E}\right)\right\|_F \leq k$, witch implies that $\|f(x)-f(y)\|_F \leq k\|x-y\|_E$ for all $x, y \in E$. Hence f is k-Lipschitz, therefore it is continuous on E.

Denote by: $L(E,F)$ the $\mathbb{K}$-vector space of all continuous linear maps from E into F; $L(E)=L(E,\mathrm{E})$ and $E^* = L(E,\mathbb{K})$, which is called the dual of E, the elements of E^* are said to be the bounded linear functionals or the continuous linear functionals.

Definition 16.4. We call the norm of $f \in L(E,F)$, any number a,b,c or d in the following lemma.

Lemma 16.4. The following numbers are equal.

$a = \sup_{(x\in E, x\neq 0)} \frac{\|f(x)\|_F}{\|x\|_E}$, $b = \sup_{x\in S(0,1)}\|f(x)\|_F$, $c = \sup_{x\in\tilde{B}(0,1)}\|f(x)\|_F$ and $d = inf\{k>0,\ such\ that\ \|f(x)\|_F \leq k\ \|x\|_E$ for all $x \in E\ \}$.

Proof. $a \leq b$. As $\|f(x)\|_F \leq\ b$ for all $x \in S(0,1)$, then $\left\|f\left(\frac{x}{\|x\|_E}\right)\right\|_F = \frac{\|f(x)\|_F}{\|x\|_E} \leq\ b$ for all $x \in E, (x \neq 0)$ it follows that $a = \sup_{\{x\in E, x\neq 0\}} \frac{\|f(x)\|_F}{\|x\|_E} \leq\ b$. Since $S(0,1) \subset \tilde{B}(0,1)$ then $\sup_{x\in S(0,1)}\|f(x)\|_F \leq \sup_{x\in\tilde{B}(0,1)}\|f(x)\|_F$, so $b \leq c$. Since $d =\ inf\{k>0,\ such\ that\ \|f(x)\|_F \leq k\|x\|_E$ for all $x \in E\ \}$, then for any $\varepsilon > 0$ it exists $k_\varepsilon > 0,\ such\ that\ \|f(x)\|_F \leq k_\varepsilon\|x\|_E$for all $x \in E$ and $k_\varepsilon <\ d+\varepsilon$ so $c = \sup_{x\in\tilde{B}(0,1)}\|f(x)\|_F \leq k_\varepsilon <\ d+\varepsilon$ when $\varepsilon \longrightarrow 0$, we have $c \leq d$. Finally as $\frac{\|f(x)\|_F}{\|x\|_E} \leq a$ for all $x \in E$ $(x \neq 0)$ then $\|f(x)\|_F \leq a\|x\|_E$ for all $x \in E$ then $d \leq a$. Therefore $a \leq b \leq c \leq d \leq a$.

Before proving that, one of the previous four numbers is a norm. Note that it is easy to check that for all $f \in L(E,F)$, $f=0$ on $E \Leftrightarrow f=0$ on $\tilde{B}(0,1)$. Let us show that $c)$ is a norm i.e. the map $\|\ \ \|_{L(E,F)}: L(E,F) \longrightarrow \mathbb{R}_+$ defined by for all $f \in L(E,F)$, $\|f\|_{L(E,F)}$=$\sup_{x\in\tilde{B}(0,1)}\|f(x)\|_F$ satisfies the conditions n_1, n_2 and n_3 in the definition 16.1.
For all $f, g \in L(E,F)$, for all $\lambda \in \mathbb{K}$ and for all $x \in \tilde{B}(0,1)$, we have:

n_1-$0 \leq \|f(x)\|_F \leq \|f\|_{L(E,F)}$=0$\Leftrightarrow$ $\|f(x)\|_F = 0 \Leftrightarrow f(x)=0 \Leftrightarrow f=0$ on $\tilde{B}(0,1) \Leftrightarrow$ $f=0$ on E.

n_2-$\|\lambda f\|_{L(E,F)}$=$\sup_{x\in\tilde{B}(0,1)}\|(\lambda f)(x)\|_F$=$\sup_{x\in\tilde{B}(0,1)}\|\lambda f(x)\|_F = \sup_{x\in\tilde{B}(0,1)}|\lambda|\|f(x)\|_F = |\lambda|\sup_{x\in\tilde{B}(0,1)}\|f(x)\|_F$=$|\lambda|\|f\|_{L(E,F)}$.

n_3-$\|f+g\|_{L(E,F)}$=$\sup_{x\in\tilde{B}(0,1)}\|(f+g)(x)\|_F$=$\sup_{x\in\tilde{B}(0,1)}\|f(x)+g(x)\|_F$

$$\leq \sup_{x\in\tilde{B}(0,1)} (\|f(x)\|_F + \|g(x)\|_F) \leq \sup_{x\in\tilde{B}(0,1)} \|f(x)\|_F + \sup_{x\in\tilde{B}(0,1)} \|g(x)\|_F$$

$$= \|f\|_{L(E,F)} + \|g\|_{L(E,F)}.$$

Hence, the map $\|\ \ \|_{L(E,F)}$ is a norm on $L(E,F)$.
Because for any $f, g \in L(E,F)$ and for any $\lambda \in \mathbb{K}$, there are k and k' in $\mathbb{R}_+^*$ such that $\|(\lambda f+g)(x)\|_F \leq |\lambda|\|f(x)\|_F + \|g(x)\|_F \leq (|\lambda|\ k + k')\|x\|_E$ for all $x \in$ E, then $\lambda f + g \in L(E,F)$. $L(E,F)$ is a $\mathbb{K}$-nvs. If, F is complete then $L(E,F)$ is complete by the theorem 15.1. Therefore E^* is complete.

Example 16.5.

a) Let $E = \mathcal{C}([0,1], \mathbb{R})$ be, the map $T: (E, \| \quad \|_\infty) \longrightarrow (E, \| \quad \|_\infty)$ defined by $(Tf)(x) = f(x) - f(0)$, for all $x \in [0,1]$ is continuous. Indeed, the linearity is obvious and because for any $f \in E$, $\|Tf\|_\infty$=$\sup_{x\in[0,1]}|f(x) - f(0)| \leq |f(0)| + \sup_{x\in[0,1]}|f(x)| = |f(0)|$+$\|f\|_\infty \leq 2\|f\|_\infty$, then T is bounded, hence it is continuous. But $T: (E, \| \quad \|_1) \longrightarrow (E, \| \quad \|_1)$ is not bounded. If not, it exists $M > 0$ such that $\|Tf\|_1 \leq M\|f\|_1$ for all $f \in E$. Thus, for the sequence f_n(x)=$(n+1)(1-x)^n$ in E, we have $\|Tf_n\|_1 = \int_0^1 |Tf_n(x)|\, dx = \int_0^1 |f_n(x) -$ $fn0\, dx$=n+1011−1−$xndx$=n for all n∈ℕ∗ and fn1=01$fn(x)dx$=n+1011−$xndx$=1. Hence, $n \leq M$ for all $n \in \mathbb{N}^*$, contradiction with the fact that, $\mathbb{N}$ is not bounded above. It follows that T is not continuous, by the theorem 16.2 d).

b) Let $E = \mathcal{C}([0,1], \mathbb{R})$ be, the map $T: (E, \| \quad \|_\infty) \longrightarrow (\mathbb{R}, | \quad |)$ defined by $T(f) = \int_0^1 f(x) sinx\, dx$ for any $f \in E$ is continuous, and $\|T\|_{E^*} = \int_0^1 sinx\, dx$. It is clear that T is linear and for any $f \in E$, $|T(f)| \leq \int_0^1 |f(x)|\, |sinx| dx \leq \left(\int_0^1 sinx\, dx\right) \|f\|_\infty = k\|f\|_\infty$ where 0<k=$\int_0^1 sinx\, dx$, then T is bounded, hence it is continuous. Furthermore, $\|T\|_{E^*} \leq \left(\int_0^1 sinx\, dx\right)$ by the definition 16.4 d). But $\|T\|_{E^*} \geq |T(f)|$ for all $f \in$ E, then for $f = 1$ in E, $\|T\|_{E^*} \geq \int_0^1 sinx\, dx$. Hence $\|T\|_{E^*} = \int_0^1 sinx\, dx$.

Corollary 16.4. If $f \in L(E, F)$ and $g \in L(F, G)$ where $(G, \| \quad \|_G)$ is a $\mathbb{K}$-nvs. Then $g \circ f \in L(E, G)$ and $\|g \circ f\|_{L(E,G)} \leq \|g\|_{L(F,G)} \|f\|_{L(E,F)}$.

Proof. Let $x, y \in E$ and $\lambda \in \mathbb{K}$ because $(g \circ f)(\lambda x + y)$=g$\big(f(\lambda x + y)\big) = \text{g}\big(f(\lambda x)\big) + \text{g}\big(f(y)\big) = \text{g}\big(\lambda f(x)\big) + \text{g}\big(f(y)\big) = \lambda \text{g}\big(f(x)\big) + \text{g}\big(f(y)\big) = \lambda(g \circ f)(x) + (g \circ f)(y)$, then $g \circ f$ is linear. And as
$\|(g \circ f)(x)\|_G$=$\big\|g\big(f(x)\big)\big\|_G \leq \|g\|_{L(F,G)} \|f(x)\|_F \leq \|g\|_{L(F,G)} \|f\|_{L(E,F)} \|x\|_E$, for all $x \in E$ then $g \circ f$ is bounded, $g \circ f \in L(E, G)$ and $\|g \circ f\|_{L(E,G)} = \sup\limits_{x \in \tilde{B}(0,1)} \|(g \circ f)(x)\|_G \leq \|g\|_{L(F,G)} \|f\|_{L(E,F)}$.

By the corollary 16.4, it follows that, if $f \in L(E)$ then, $f^n \in L(E)$ and $\|f^n\|_{L(E)} \leq \left(\|f\|_{L(E)}\right)^n$ for every $n \in \mathbb{N}^*$.

In mathematics, a hyperplane H is a linear subspace of the $\mathbb{K} - vs$ E, such that the basis of its complementary has cardinality one. In the case when E is an n-dimensional vector space ($n \in \mathbb{N}^*$), then H is an $(n-1)$-dimensional subspace. Examples of hyperplanes: the space $\{0\}$ in 1-dimension space, any straight line through the origin in 2-dimensions, any plane containing the origin in 3-dimensions. In higher dimensions, it is useful to think of a hyperplane as member of an affine family of $(n-1)$-dimensional subspaces (affine spaces look and behavior very similar to linear spaces but they are not required to contain the origin), such that the entire space is partitioned into these affine subspaces. This family will be stacked along the unique vector (up to sign) that is perpendicular to the original hyperplane. This "visualization" allows one to easily understand that a hyperplane always divides the parent vector space into two regions. In general $\mathbb{K}$-nvs E the definition is given by.

Definition 16.5. A subset H of a $\mathbb{K}$-nvs E is said to be an affine hyperplane, if it exists a linear form $f \not\equiv 0$ (f non identiquely equal to 0 on E) and a constant $b \in \mathbb{R}$ such that $H = \{x \in E, f(x) = b\}$. We say that H is the hyperplane of the equation $[f = b]$.

Remark 16.6.

a) $H = Kerf$, when $f = 0$.

b) $H^C \neq \emptyset$. Indeed, if $H^C = \emptyset$ then, for all $x \in E$, $f(x) = b$ so $f(0) = 0 = b$ hence for all $x \in E$, $f(x) = 0$ i.e. $f \equiv 0$ on E, contradiction.

c) The map f is not necessary continuous.

d) H is not necessary containing 0.

e) For the given $a=(a_1,\dots,a_i,\dots,a_n)\in\mathbb{R}^n$ $(a\neq 0)$ and $b\in\mathbb{R}$ and for any $x=(x_1,\dots,x_i,\dots,x_n)\in\mathbb{R}^n$, the hyperplane H in $\mathbb{R}^n$ takes the form $H=\{x\in\mathbb{R}^n,\sum_1^n a_i\,x_i=b\}$. Specifically, when $n=2$ and $a_2\neq 0$, $H=\{x\in\mathbb{R}^2,\sum_1^2 a_i\,x_i=b\}=\{x\in\mathbb{R}^2, a_1x_1+a_2x_2=b=x\in\mathbb{R}2, x2=ax1+b$.

Lemma 16.3. The hyperplane H of the equation $[f=b]$ is closed$\Leftrightarrow f$ is continuous.

Proof. By the continuity of f and the closure of the segleton $\{b\}$ in $\mathbb{R}$ and as $H=f^{-1}(\{b\})$, then H is closed. Conversely, let $x_0\in H^C$ which is open, then it exists $r>0$ such $B(x_0,r)\subset H^C$. We can assert that : i) if $f(x_0)<b$, then $f(y)<b$ for all $y\in B(x_0,r)$ and ii) if $f(x_0)>b$, then $f(y)>b$ for all $y\in B(x_0,r)$. Let us check i) (the verification of ii) is done in the same way). Suppose, it exists $x_1\in B(x_0,r)$ such that $f(x_1)>b>f(x_0)$. $B(x_0,r)$ being convex and $t=\frac{b-f(x_0)}{f(x_1)-f(x_0)}\in]0,1[$ then, $f(z)=0$ where $z=tx_1+(1-t)x_0\in B(x_0,r)$, so $f(z)=0$ then $z\in H=Kerf$ contradiction. Because $\tilde{B}\left(x_0,\frac{r}{2}\right)\subset B(x_0,r)$ then $f(y)<b$ for all $y\in\tilde{B}\left(x_0,\frac{r}{2}\right)$, as $\tilde{B}\left(x_0,\frac{r}{2}\right)=x_0+\frac{r}{2}\tilde{B}(0,1)$ then $f\left(x_0+\frac{r}{2}x\right)<b$ for all $x\in\tilde{B}(0,1)$, hence $f(x_0)+\frac{r}{2}f(x)<b$ for all $x\in\tilde{B}(0,1)$. By the linearity of f and $-x\in\tilde{B}(0,1)$ we have $-\frac{2}{r}\left(b-f(x_0)\right)<f(x)<\frac{2}{r}\left(b-f(x_0)\right)$ for all $x\in\tilde{B}(0,1)$, so $\left|f\left(\frac{x}{\|x\|}\right)\right|<\left[\frac{2}{r}\left(b-f(x_0)\right)\right]$ for all $x\in\tilde{B}(0,1)$ $(x\neq 0)$, therefore $|f(x)|\leq\left[\frac{2}{r}\left(b-f(x_0)\right)\right]\|x\|$ for all $x\in\tilde{B}(0,1)$, hence f is continuous by the theorem 16.2 e).

Corollary 16.5. $Kerf$ is a $\mathbb{R}$-vector subspace of E whith codimension one.

Proof. It is clear that $Kerf$ is a $\mathbb{R}$-vector subspace of E. As $f\not\equiv 0$ it exists $x_0\in E$ such that $f(x_0)\neq 0$. It is clear that, $f\left(x-\frac{f(x)}{f(x_0)}x_0\right)=0$ for every $x\in E$, so $x\in Kerf+\frac{f(x)}{f(x_0)}x_0$ and $E=Kerf+\frac{f(x)}{f(x_0)}x_0=Kerf+\mathbb{R}x_0$ where $\mathbb{R}x_0$ is $\mathbb{R}$-vector subspace of E enjoyed by x_0 $(dim\mathbb{R}x_0=1)$. If now $x\in Kerf\cap\mathbb{R}x_0$, it exists λ in $\mathbb{R}$ such that $x=\lambda x_0$ and $f(\lambda x_0)=\lambda f(x_0)=0$ then $\lambda=0$ therefore $x=0$ and $E=Kerf\oplus\mathbb{R}x_0$ i.e. $\mathbb{R}x_0$ is a supplementary algebraic of $Kerf$. Thus codimension of $Kerf$ is one.

Corollary 16.6. $Kerf$ is closed or everywhere dense in the $\mathbb{R}$-nvs E.

Proof. If f is continuous then $Kerf$ is closed. If f is not continuous, $Kerf$ is a $\mathbb{R}$-subvector space of a $\mathbb{R}$-nvs E and $Kerf\not\subseteq cl(Kerf)$ which is also a $\mathbb{R}$-subvector space of a $\mathbb{R}$-nvs E. Then codimension of $cl(Kerf)=0$ so $cl(Kerf)=E$.

17-Fundamental theorems of functional analysis

17.1-Hahn Banach theorems

Let E be a $\mathbb{R}$-nvs. The answer to the next question is yes: is there "enough" continuous linear functionals on E which separate the points of E?. (This result is a kind of analogue of the Urysohn's theorem 8.1, for continuous function over a normal topological space). We are going to prove an extension theorem for continuous linear functional defined on a proper linear subspace G of E i.e. $G\not\subseteq E$ (this result is a kind of analogue of Tietze's-Urysohon extension theorem 8.2, for the continuous functions defined on a proper closed subset, of a normal topological space E). The important fact here is that the continuous linear extension preserves the norm see corollary 17.1. Note also that here (unlike Tietze's-Urysohon extension theorem) the subspace G does not need to be closed. Indeed, from the corollary 17.4, a continuous linear functional can always be extended continuously from G to $cl(G)$. So, it makes no difference whether G is closed or not. To simplify, we will only prove the Hahn-Banach theorems in the real case.

Substantially, there are three fundamental forms of Hahn Banach's theorems: algebraic form, topological form and geometric form or separation form. To establish the algebraic form we need in addition to the Zorn's lemma a map p defined on the $\mathbb{R}$-vs E into $\mathbb{R}$ satisfying for all $x, y \in E$ and for all $\lambda \in \mathbb{R}_+^*$,

$p(\lambda x) = \lambda p(x)$ (p is positively homogeneous); (1)

$p(x+y) \leq p(x) + p(y)$ (p is subadditive). (2)

Theorem 17.1 (Algebrical form of Hahn Banach theorem). If g is a linear function from a proper linear subset G of E into $\mathbb{R}$ satisfying: $g(x) \leq p(x)$ for all $x \in G$. (3)

Then, there is a linear function f from E into $\mathbb{R}$ satisfying:

$f(x) = g(x)$ for all $x \in G$ and $f(x) \leq p(x)$ for all $x \in E$. (4)

Proof. By stapes:

Stape 1. Let $G + \mathbb{R}x_0$ be the linear subset of E, where $x_0 \in G^C$. We will proof that, it exists a linear function h, from $G + \mathbb{R}x_0$ into $\mathbb{R}$ which satisfies (4) on H. By the linearity of g and (2),(3), $g(x) - g(y) = g(x-y) \leq p(x-y) \leq p(x+x_0) + p(-x_0-y)$ for every $\mathrm{x, y} \in G$. Hence $-g(y) - p(-x_0 - y) \leq p(x+x_0) - g(x)$, for every $\mathrm{x, y} \in G$ (5).

For a fixed x in (5), the set $Y = \{-g(y) - p(-x_0 - y),\ \mathrm{y} \in G\}$ is bounded above, and for a fixed y in (5), the set $X = \{p(x+x_0) - g(x),\ \mathrm{x} \in G\}$ is bounded bellow. Therefore, it exists $\alpha \in \mathbb{R}$ such that: for all $\mathrm{z} \in G$

$-g(z) - p(-z - x_0) \leq \sup_{\mathrm{y} \in G} Y \leq \alpha \leq \inf_{\mathrm{x} \in G} X \leq p(z + x_0) - g(z)$ (6).

The function h from $G + \mathbb{R}x_0$ into $\mathbb{R}$, defined by: for all $x \in G$ and for all $t \in \mathbb{R}$, $h(x + tx0 = g(x) + t\alpha$, satisfies for $t = 0$, $hx = g(x)$.on G, and for any $\lambda \in \mathbb{R}$, $x, y \in G$ and $t, s \in \mathbb{R}$, $h[\lambda(x + tx_0) + y + sx_0] = h[\lambda x + y + (\lambda t + s)x_0] = g(\lambda x + y) + (\lambda t + s)\alpha = \lambda\,(g(x) + t\alpha + gy + \mathrm{s}\ \alpha = \lambda hx + tx0 + hy + sx0$, then h is linear. It remains to verify that for all $x \in G$ and $t \in \mathbb{R}^*$, $h(x + tx_0) \leq p(x + tx_0)$. Let $x \in G$ and $t \in \mathbb{R}^*$ are, if $t > 0$, by the right side of (6) and (1) we have $\alpha \leq p\left(\frac{x}{t} + x_0\right) - g\left(\frac{x}{t}\right) \leq \frac{1}{t}p(x + tx_0) - \frac{1}{t}g(x)$, equivalently $g(x) + t\alpha \leq p(x + tx_0)$ thus $h(x + tx_0) \leq p(x + tx_0)$ for all $x \in G$ and all $t > 0$. If $t < 0$ then $-g\left(\frac{x}{t}\right) - p\left(-\frac{x}{t} - x_0\right) \leq \alpha$ by the left side of (6) and (1), we have $t\alpha \leq -\mathrm{t}g\left(\frac{x}{t}\right) - \mathrm{t}p\left(-\frac{x}{t} - x_0\right) = -g(x) + p(x + tx_0)$, hence $g(x) + t\alpha \leq p(x + tx_0)$. Thus, $h(x + tx_0) \leq p(x + tx_0)$ for all $x \in G$ and all $t \in \mathbb{R}^*$.

Stape 2. In this step, we use Zorn's lemma (just before the lemma 10.6) and the step 1: Let $\mathcal{H}$ be the set of all functions h defined from G_h into $\mathbb{R}$, where G_h is a subspace of E containing G, with $h = g$ on G and $g \leq p$ on G_h. As $\mathcal{H} \neq \emptyset$ since $g \in \mathcal{H}$, we define the relation $\leq$ on $\mathcal{H}$ by: for any $h_1, h_2 \in \mathcal{H}$, $(h_1 \leq h_2) \Leftrightarrow (G_{h_1} \subset G_{h_2}$ and $h_1 = h_2$ on $G_{h_1})$. Let $\mathcal{I} = \{h_\alpha, \alpha \in \Delta\}$ be any totally ordered collection in $\mathcal{H}$. Check that $(\mathcal{I}, \leq)$ is bounded above. Let $G_h = \cup_{\alpha \in \Delta} G_{h_\alpha}$ be, it is clear that G_h is a subspace of E, consider the function h from G_h into $\mathbb{R}$ defined by $h = h_\alpha$ on G_{h_α} for all $\alpha \in \Delta$. It is clear that $h \in \mathcal{H}$ and $h_\alpha \leq h$ for all $\alpha \in \Delta$, thus h is an upper bound of $\mathcal{I}$. By the Zorn's lemma, $\mathcal{H}$ has a maximal element f. Let us show that $E \subset G_f$. Assume that, there is $x_0 \in E$ and $x_0 \notin G_f$, by the stape 1, it exists a function h from $G_f + \mathbb{R}x_0$ into $\mathbb{R}$ such that $h = f$ on G_f, $h \leq p$ on $G_f + \mathbb{R}x_0$. Therefore $h \in \mathcal{H}$ and $f \leq h$, because f is maximal, then $f = h$ on $G_f + \mathbb{R}x_0$, contradiction with $h = f + t\alpha$ on $G_f + \mathbb{R}x_0$ for any $t \in \mathbb{R}^*$.

As a first consequence of the theorem 17.1, we will state the topological form of Hahn Banach theorems. For all $x \in E$ and for all $f \in E^*$, $< f, x >_{E^*,E}$, denotes $f(x)$ and it is said to be the inner product in the duality E^*, E.

Corollary 17.1 (Topological form of Hahn Banach theorem). Let G be a proper linear subspace of E. For any $g \in G^*$, it exists $f \in E^*$such that $f = g$ on G and $\|f\|_{E^*} = \|g\|_{G^*}$.

Proof. As $g \in G^*$, by the lemma 16.4 $d)$ $\left|< g, x >_{G^*,G}\right| \leq \|g\|_{G^*}\|x\|_G = p(x)$ for all $x \in G$. Clearly the function p satisfies (1) and (2). By the theorem 17.1, there is a linear function f from E into $\mathbb{R}$ such that $f(x) \leq p(x) = \|g\|_{G^*}\|x\|_E$ for all $x \in E$, then $f(-x) \leq \|g\|_{G^*}\|-x\|_E$ so $-\|g\|_{G^*}\|x\|_E \leq f(x) \leq \|g\|_{G^*}\|x\|_E$ for all $x \in E$, hence $|f(x)| \leq \|g\|_{G^*}\|x\|_E$ for all $x \in E$ i.e. f is bounded on E and $\|f\|_{E^*} \leq \|g\|_{G^*}$ (7), by lemma 16.4 $d)$, here $\|g\|_{G^*} = \sup_{\{x \in G, \|x\|_G \leq 1\}} < g, x >_{G^*,G}$. Therefore, f is continuous on E by the theorem 16.2 $d)$, so $f \in E^*$. In the other hand, $|f(x)| \leq \|f\|_{E^*}\|x\|_E$ for all $x \in E$, then $|f(x)| \leq \|f\|_{E^*}\|x\|_G$ for all $x \in G$, so $\|g\|_{G^*} \leq \|f\|_{E^*}$ (8) by the lemma 16.4 $d)$. From (7) and (8) we have $\|g\|_{G^*} = \|f\|_{E^*}$.

Corollary 17.2. For any nonzero $x \in E$, it exists $f \in E^*$ such that $< f, x >_{E^*,E} = \|x\|_E^2$ and $\|f\|_{E^*} = \|x\|_E$.

Proof. Let g be the function from $G = \mathbb{R}x$ into $\mathbb{R}$, defined by: for all $t \in \mathbb{R}$; $g(tx) = t\|x\|_E^2$,. Because, for any $\lambda, t, s \in \mathbb{R}$,
$g[\lambda(tx) + sx] = g[(\lambda t + s)x] = (\lambda t + s)\|x\|_E^2 = \lambda(t\|x\|_E^2) + s\|x\|_E^2 = \lambda g(tx) + g(sx)$
then, g is linear. As for any $t \in \mathbb{R}$, $|g(tx)| = |t|\|x\|_E^2 = \|x\|_E\|tx\|_G = k\|tx\|_G$ for all $t \in \mathbb{R}$, where $k = \|x\|_E > 0$. Then g is bounded on G, therefore $g \in G^*$. By the corollary 17.1, it exists $f \in E^*$ satisfying: $f(tx) = g(tx)$ for any $t \in \mathbb{R}$. Hence, for all $t \in \mathbb{R}$, $tf(x) = g(tx) = t\|x\|_E^2$. Then, for the nonzero t, $< f, x >_{E^*,E} = \|x\|_E^2$, thus
$\|f\|_{E^*} = \sup_{\{x \in E, x \neq 0\}} \frac{< f, x >_{E^*,E}}{\|x\|_E} = \|x\|_E$.

$\tilde{B}_{E^*}(0,1)$ denotes the closed unit ball in E^*.

Corollary 17.3. For all $x \in E$, $\|x\|_E = \max_{f \in \tilde{B}_{E^*}(0,1)}\left|< f, x >_{E^*,E}\right|$.

Proof. As, for all $x \in E$ and for all $f \in E^*$, $\left|< f, x >_{E^*,E}\right| \leq \|f\|_{E^*}\|x\|_E$, then $\sup_{f \in \tilde{B}_{E^*}(0,1)}\left|< f, x >_{E^*,E}\right| \leq \|x\|_E$ for all $x \in E$. By the corollary 17.2, for nonzero $x \in E$, it exists $f \in E^*$ such that $< f, x >_{E^*,E} = \|x\|_E^2$ and $\|f\|_{E^*} = \|x\|_E$. Setting $h = \frac{f}{\|x\|_E}$, then $h \in E^*$, $\|h\|_{E^*} = 1$ and $\|x\|_E = < h, x >_{E^*,E}$. So $h \in \tilde{B}_{E^*}(0,1)$ and $\|x\|_E \leq \sup_{f \in \tilde{B}_{E^*}(0,1)}\left|< f, x >_{E^*,E}\right|$ for all nonzero $x \in E$, thus $\|x\|_E = \max_{f \in \tilde{B}_{E^*}(0,1)}\left|< f, x >_{E^*,E}\right|$ for all $x \in E$.

In order to give the geometric forms of Hahn Banach's theorems, or convex separation theorems. We need some simple properties of convex sets. Recall that the set C in the $\mathbb{R}$-vs E is said to be **convex** if $tx + (1 - t)y \in C$, for all $t \in [0,1]$ and for all $x, y \in C$. By convention $\emptyset$ is convex.

Example 17. 1: It is easy to verify that:

$a)$ The singletons, the balls and the $\mathbb{R}$-vector spaces of E are convex.

$b)$ The any intersection of the convex sets is convex.

$c)$ If $\{C_n\}$ is an increasing sequence of convex sets then $\cup_{n \in \mathbb{N}} C_n$ is convex.

$d)$ If C and C' are convex, then $C + C'$ and for all $\lambda \in \mathbb{R}$, λC are convex.

$e)$ If C is convex then $cl(C)$ is convex and $C + C = 2C$.

$f)$ If f is a linear map from the $\mathbb{R}$-vs E into the $\mathbb{R}$-vs F and C is a convex in E then $f(C)$ is a convex in F.

Definition 17.1. The hyperplane H of the equation $[f = b]$ is said to be:

$a)$ Separates the sets A and B, if $f(x) \leq b$ for all $x \in A$ and $b \leq f(x)$ for all $x \in B$.

b) Strictly Separates the sets A and B, if it exists $\delta > 0$ such that $f(x) \leq b - \delta$ for all $x \in A$ and $b + \delta \leq f(x)$ for all $x \in B$.

Definition 17.2. The Minkowski function of the subset A of E, is the function p_A from E into $\mathbb{R}_+^* \cup \{+\infty\}$ defined by for all $x \in E$, $p_A(x) = inf\{\alpha > 0,\ \alpha^{-1}x \in A\}$. By convention $inf\emptyset = +\infty$.

Lemma 17.1. The Minkowski function p_C, of the open convex subset $C$of E containing 0, satisfies:

a) It exists $M > 0$ such that, $0 \leq p_C(x) \leq M\|x\|_E$ for all $x \in E$.

b) $C = \{x \in E, p_C(x) < 1\}$.

c) For all $\lambda \in \mathbb{R}_+^*$ and for all $x, y \in E$, $p_C(\lambda x)$=$\lambda p_C(x)$ and $p_C(x + y) \leq p_C(x) + p_C(y)$.

Proof. a) As 0 is the lower bound of the set $\{\alpha > 0, \alpha^{-1}x \in C\}$ for all $x \in E$, then $0 \leq p_C(x)$ for all $x \in E$. Because $0 \in C$ and C is open, it exists $r > 0$ such that $\tilde{B}\left(0, \frac{r}{2}\right) \subset B(0, r) \subset C$. Let $\rho = \frac{r}{2}$ be, as, for all nonzero $x \in E$, $\frac{x}{\|x\|_E}\rho \in \tilde{B}(0, \rho)$ and $\frac{\rho}{\|x\|_E}x = \left(\frac{\|x\|_E}{\rho}\right)^{-1} x \in C$, then $0 \leq p_C(x) \leq \frac{\|x\|_E}{\rho} = M\|x\|_E$ for all $x \in E$, where $M = \frac{1}{\rho}$. b) Let $x \in C$ be, it exists $r > 0$ such that $x + \rho\tilde{B}(0,1) = \tilde{B}(x, \rho) \subset B(x, r) \subset C$, where $\rho = \frac{r}{2}$. Then, for all $z \in \tilde{B}(0,1)$, $x + \rho z \in C$, hence for z=$\frac{x}{\|x\|_E} \in \tilde{B}(0,1)$, $\left(1 + \frac{\rho}{\|x\|_E}\right)x \in C$ it follows that $p_C(x) \leq \frac{1}{1+\frac{\rho}{\|x\|_E}} < 1$ for all nonzero $x \in C$. If now $x \in E$ satisfies $p_C(x) < 1$, there is $\alpha \in \mathbb{R}_+^*$ betwin $p_C(x)$ and 1, thus $\alpha^{-1}x \in C$, if not $\alpha \leq p_C(x) < \alpha < 1$ contradiction, hence $\alpha(\alpha^{-1}x) + (1 - \alpha)0 = x \in C$, by the convexity of C and $0 \in C$. Therefore, $C = \{x \in E, p_C(x) < 1\}$. c) Let $\lambda > 0$ be and $x \in E$, then $\lambda p_C(x) = \lambda inf\{\alpha > 0,\ \alpha^{-1}x \in C\} = inf\{\lambda\alpha > 0,\ (\lambda\alpha)^{-1}(\lambda x) \in C\}$=$p_C(\lambda x)$. In the other hand, for any $\varepsilon > 0$ and any $x, y \in E$, $p_C\left(\frac{2x}{2p_C(x)+\varepsilon}\right)$=$\frac{2p_C(x)}{2p_C(x)+\varepsilon} < 1$ and $p_C\left(\frac{2y}{2p_C(y)+\varepsilon}\right)$=$\frac{2p_C(y)}{2p_C(y)+\varepsilon} < 1$, so $\frac{2x}{2p_C(x)+\varepsilon}, \frac{2y}{2p_C(y)+\varepsilon} \in C$. But $0 < t = \frac{2p_C(x)+\varepsilon}{2p_C(x)+2p_C(y)+2\varepsilon} < 1$, then $t\frac{x}{p_C(x)+\frac{\varepsilon}{2}} + (1 - t)\frac{y}{p_C(y)+\frac{\varepsilon}{2}}$=$\frac{x+y}{p_C(x)+p_C(y)+\varepsilon} \in C$, hence $p_C\left(\frac{x+y}{p_C(x)+p_C(y)+\varepsilon}\right) = \frac{1}{p_C(x)+p_C(y)+\varepsilon}p_C(x + y) < 1$, thus $p_C(x + y) < p_C(x) + p_C(y) + \varepsilon$, and when $\varepsilon \longrightarrow 0$, $p_C(x + y) \leq p_C(x) + p_C(y)$, for all $x, y \in E$.

Lemma 17.2. If, C is a nonempty open convex subset, of the $\mathbb{R}$-nvs E and $x_0 \in C^C$. Then, it exists $f \in E^*$ such that $f(x) < f(x_0)$ for all $x \in C$, i.e. the hyperplane H of the equation $[f = f(x_0)]$ strictly separates the two convex C and $\{x_0\}$.

Proof. We assume that $0 \in C$, if not there is $x \in C$ such that $0 \in -x + C$ which is convex. The linear function g from $G = \mathbb{R}x_0$ into $\mathbb{R}$, defined by $g(tx_0) = t$ for all $t \in \mathbb{R}$ satisfies: for $t > 0$ $x_0 \notin C$, then $p_C(x_0) \geq 1$ it follows that $\frac{t}{t} \leq p_C(x_0)$ or $g(tx_0) \leq tp_C(x_0) = p_C(tx_0)$. For $g(tx_0) = t \leq 0 \leq p_C(tx_0)$. From the theorem 17.1, it exists a linear function f from E into $\mathbb{R}$ such that $f = g$ on G in particular $g(x_0) = f(x_0) = 1$ and $f(x) \leq p_C(x) \leq M\|x\|_E$ for all $x \in E$. Hence $f \in E^*$ and $f(x) < 1 = f(x_0)$.

Before giving the first geometric form of Hahn Banach's theorems, which is the generalization of the lemma 17.2. Note that if O is an open in E and $a \in E$, then for all $\lambda \in \mathbb{R}_+^*$, $a + \lambda O$ is open. In fact, if $x \in a + \lambda O$ then $\frac{x-a}{\lambda} \in O$ so, it exists $r > 0$ such that $B\left(\frac{x-a}{\lambda}, r\right) = \frac{x-a}{\lambda} + B(0, r) \subset O$ equivalently $x - a + \lambda B(0, r) \subset \lambda O$ or $x + \lambda B(0, r) \subset a + \lambda O$, hence $B(x, \rho) \subset a + \lambda O$ where $\rho = \lambda r > 0$, hence $a + \lambda O$ is open.

Theorem 17.2 (First geometric form of Hahn Banach's theorem). Let A and B are two nonempty convex subsets of E. If A is open and $A \cap B = \emptyset$. Then, it exists a closed hyperplane H of equation $[f = b]$ which separates A and B.
Proof. The set $C = A - B = \cup_{y\in B} (A - y)$ is an open convex subset of E, with nonzero element. If not, it exist $a \in A$ and $b \in B$ such that, $0 = a - b$ then $a = b$ contradiction with $A \cap B = \emptyset$. By the lemma 17.2, it exists $f \in E^*$ such that $f(z) < f(0) = 0$ for all $z \in C$ i.e. the hyperplane H of the equation $[f = 0]$ strictly separates C and $\{0\}$. Hence $f(x - y) < 0$ for all $x \in A$ and for all $y \in B$ equivalently $f(x) < f(y)$ for all $x \in A$ and for all $y \in B$. For a fixed y in B the set $\{ f(x), x \in A\}$ is bounded above and for a fixed x in A the set $\{ f(y), y \in B$ is bounded below, hence, it exists $b\in\mathbb{R}$ such that $\sup_{x\in A} f(x) \le b \le \inf_{y\in B} f(y)$. By the lemma 16 3, the closed hyperplane H of equation $[f = b]$ separates A and B.

We will now, state and demonstrate, the second geometric form of Hahn Banach's theorems.
Theorem 17.3 (Second geometric form of Hahn Banach's theorem). Let A and B two nonempty convex subset of E, where A is closed, B is compact and $A \cap B = \emptyset$. Then, it exists a closed hyperplane H of equation $[f = b]$, which strictly separates A and B.
Proof. Setting for a fixed $n_0 \in \mathbb{N}^*$, $A_{n_0} = A + \frac{1}{2n_0}B(0,1) = \cup_{x\in A}\left(x + \frac{1}{2n_0}B(0,1)\right)$ and $B_{n_0} = B + \frac{1}{2n_0}B(0,1)$. Then, A_{n_0} is an open convex and B_{n_0} is convex. Moreover $A_{n_0} \cap B_{n_0} = \emptyset$. Indeed if, for all n$\in \mathbb{N}^*$, $A_n \cap B_n \neq \emptyset$, there are $a_n \in A$, $b_n \in B$ and there are $s, t \in B(0,1)$ such that $a_n + \frac{1}{2n}t = b_n + \frac{1}{2n}s$, so $a_n - b_n = \frac{1}{2n}(s - t)$ hence $0 \le \|a_n - b_n\| = \frac{1}{2n}\|s - t\| \le \frac{1}{2n}(\|s\| + \|t\|) < \frac{1}{n}$. When $n \longrightarrow \infty$, $a_n - b_n \longrightarrow 0$, as $a_n - b_n \in A - B$ and $A - B$ is closed by the lemma 16.1, then $0 \in A - B$ which implies that $A \cap B \neq \emptyset$, contradiction. By the theorem 17.2, it exists a closed hyperplane H of equation $[f = b]$ which separates $A_{n_0}$and B_{n_0} i.e. $f(x + t) \le b$ for all $x \in A$, and for all $t \in B(0, \frac{1}{2n_0})$, and $b \le f(y + t)$, for all $y \in B$ and for all $t \in B\left(0, \frac{1}{2n_0}\right)$. Thus, $f(x + t) \le b$ for all $x \in A$, and for all $t \in \tilde{B}(0, \frac{1}{3n_0})$, and $b \le f(y + t)$, for all $y \in B$ and for all $t \in \tilde{B}\left(0, \frac{1}{3n_0}\right)$. Therefore, $f\left(x + \frac{1}{3n_0}z\right) = f(x) + \frac{1}{3n_0}f(z) \le b$ for all $x \in A$, and for all $z \in \tilde{B}(0,1)$, and $b \le f\left(y + \frac{1}{3n_0}z\right) = f(y) + \frac{1}{3n_0}f(z)$, for all $y \in B$ and for all $z \in \tilde{B}(0,1)$. By the lemma 16 4 c) $f(x) + \frac{1}{3n_0}\|f\|_{E^*} \le b$ for all $x \in A$, hence $f(x) \le b - \delta$ for all $x \in A$ where $\delta = \frac{1}{3n_0}\|f\|_{E^*} > 0$ $(f \not\equiv 0)$, and $b \le f(y) + \frac{1}{3n_0}f(-z) = f(y) - \frac{1}{3n_0}f(z)$ for all $y \in B$ and for all $z \in \tilde{B}(0,1)$ or $b + \frac{1}{3n_0}f(z) \le f(y)$, for all $y \in B$ and for all $z \in \tilde{B}(0,1)$. Thus, $b + \delta \le f(y)$ for all $y \in B$. Therefore, a closed hyperplane H of equation $[f = b]$ strictly separates A and B.
Remark 17.1.
a) We obtain the theorem 17.2, if we assume B is open instead of A is open.
b) In the finite dimension space, we obtain the theorem 17.2, even if A is not open.
Corollary 17.4. Let F be a linear subspace of a $\mathbb{R}$-nvs E. If, $f(x) = 0$ on F for any $f \in E^*$, implies $f(x) = 0$ on E. Then $cl(F) = E$. Equivalently, if $cl(F) \neq E$, it exists $f \in E^*$ $(f \neq 0)$ such that $f(x) = 0$ on F.
Proof. Let $cl(F) = G$ be, by corollary 16.1 G is a linear subspace of a $\mathbb{R}$-nvs E. Assume that $G \neq E$ and let $x_0 \in G^C$ be. It is clear that, $A = \{x_0\}$ and $G = B$ satisfy the assumptions of the theorem 17 3. Then, there are $b \in \mathbb{R}$ and $f \in E^*$ $(f \neq 0)$ such that, $f(x) < b < f(x_0)$ for all

$x \in cl(F)$. Hence $f(x) < b < f(x_0)$ for all $x \in F$, as $0 \in F$ then $f(0) = 0 < b$ and as $-x \in F$ then $-f(x) < b$ hence $-b < f(x) < b$ or $0 \leq |f(x)| < b$ for all $x \in F$. As for all $n \in \mathbb{N}^*$ and for all $x \in F$, $nx \in F$ then, $0 \leq |f(nx)| < b$ for all $x \in F$ and for all $n \in \mathbb{N}^*$. So $0 \leq |f(x)| < \frac{1}{n} b$ for all $x \in F$ and for all $n \in \mathbb{N}^*$. When $n \longrightarrow \infty$, we have $|f(x)| = 0 \Leftrightarrow f(x) = 0$ for all $x \in F$.

17.2-Banach-Stienhaus theorem, open map theorem, the closed graph theorem

Let $(E, \|\ \ \|_E)$ and $(F, \|\ \ \|_F)$ are two $\mathbb{K}$-nvs and $\|T\|_{L(E,F)} = \sup_{x \in \tilde{B}(0,1)} \|T(x)\|_F$ the norm of any $T \in L(E, F)$. Another fundamental theorem, of functional analysis is the Banach–Steinhaus theorem, which is known as the uniform boundedness principle. It is based on the Baire lemma 14.6. Let $\{T_\alpha, \alpha \in \Delta\}$ be a collection of the elements of $L(E, F)$. We write: $\sup_{\alpha \in \Delta} \|T_\alpha(x)\|_F < +\infty$ for all $x \in E$ (**pointwice boundedness or strong boundedness**), if it exists $K > 0$ such that $\|T_\alpha(x)\|_F \leq K$, for all $\alpha \in \Delta$ and for all $x \in E$, and we write $\sup_{\alpha \in \Delta} \|T_\alpha\|_{L(E,F)} < +\infty$ (**uniform boundedness**), if it exists $M > 0$ such that $\|T_\alpha\|_{L(E,F)} \leq M$, for all $\alpha \in \Delta$.

Theorem 17.4. (Banach-Steinhaus theorem). If E is a Banach space, F is a normed space and $\{T_\alpha, \alpha \in \Delta\}$ is a collection of the elements of $L(E, F)$ such that: $\sup_{\alpha \in \Delta} \|T_\alpha(x)\|_F < +\infty$ for all $x \in E$. Then, $\sup_{\alpha \in \Delta} \|T_\alpha\|_{L(E,F)} < +\infty$.

Proof. Let for all $\alpha \in \Delta$, $F_n = \{x \in E, \|T_\alpha(x)\|_F \leq n\} = (\|\ \ \|_F \circ T_\alpha)^{-1}(]-\infty, n])$ be, where $n \in \mathbb{N}^*$, the sequence of closed subsets of E. By assumption, it exists $M > 0$ such that $\|T_\alpha(x)\|_F \leq M$ for all $\alpha \in \Delta$ and for all $x \in E$, thus there is $m \in \mathbb{N}^*$ such that $\|T_\alpha(x)\|_F \leq m$ for all $\alpha \in \Delta$ and for all $x \in E$ by Archimedean axiom, then $x \in F_m$ and $E = \cup_{n \in \mathbb{N}^*} F_n$. Using Baire's lemma 14.6, it exists $n_0 \in \mathbb{N}^*$ such that $int(F_{n_0}) \neq \emptyset$. Therefore, for $x_0 \in int(F_{n_0})$, it exists $r > 0$ such that $\tilde{B}(x_0, r) \subset int(F_{n_0}) \subset F_{n_0}$, hence $\|T_\alpha(x_0 + rz)\|_F \leq n_0$ and $\|T_\alpha(x_0)\|_F \leq n_0$ for all $\alpha \in \Delta$ and for all $z \in \tilde{B}(0,1)$. Because $\|T_\alpha(rz)\|_F = r\|T_\alpha(z)\|_F = \|T_\alpha(x_0 + rz - x_0)\|_F = \|T_\alpha(x_0 + rz) - T_\alpha(x_0)\|_F \leq \|T_\alpha(x_0 + rz)\|_F + \|T_\alpha(x_0)\|_F \leq 2n_0$ for all $\alpha \in \Delta$ and for all $z \in \tilde{B}(0,1)$, then $\|T_\alpha(z)\|_F \leq \frac{2n_0}{r}$ for all $\alpha \in \Delta$ and for all $z \in \tilde{B}(0,1)$, hence $\|T_\alpha\|_{L(E,F)} \leq \frac{2n_0}{r}$ for all $\alpha \in \Delta$. So, it exists $K = \frac{2n_0}{r} > 0$ such that, $\sup_{\alpha \in \Delta} \|T_\alpha\|_{L(E,F)} \leq K$.

As a direct consequence of the theorem 17.4, we have:

Corollary 17.5. Let E and F are Banach spaces, and let $\{T_n\}$ be a sequence in $L(E, F)$. If for any $x \in E$, the sequence $\{T_n(x)\}$ converges to the limit $y = T(x) \in F$. Then:

$i)$ $\sup_{n \in \mathbb{N}^*} \|T_n\|_{L(E,F)} < +\infty$.

$ii)$ $T \in L(E, F)$.

$iii)$ $\|T\|_{L(E,F)} \leq \liminf_{n \to +\infty} \|T_n\|_{L(E,F)}$.

Proof. $i)$ As the sequence $\{T_n(x)\}$ converges to the limit $y = T(x) \in F$, it is bounded. So, it exists $M > 0$ such that $\|T_n(x)\|_F \leq M$ for all $n \in \mathbb{N}^*$ and for all $x \in E$, then $\sup_{n \in \mathbb{N}^*} \|T_n(x)\|_F \leq M$ for all $x \in E$. From the theorem 17.4, it exists $K > 0$ such that, $\sup_{n \in \mathbb{N}^*} \|T_n\|_{L(E,F)} \leq K$. $ii)$ Because, for all $x, y \in E$ and for all $\lambda \in \mathbb{R}$, $T_n(\lambda x + y) = \lambda T_n(x) + T_n(y) \longrightarrow T(\lambda x + y) = \lambda T(x) + T(y)$ and it exists $M > 0$ such that $\|T_n(x)\|_F \leq M\|x\|_E$, for all $x \in E$ and for all $n \in \mathbb{N}^*$, hence $\lim_{n \to +\infty} \|T_n(x)\|_F = \|T(x)\|_F \leq M\|x\|_E$ for all $x \in E$, so $T \in L(E, F)$. $iii)$ As $\|T_n\|_{L(E,F)} \leq K$ for all $n \in \mathbb{N}^*$, from the Weierstrass-Bolzano theorem, $\liminf_{n \to +\infty} \|T_n\|_{L(E,F)}$ exists. Because $\|T_n(x)\|_F \leq \|T_n\|_{L(E,F)}$ for all $x \in \tilde{B}(0,1)$ and for all $n \in \mathbb{N}^*$, so $\lim_{n \to +\infty} \|T_n(x)\|_F = \|T(x)\|_F \leq \liminf_{n \to +\infty} \|T_n\|_{L(E,F)}$ for all $x \in \tilde{B}(0,1)$, thus $\|T\|_{L(E,F)} \leq \liminf_{n \to +\infty} \|T_n\|_{L(E,F)}$.

Corollary 17.6. If G is a Banach space and B is a subset of G such that for all $f \in G^*$, $f(B)$ is bounded. Then B is bounded.
Proof. Let $E = G^*$ and $F = \mathbb{R}$ are and let $\{T_b, b \in B\}$ be a collection of the elements of $L(E,F)$ defined by: for all $b \in B$ and for all $f \in E$, $T_b(f) = f(b)$. As for all $f \in E$, $f(B)$ is bounded, then $\sup_{b\in B}|f(b)|<+\infty$ for all $f \in E$, equivalently $\sup_{b\in B}|T_b(f)|<+\infty$ for all $f \in E$, hence $\sup_{b\in B}\|T_b\|_{L(E,F)}<+\infty$ by the theorem 17.4. It exists $K > 0$ such that $\|T_b\|_{L(E,F)} \le K$ for all $b \in B$. Therefore, $|T_b(f)| \le K$ for all $b \in B$ and for all $f \in \tilde{B}_E(0,1)$. Then $\sup_{f\in\tilde{B}_E(0,1)}|T_b(f)|=\sup_{f\in\tilde{B}_E(0,1)}|f(b)| \le K$, for all $b \in B$. Because, $\sup_{f\in\tilde{B}_E(0,1)}|f(b)| = \|b\|_G$ for all $b \in B$ by the corollary 17.3, hence B is bounded.

Another fundamental theorem, of functional analysis, is the open mapping theorem, also known as **Banach-Schauder theorem**, whose the proof is a direct consequence of the following two lemmas:
Lemma 17.3. If F is a Banach space and if, the map T from E into F is surjective and linear. Then, there is $r > 0$ such that $B_F(0,2r) \subset cl\left(T\big(B_E(0,1)\big)\right)$.
Proof. Setting $F_n = ncl\left(T\big(B_E(0,1)\big)\right)$, for all $n \in \mathbb{N}^*$. It is clear that, the elements of the sequence $\{F_n\}$ are closed in F. Because $\forall x \in E$, it exists $n_0 \in \mathbb{N}^*$ such that $\|x\|_E < n_0$ by Archimedean axiom, i.e. $x \in n_0 B_E(0,1)$, then $E = \bigcup_{n\in\mathbb{N}^*} nB_E(0,1)$ and as T is surjective and linear $T(E) = F = \bigcup_{n\in\mathbb{N}^*} nT\big(B_E(0,1)\big) \subset \bigcup_{n\in\mathbb{N}^*} ncl\left(T\big(B_E(0,1)\big)\right)=\bigcup_{n\in\mathbb{N}^*} F_n$, hence $F = \bigcup_{n\in\mathbb{N}^*} F_n$. As F is a Banach space, by the Baire's lemma 14.6, it exists $n_0 \in \mathbb{N}^*$ such that $int(F_{n_0}) \neq \emptyset$, thus $\frac{1}{4}int\left(cl\left(T\big(B_E(0,1)\big)\right)\right) \neq \emptyset$. Let $y_0 \in \frac{1}{4}int\left(cl\left(T\big(B_E(0,1)\big)\right)\right)$ be, there is $r > 0$ such that $B_F(y_0, 4r) \subset int\left(cl\left(T\big(B_E(0,1)\big)\right)\right) \subset cl\left(T\big(B_E(0,1)\big)\right)$, so $y_0 \in cl\left(T\big(B_E(0,1)\big)\right)$, it exists a sequence $\{x_n\}$ in $B_E(0,1)$ such that $T(x_n) \longrightarrow y_0$. Because the sequence $\{-x_n\}$ is in $B_E(0,1)$ then $T(-x_n) = -T(x_n) \longrightarrow -y_0 \in cl\left(T\big(B_E(0,1)\big)\right)$. By the example17.1), d), e) and f), $cl\left(T\big(B_E(0,1)\big)\right)$ is convex, $-y_0 + B_F(y_0, 4r) = 2B_F(0,2r) \subset cl\left(T\big(B_E(0,1)\big)\right) + cl\left(T\big(B_E(0,1)\big)\right)=2\,cl\left(T\big(B_E(0,1)\big)\right)$, hence $B_F(0,2r) \subset cl\left(T\big(B_E(0,1)\big)\right)$.
Lemma 17.4. If E and F are Banach spaces, and if the map $T \in L(E,F)$ is surjective. Then, there is $r > 0$ such that $B_F(0,r) \subset T\big(B_E(0,1)\big)$.
Proof. As by the lemma 17.3, it exists $r > 0$ such that $B_F(0,2r) \subset cl\left(T\big(B_E(0,1)\big)\right)$, then $B_F(0,r) \subset cl\left(T\left(B_E\left(0,\frac{1}{2}\right)\right)\right)$. By the proposition 13.4 a) for any $y \in B_F(0,r)$ and for $\frac{1}{2}r > 0$, it exists $z_1 \in E$ with $\|z_1\|_E < \frac{1}{2}$ such that $\|y - T(z_1)\|_F < \frac{1}{2}r$, then $y - T(z_1) \in \frac{1}{2}B_F(0,r) \subset cl\left(T\left(B_E\left(0,\frac{1}{2^2}\right)\right)\right)$. Hence, for $\frac{1}{2^2}r > 0$, it exists $z_2 \in E$ with $\|z_2\|_E < \frac{1}{2^2}$ such that $\|y - T(z_1 + z_2)\|_F < \frac{1}{2^2}r$. By iteration up to order n, for $\frac{1}{2^n}r > 0$, it exists $z_n \in E$, with $\|z_n\|_E < \frac{1}{2^n}$ such that $\|y - T(z_1 + z_2 + \cdots + z_n)\|_F < \frac{1}{2^n}r$, for all $n \in \mathbb{N}^*$. Setting $x_n = z_1 + z_2 + \cdots + z_n$, for all $n \in \mathbb{N}^*$ because $\|x_{n+1} - x_n\|_E=\|z_{n+1}\|_E < \frac{1}{2^{n+1}}$ for all $n \in \mathbb{N}^*$then $0 \le \|x_n - x_m\|_E \le \|x_n - x_{n+1}\|_E + \|x_{n+1} - x_{n+2}\|_E + \cdots + \|x_{m-1} - x_m\|_E < \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}+\ldots+\frac{1}{2^{n+(m-n)}}=\frac{1}{2^n}\left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m}\right] = \frac{1}{2^m}\left[1 - \frac{1}{2^{(m-n)}}\right]<\frac{1}{2^n}$ for all $n, m \in \mathbb{N}^*(m > n)$, so the sequence $\{x_n\}$ is a Cauchy in the Banach E, hence it converges to the series $x =$

$\sum_{n\geq 1} z_n \in E$. By the continuity of T and the uniqueness of the limit $T(x_n) \longrightarrow T(x) = y$. Because $\|x\|_E$=$\|\sum_{n\geq 1} z_n\|_E \leq \sum_{n\geq 1}\|z_n\|_E < \sum_{n\geq 1}\frac{1}{2^n} = 1$, it follows that $x \in B_E(0,1)$, therefore $T(x) = y \in T\big(B_E(0,1)\big)$.

Theorem 17.5 (open map theorem). If E and F are Banach spaces and, if the map $T \in L(E,F)$ is surjective. Then T is open.

Proof. Let O be an open in E and $y \in T(O)$, it exists $x \in O$ such that $y = T(x)$. Thus, it exists $\rho > 0$ where $B_E(x,\rho) = x + \rho B_E(0,1) \subset O$, then $T\big(B_E(x,\rho)\big) = y + \rho T\big(B_E(0,1)\big) \subset T(O)$. From the lemma 17.4, it exists $r > 0$ such that $B_F(0,r) \subset T\big(B_E(0,1)\big)$, then $B_F(0,\rho r) \subset \rho T\big(B_E(0,1)\big)$, hence $B_F(y,\rho r) \subset T(O)$ and $T(O)$ is open. Thus T is open.

Corollary 17.6 (the inverse bounded theorem)**.** If E and F are Banach spaces and if, the map $T \in L(E,F)$ is bijective. Then, the inverse map $T^{-1} \in L(F,E)$.

Proof. By the lemma 17.4, it exists $r > 0$ such that $B_F(0,r) \subset T\big(B_E(0,1)\big)$, then $T^{-1}\big(B_F(0,r)\big) \subset B_E(0,1)$, hence for all $x \in E$ satisfying $\|T(x)\|_F < r$, we have $\|x\|_E$<1. As, for any nonzero $x \in E$ $\left\|T\left(\frac{x}{\|T(x)\|_F}\frac{r}{2}\right)\right\|_F = \frac{r}{2}$<r, then $\left\|\frac{x}{\|T(x)\|_F}\frac{r}{2}\right\|_E$<1, for any nonzero $x \in E$. Therefore, $\|x\|_E \leq \frac{2}{r}\|T(x)\|_F$ for all $x \in E$. As T^{-1} is obviously linear and by assumption T is bijective, for any $y \in F$, there is a unique $x \in E$ such that, y = T(x) $\Leftrightarrow$ $T^{-1}(y)$ = x, then $\|T^{-1}(y)\|_E \leq \frac{2}{r}\|y\|_F$ for all $y \in F$, ultimately $T^{-1} \in L(E,F)$.

Corollary 17.7. If $(E,\| \quad \|_1)$ and $(E,\| \quad \|_2)$.are two Banach space and if, it exists $\alpha > 0$ such that $\|x\|_2 \leq \alpha\|x\|_1$ for all $x \in E$. Then $\| \quad \|_1$ and $\| \quad \|_2$ are equivalent.

Proof. Consider the identity map id from $E = (E,\| \quad \|_1)$ into $F = (E,\| \quad \|_2)$. It is clear that id satisfies the conditions of the corollary 17.6. Then, $id^{-1} \in L(F,E)$, it exists $\beta > 0$ such that $\|x\|_1 \leq \beta\|x\|_2$ for all $x \in E$, so $\frac{1}{\alpha}\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2$ for all $x \in E$, hence $\| \quad \|_1$ and $\| \quad \|_2$ are equivalent. G_T=$\{(x,y) \in E \times F;\ y = T(x)\}$, denotes the graph of the map $T: E \longrightarrow F$.

Theorem 17.6 (the closed graph theorem). Let E, F are Banach spaces, and the $T: E \longrightarrow F$ a linear map. Then, T is continuous iffy the graph of T is closed in the Banach $E \times F$.

Proof. Let $(x,y) \in cl(G_T)$ be, it exists (x_n,y_n) in G_T which converges to $(x,y) \in E \times F$, as $x_n \longrightarrow x$ and T is continuous $T(x_n) = y_n \longrightarrow T(x)$, but $y_n \longrightarrow y$ and the limit is unique in F, then $T(x) = y$, hence $(x,y) \in G_T$ and G_T is closed. Conversely, define the two norms $\| \quad \|_1$ and $\| \quad \|_2$ on E by for all $x \in E$, $\|x\|_2 = \|x\|_E$ and $\|x\|_1 = \|x\|_2 + \|T(x)\|_F$. Show that $(E,\|x\|_1)$ is a Banach. Let $\{x_n\}$ be a Cauchy in $(E,\|x\|_1)$ then $\{x_n\}$ is a Cauchy in $(E,\| \quad \|_E)$ and $\{T(x_n)\}$ is a Cauchy in $(F,\| \quad \|_F)$, there is $(x,y) \in E \times F$ such that $(x_n,T(x_n)) \longrightarrow (x,y)$. Because G_T is closed in the $E \times F$, then $(x,y) \in G_T$, hence $T(x) = y$. Since $\|x_n -$ $x1=xn-x2+Txn-xF=xn-x2+Txn-TxF$ when $n\longrightarrow+\infty$, $xn\longrightarrow x$ in E,$x1$, hence E,$x1$ is a Banach. In view of, $(E,\|x\|_1)$ and $(E,\|x\|_2)$ are banach and $\|x\|_2 \leq \|x\|_1$ for all $x \in E$. By the corollary 17.7, it exists $\alpha > 0$ such that $\|x\|_1 \leq \alpha\|x\|_2$ for all $x \in E$, therefore $\|T(x)\|_F \leq \alpha\|x\|_E$ for all $x \in E$, and T is continuous.

17.3-Weak topologie in the general case

In this section, we are given a set E, a collection of topological spaces $(F_\alpha)_{\alpha\in\Delta}$ and a collection of maps $(\varphi_\alpha)_{\alpha\in\Delta}$ such that each φ_α maps E into F_α . We wish to define a topology on E that makes all the φ_α's continuous. And that this topology is the least fine, that is: with a minimum of open sets. Obviously, all the $\varphi_\alpha^{-1}(U_\alpha)$, where U_α is an open set in F_α should be open in E. Then, finite intersections of those should also be open. And then any union of finite intersections should be open. By this process, we have created as few open sets as required. Denote by $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$ the collection of the sets of E of the form $\bigcup_{any}\left(\bigcap_{finite}\varphi_\alpha^{-1}(U_\alpha)\right)$. Then, $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$ is the desired topology. Indeed, it is clear that $\emptyset$, E and any union belong to $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$. It remains to check that the finite intersection is in $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$. Let O_1 and O_2 are in $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$, there exist two finite families, I in Δ and j in ∇ such that: $O_1=\bigcup_{\alpha\in\Delta}(\bigcap_{\alpha\in I}\varphi_\alpha^{-1}(U_\alpha))$ and $O_2=\bigcup_{\beta\in\nabla}\left(\bigcap_{\beta\in J}\varphi_\beta^{-1}(U_\beta)\right)$, where Δ and ∇ are a families of index, then $O_1\cap O_2$=$\bigcup_{(\alpha,\beta)\in\Delta\times\nabla}\left[\bigcap_{\alpha\in I}\varphi_\alpha^{-1}(U_\alpha)\cap\bigcap_{\beta\in J}\varphi_\beta^{-1}(U_\beta)\right]=\bigcup_{(\alpha,\beta)\in\Delta\times\nabla}\left[\bigcap_{(\alpha,\beta)\in I\times J}\left(\varphi_\alpha^{-1}(U_\alpha)\cap\varphi_\beta^{-1}(U_\beta)\right)\right]$. Assuming that, the family $\{\varphi_\alpha^{-1}(U_\alpha),\alpha\in\Delta\}$ is closed under finite intersections, then it containing $\varphi_\alpha^{-1}(U_\alpha)\cap\varphi_\beta^{-1}(U_\beta)$ i.e. it exists $\gamma\in\Delta$ such that $\varphi_\alpha^{-1}(U_\alpha)\cap\varphi_\beta^{-1}(U_\beta)=\varphi_\gamma^{-1}(U_\gamma)$ where, U_γ is an open in F_γ, so $O_1\cap O_2\in\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$. By induction, $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$ is closed under finite intersections. The topology $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$ is called **the weak topology** on E generated by the $(\varphi_\alpha)_{\alpha\in\Delta}$'s. By definition, the functions $(\varphi_\alpha)_{\alpha\in\Delta}$ are continuous for this topology, then the collection $\{\varphi_\alpha^{-1}(U_\alpha),\alpha\in\Delta\}$ is contained in $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$. It is easy to check that, a basis of neighborhoods of $x\in E$, for the weak topology is given by the collection of sets of the form $\bigcap_{\alpha\in I}\varphi_\alpha^{-1}(N_\alpha)$, where I is a finite subset of Δ and $N_\alpha\in\mathcal{N}\left(\varphi_\alpha(x)\right)$.

Proposition 17.1. Let $\{x_n\}$ be a sequence in E. Then, $\{x_n\}$ converges in the topology $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$ to some $x\in E$ iffy $\forall\alpha\in\Delta$, $\lim_{n\to\infty}\varphi_\alpha(x_n)=\varphi_\alpha(x)$.

Proof. As, $x_n\longrightarrow x$ in $\left(E,\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})\right)$, and $\forall\alpha\in\Delta$, $\varphi_\alpha:\left(E,\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})\right)\longrightarrow F_\alpha$ is continuous, then $\forall\alpha\in\Delta$, $\lim_{n\to\infty}\varphi_\alpha(x_n)=\varphi_\alpha(x)$. Conversely, let $N=\bigcap_{\alpha\in I}\varphi_\alpha^{-1}(N_\alpha)\in$ 𝒩𝑥 be, where 𝐼 is a finite subset of 𝛥 and 𝑁𝛼∈𝒩𝜑𝛼𝑥. As, for all 𝛼∈𝐼, 𝜑𝛼𝑥∈𝑁𝛼 and $\lim_{n\to\infty}\varphi_\alpha(x_n)=\varphi_\alpha(x)$, it exists $n_\alpha\in\mathbb{N}$, such that for all $n\in\mathbb{N}$, $n>n_\alpha$ implies $x_n\in$ 𝜑𝛼−1𝑁𝛼, so for 𝑚=max𝛼∈𝐼𝑛𝛼, and for all 𝑛>𝑚, 𝑥𝑛∈𝑁, it follows that lim𝑛⟶∞𝑥𝑛=𝑥 for $\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})$.

Proposition 17.2. Let (G,τ) be a topological space, then the map $\psi:(G,\tau)\longrightarrow\left(E,\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})\right)$ is continuous iffy for all $\alpha\in\Delta$, $\varphi_\alpha\circ\psi$ is continuous.

Proof. As $\forall\alpha\in\Delta$, $\varphi_\alpha:\left(E,\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})\right)\longrightarrow F_\alpha$ is continuous and $\psi:(G,\tau)\longrightarrow\left(E,\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})\right)$, is continuous then, for all $\alpha\in\Delta$, $\varphi_\alpha\circ\psi$ is continuous (the composition of two continuous functions is a continuous function). Reciprocally, demonstrate that $\psi:(G,\tau)\longrightarrow\left(E,\sigma(E,(\varphi_\alpha)_{\alpha\in\Delta})\right)$ is continuous. Let $N=\bigcap_{\alpha\in I}\varphi_\alpha^{-1}(N_\alpha)\in\mathcal{N}(x)$ be, where I is a finite subset of Δ and $N_\alpha\in\mathcal{N}\left(\varphi_\alpha(x)\right)$, as $\psi^{-1}(N)=\bigcap_{\alpha\in I}\psi^{-1}\left(\varphi_\alpha^{-1}(N_\alpha)\right)=$ 𝛼∈𝐼𝜑𝛼∘𝜓−1𝑁𝛼 and as for all 𝛼∈𝛥, 𝜑𝛼∘𝜓 is continuous, then 𝜑𝛼∘𝜓−1𝑁𝛼∈𝒩𝑥, for all 𝛼∈𝐼, therefore $\psi^{-1}(N)=\bigcap_{\alpha\in I}(\varphi_\alpha\circ\psi)^{-1}(N_\alpha)\in\mathcal{N}(x)$, so ψ is continuous.

17.4 The weak topology $\sigma(E,E^*)$ in the $\mathbb{R}$-nvs E

In the sequel, E is a $\mathbb{R}$-nvs, E^* it's dual, $\left(\varphi_f\right)_{f\in E^*}$ is a collection of functions from E into $\mathbb{R}$, defined by: $\varphi_f(x)=\langle f,x\rangle_{E^*,E}$ for all $x\in E$ and all $f\in E^*$.

Definition 17.3. The weak topology in the $\mathbb{R}$-nvs E, is the the topologie $\sigma\left(E,\left(\varphi_f\right)_{f\in E^*}\right)$ i.e. the least fine topology, which makes all the functions $\left(\varphi_f\right)_{f\in E^*}$ continuous. We will note it $\sigma(E,E^*)$.

Proposition 17.2. The topological space $\left(E,\sigma(E,E^*)\right)$ is Hausdorff.

Proof. Let $x_0,y_0\in E$ be with $x_0\neq y_0$. Apply Hahn Banach's theorem 17.3, for $A=\{x_0\}$ and $B=\{y_0\}$, it exists $f\in E^*$ and $b\in\mathbb{R}$ such that $\langle f,x_0\rangle_{E^*,E}<b<\langle f,y_0\rangle_{E^*,E}$. Because x_0 belongs to the weak open O_{x_0}=$\left\{x\in E,\langle f,x_0\rangle_{E^*,E}<b\right\}=\varphi_f^{-1}(]-\infty,b[)$, y_0 belongs to the

weak open O_{y_0}=$\{x \in E, \langle f, y_0 \rangle_{E^*,E} > b\} = \varphi_f^{-1}(]b, +\infty[)$ and $O_{x_0} \cap O_{y_0} = \emptyset$, then $(E, \sigma(E, E^*))$ is Hausdorff.

Proposition 17.3. Let $(E, \sigma(E, E^*))$ and $x_0 \in E$ are. The collection of the subsets B of E defined by: $x \in B$ iffy, it exist $\varepsilon > 0$ and n elements $\{f_1, \dots, f_i, \dots, f_n\}$ of E^*, such that $|\langle f_i, x - x_0 \rangle_{E^*,E}| < \varepsilon$, for all $i \in \{1, \dots, n\}$, is a basis of neighborhoods of x_0.

Proof. Let $N \in \mathcal{N}(x_0)$ be a weak neighborhoods of x_0, it exists a weak open set $O = \bigcap_{i\in\{1,\dots,n\}} \varphi_{f_i}^{-1}(U_i)$, such that $x_0 \in O \subset N$, where for all $i \in \{1, \dots, n\}$, $f_i \in E^*$ and U_i is an open in $\mathbb{R}$ containing $a_i = \langle f_i, x_0 \rangle$. Then, for all $i \in \{1, \dots, n\}$, it exists $\varepsilon_i > 0$, such that $]a_i - \varepsilon_i, a_i + \varepsilon_i[\subset U_i$. Thus, $\varphi_{f_i}^{-1}(]a_i - \varepsilon_i, a_i + \varepsilon_i[) \subset \varphi_{f_i}^{-1}(U_i)$ for all $i \in \{1, \dots, n\}$. Therefore for $\varepsilon = \min_{i\in\{1,\dots,n\}} \varepsilon_i$,
$x_0 \in B = \bigcap_{i\in\{1,\dots,n\}} \varphi_{f_i}^{-1}(]a_i - \varepsilon, a_i + \varepsilon[) \subset \bigcap_{i\in\{1,\dots,n\}} \varphi_{f_i}^{-1}(U_i) = O \subset N$.

In the following proposition, we will summarized some easy results comparing the weak topology and the norm (also called strong) topology on E.

Proposition 17 4.
a) Every weakly open (respectively closed) set is strongly open (respectively closed).
b). A sequence $\{x_n\}$ converges weakly to $x \in E$, iffy for all $f \in E^*$, $\langle f, x_n \rangle_{E^*,E} \longrightarrow \langle f, x \rangle_{E^*,E}$.
c). A strongly converging sequence converges weakly.
d). If $\{x_n\}$ is a sequence in E converging weakly to $x \in E$, then the sequence $\{x_n\}$ is bounded and $\|x\|_E \leq \liminf_{n\to\infty} \|x_n\|_E$.
e). If $\{x_n\}$ is a sequence in E converging weakly to $x \in E$ and $\{f_n\}$ is a sequence in E^* converging strongly to $f \in E^*$, then $\langle f_n, x_n \rangle_{E^*,E} \longrightarrow \langle f, x \rangle_{E^*,E}$.

Proof. a) Because the elements of E^* are continuous for the strong topology and the weak topology is the weakest with this property, it is weaker than the strong topology. So every weakly open set is strongly open, and by taking complements, every weakly closed set is strongly closed. b) It is just a restatement of the proposition 17.1, in the particular case of the weak topology on E. c) Suppose that the sequence $\{x_n\}$ converges strongly to $x \in E$. Because for any $f \in E^*$, $|\langle f, x_n \rangle_{E^*,E} - \langle f, x \rangle_{E^*,E}| = |\langle f, x_n - x \rangle_{E^*,E}| \leq \|f\|_{E^*} \|x_n - x\|_E$, when $n \longrightarrow \infty$ $\langle f, x_n \rangle_{E^*,E} \longrightarrow \langle f, x \rangle_{E^*,E}$. d) Because when $n \longrightarrow \infty$, $x_n \longrightarrow x$ weakly, for every $f \in E^*$ $\langle f, x_n \rangle_{E^*,E} \longrightarrow \langle f, x \rangle_{E^*,E}$ by b). Then, for every $f \in E^*$ the sequence $\langle f, x_n \rangle_{E^*,E}$ is bounded in $\mathbb{R}$, hence for every $f \in E^*$, $f(B)$ is bounded in $\mathbb{R}$, here B=$\{x_n\}$. By the corollary 17.6, the sequence $B = \{x_n\}$ is bounded, therefore $\liminf_{n\to\infty} \|x_n\|_E$ exists. As, $|\langle f, x_n \rangle_{E^*,E}| \leq \|f\|_{E^*} \|x_n\|_E$ for all $f \in E^*$, when $n \longrightarrow \infty$, $|\langle f, x \rangle_{E^*,E}| \leq \|f\|_{E^*} \liminf_{n\to\infty} \|x_n\|_E$ for all $f \in E^*$, using corollary 17.3, we have $\|x\|_E$=$\sup_{f\in\tilde{B}_{E^*}(0,1)} |\langle f, x \rangle_{E^*,E}| \leq \|f\|_{E^*} \liminf_{n\to\infty} \|x_n\|_E$. e) Since, $0 \leq |\langle f_n, x_n \rangle_{E^*,E} -$ $f,xE*,E=fn-f,xnE*,E+f,xn-xE*,E\leq fn-fE*xnE+f,xn-xE*,E$; $fn-fE*\longrightarrow 0$; $|\langle f, x_n - x \rangle_{E^*,E}| \longrightarrow 0$ when $n \longrightarrow \infty$ and, the sequence $\{x_n\}$ is bounded, when $n \longrightarrow \infty$, $|\langle f_n, x_n \rangle_{E^*,E} - \langle f, x \rangle_{E^*,E}| \longrightarrow 0$ and $\langle f_n, x_n \rangle_{E^*,E} \longrightarrow \langle f, x \rangle_{E^*,E}$.

Proposition 17.5. In the case when, the $\mathbb{R}$-nvs E is finite dimensional, both weak and strong topologies on E coincide.

Proof. We have seen in proposition 17.4 a) that in the infinite dimension, the weak topology is contained in the strong topology. Conversely assume that $dimE = n$. Let O be any strong open set in E, since all the norms defined on E are equivalent by proposition 16.7, for any x$\in O$, it exists $\varepsilon > 0$ such that $B_\infty(x, \varepsilon) = \{y \in E, \|x - y\|_\infty < \varepsilon\} \subset O$. If we show that $B_\infty(x, \varepsilon)$ is weakly open, then O is weakly open. Let $\{e_1, \dots, e_i, \dots, e_n\}$ be a basis of E, for any $x \in E$ there are n-components $\{x_1, \dots, x_i, \dots, x_n\}$ in $\mathbb{R}$ such that, $x = \sum_1^n x_i e_i$. Obviously, the functions $\{f_1, \dots, f_i, \dots, f_n\}$ defined by: for any $i \in \{1, \dots, n\}$ and for any $x \in E$, $\langle f_i, x \rangle_{E^*,E} = x_i$

are in E^*. As, for all $y \in B_\infty(x,\varepsilon)$, $y \in E$ and $\|x-y\|_\infty = \max_{i\in\{1,\dots,n\}}|x_i - y_i| < \varepsilon$ where $\{y_1,\dots,y_i,\dots,y_n\}$ are the components of y. Hence, $|x_i - y_i| = \left|\langle f_i, x\rangle_{E^*,E} - \langle f_i, y\rangle_{E^*,E}\right| = \left|\langle f_i, x-y\rangle_{E^*,E}\right| < \varepsilon$, for all $i \in \{1,\dots,n\}$. Conclusion, $B_\infty(x,\varepsilon) = \left\{y \in E, \left|\langle f_i, x-y\rangle_{E^*,E}\right| < \varepsilon, \text{for all } i \in \{1,\dots,n\}\right\}$, then it is weakly open, therefore O is weakly open.

Corollary 17.8. The nonempty strongly closed convex subset C of the $\mathbb{R}$-nvs E is weakly closed.

Proof. As C is strongly closed, his complementary C^C is stongly open. Let $A = C$ and $B = \{x_0\}$ are, where $x_0 \in C^C$, by Hahn Banach's theorem 17.3, there are a nonzero $f \in E^*$ and $b \in \mathbb{R}$ such that, $\langle f, x_0\rangle_{E^*,E} < b < \langle f, y\rangle_{E^*,E}$ for all $y \in C$. It is clear that, the weak neighborhood $V = \left\{x \in E, \left|\langle f, x_0\rangle_{E^*,E}\right| < b\right\} = f^{-1}(]-b,b[)$ contains x_0 and $V \cap C = \emptyset$, therefore $V \subset C^C$, thus C^C is weakly open equivalently C is weakly closed.

Remark 17.2. The reverse of $a)$ in the proposition 17.4 is not true. For example:

$a)$ The strong closed unit ball $\bar{B}(0,1)$=$\{x \in E, \|x\|_E \le 1\}$ is exactly the weak closure of the strongly closed unit sphere S=$\{x \in E, \|x\|_E = 1\}$. Indeed, by corollary 17.8, $\bar{B}(0,1)$ is weakly closed, as $S \subset \bar{B}(0,1)$, then the weak closure of S is contained in $\bar{B}(0,1)$.

It remains to show that, $\bar{B}(0,1)$ is contained in the weak closure of S. Let x_0 be any element of $\bar{B}(0,1)$ and let V be any weak neighborhood of x_0, there are $\varepsilon > 0$ and n-functions $f_1,\dots,f_i,\dots,f_n$ in E^* such that $V = \left\{x \in E, \left|\langle f_i, x-x_0\rangle_{E^*,E}\right| < \varepsilon\right\}$ for all $i \in \{1,\dots,n\}$. The function $\Phi: E \longrightarrow \mathbb{R}^n$ defined by: for all $y \in E$,
$\Phi(y) = \{\langle f_1, y\rangle_{E^*,E},\dots,\langle f_i, y\rangle_{E^*,E},\dots,\langle f_n, y\rangle_{E^*,E}\}$ is clearly linear and $\mathrm{Ker} f_i = \{y \in$ $E, fi,yE*,E=0$ so KerΦ=1nKerfi. As, it exists $y0\in E$ such that $y0\neq 0$ and $fi,y0E*,E=0$ for all $i \in \{1,\dots,n\}$. If not the function Φ: E $\longrightarrow$ Φ(E) is bijective and bicontinuous. So, Φ is a homeomorphism, thus $\dim E = \dim \mathrm{Im}\Phi \le$ n, contradiction. Because, for all $\lambda \in \mathbb{R}$ and for all $i \in \{1,\dots,n\}$, $\left|\langle f_i, x_0 + \lambda y_0 - x_0\rangle_{E^*,E}\right| = |\lambda|\left|\langle f_i, y_0\rangle_{E^*,E}\right| = 0 < \varepsilon$, then $x_0 + \lambda y_0 \in V$ for all $\lambda \in \mathbb{R}$ (in infinite dimension, any weak neighborhood of x_0 contains the line passing through x_0). It is obvious the the function $g: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ define by: $\mathrm{g}(\lambda) = \|x_0 + \lambda y_0\|_E$ for all $\lambda \in \mathbb{R}$ satisfies $\mathrm{g}(0) < 1$ and $\mathrm{g}(\lambda) \longrightarrow +\infty$ when $\mathrm{g}(\lambda) \longrightarrow +\infty$, therefore it exists $\lambda_0 > 0$ such that $\mathrm{g}(\lambda_0) = 1$, hence $x_0 + \lambda_0 y_0 \in V \cap S$. Finally x_0 is contained in the weak closure of S.

$b)$ We can also check that, the weak $int(B(0,1))$=$\emptyset$. Indeed, if it exists x_0 in the weak $int(B(0,1))$, it exists a weak neighborhood V of x_0, such that $V \subset B(0,1)$. As in $a)$ there is a nonzero y_0 $\in$KerΦ and $\lambda_0 > 0$ such that, $\|x_0 + \lambda y_0\|_E = 1$ and $x_0 + \lambda_0 y_0 \in V \subset B(0,1)$, contradiction.

Theorem 17.7. If E and F are two Banach spaces, and $T: E \longrightarrow F$ is a linear map. Then T is strongly continuous iffy T is weakly continuous.

Proof. Assume that, T is strongly- strongly continuous linear map. Let $\mathrm{g} \in \mathrm{F}^*$ be, as the function $f = g \circ T \in E^*$ it is weak-weak continous, then T is weakly continuous by the proposition 17.2. If now the map T is weakly-weakly continuous and linear, by the closed graph theorem 17.6, the graph $G(T)$ of T is weakly closed in $E \times F$ and a fortiori $G(T)$ is strongly closed, so T is strong-strong continuous.

Remark 17.3.

$a)$ By the same argument used in the proof of theorem 17.7, we prove that if T is linear and strong-weak continues it is strong-strong continuous.

$b)$ The linearity of T in theorem 17.7 plays an essential role in both sens. Without linearity, the theorem fails.

17.5 The weak∗ topology $\boldsymbol{\sigma(E^*,E)}$ in the ℝ-nvs $\boldsymbol{E^*}$

In the ℝ-nvs E^* two topologies are defined: the strong topology $\tau_{\|.\|_{E^*}}$ and the weak topology $\sigma(E^*,E)$. In this section we will define a third topology on E^* as follows. Let E^{**} be the dual of E^*, also called the bidual of ℝ-nvs E. The norm of any element $\xi \in E^{**}$ is defined by: $\|\xi\|_{E^{**}} = \sup_{\{f\in E^*,\|f\|_{E^*}\leq 1\}}|\langle \xi, f\rangle_{E^{**},E^*}|$, where $\langle \xi, f\rangle_{E^{**},E^*} = \xi(f)$, for $\tau_{\|.\|_{E^*}}$all $f \in E^*$. Note that the canonical injection $\pi: E \longrightarrow E^{**}$ defined by: $\langle \pi(x), f\rangle_{E^{**},E^*} = \langle f, x\rangle_{E^*,E}$ for all $x \in E$ and all $f \in E^*$ is continuous linear isometric and injective from E into E^{**}. Indeed, it is clearly continuous linear and for all $x \in E$, $\|\pi(x)\|_{E^{**}} = \sup_{\{f\in E^*,\|f\|_{E^*}\leq 1\}}|\langle \pi(x), f\rangle_{E^{**},E^*}| = \sup_{\{f\in E^*,\|f\|_{E^*}\leq 1\}}|\langle f, x\rangle_{E^*,E}| = \|x\|_E$ by the corollary 17.3, then π is isometric, thus it is injective. Therefore π is bijective between E and $\pi(E)$, which allows us to identify E and $\pi(E) \subset E^{**}$, and consider E as a subset of E^{**}. In the case when $\pi(E) = E^{**}$, then $E = E^{**}$ and E is said to be **reflexive**. Consider the collection of the functions $(\varphi_x)_{x\in E}$ defined from E^* into ℝ by: $\varphi_x(f) = \langle f, x\rangle_{E^*,E}$ for all $x \in E$ and all $f \in E^*$. Note that for a fixed $x \in E$, the φ_x satisfies the same properties of π.

Definition 17.4. The weak∗ topology in the ℝ-nvs E^* is the topology $\sigma(E^*,(\varphi_x)_{x\in E})$, which will be noted by $\sigma(E^*,E)$.

Remark 17.4.

a) As, $E \subset E^{**}$, then $\sigma(E^*,E) \subset \sigma(E^*,E^{**}) \subset \tau_{\|.\|_{E^*}}$ i.e. in E^*, the weak∗ topology $\sigma(E^*,E)$ is weaker than the weak topology $\sigma(E^*,E^{**})$, which is weaker than the strong topology $\tau_{\|.\|_{E^*}}$. Therefore, the weak∗ topology $\sigma(E^*,E)$ offers more compacts than $\sigma(E^*,E^{**})$. If a topology has fewer open sets, it has more compact sets. However, compact sets play a fundamental role when we seek to establish existence theorems. Hence the importance of introducing the weak∗ topology $\sigma(E^*,E)$.

b) In the finite dimensional all the topologies are identical. Since in this case dim E =dimE^* =dimE^{**}, therefore the canonical injection $\pi: E \longrightarrow E^{**}$is surjective, so $E = E^{**}$ and $\sigma(E^*,E^{**}) = \sigma(E^*,E)$.

c) Given the two families $\{x_n, x\}$ in E and $\{f_n, f\}$ in E^*. We often use:

$x_n \longrightarrow x$ to express that the sequence $\{x_n\}$ converges strongly to x i.e. $\|x_n - x\|_E \longrightarrow 0$.
$x_n \rightharpoonup x$ to express that the sequence $\{x_n\}$ converges weakly to x.
$f_n \longrightarrow f$ to express that the sequence $\{f_n\}$ converges strongly to f i.e. $\|f_n - f\|_{E^*} \longrightarrow 0$.
$f_n \rightharpoonup f$ to express that the sequence $\{f_n\}$ converges weakly to f.
$f_n \rightharpoonup^* f$ to express that the sequence $\{f_n\}$ converges weakly∗ to f.

The propositions 17.6-17.8 below, whose verification is simple, summarize the usual properties of the weak∗ topology.

Proposition 17.6. Let $(E^*,\sigma(E^*,E))$ and $f_0 \in E^*$ are. The collection of the subsets B of E^* defined by: $f \in B$ iffy, it exist $\varepsilon > 0$ and n elements $\{x_1,\dots,x_i,\dots,x_n\}$ of E, such that $|\langle f - f_0, x_i\rangle_{E^*,E}| < \varepsilon$, for all $i \in \{1,\dots,n\}$, is a basis of neighborhoods of f_0.

Proposition 17.7. The topological space $(E^*,\sigma(E^*,E))$ is Hausdorff.

Proof: Let f and g are distinct elements of E^*. Thus, there exists $x_0 \in E$ such that $\langle f, x_0\rangle_{E^*,E} \neq \langle g, x_0\rangle_{E^*,E}$. Assuming for example that, $\langle f, x_0\rangle_{E^*,E} < \langle g, x_0\rangle_{E^*,E}$., we can find a real number b such that $\varphi_{x_0}(f) = \langle f, x_0\rangle_{E^*,E} < b < \langle g, x_0\rangle_{E^*,E} = \varphi_{x_0}(f)$. Therefore $f \in V_f = \varphi_{x_0}^{-1}(]-\infty, b[)$ and $g \in V_g = \varphi_{x_0}^{-1}(]b, +\infty[)$. Those are two disjoint weak∗ open sets that separate f and g.

Proposition 17.8. In E^*, we have:

a). A sequence $\{f_n\}$ converges weakly∗ to $f \in E^*$, iffy for all $x \in E$, $\langle f_n, x\rangle_{E^*,E} \longrightarrow \langle f, x\rangle_{E^*,E}$.

b) A stronly converging sequence converges weakly.

c). A weakly converging sequence converges weakly$*$.
d). If $\{f_n\}$ is a sequence in E^* converging weakly$*$ to $f \in E^*$, then the sequence $\{f_n\}$ is bounded and $\|f\|_{E^*} \leq \liminf_{n\to\infty}\|f_n\|_{E^*}$.
e). If $\{f_n\}$ is a sequence in E^* converging weakly$*$ to $f \in E$ and $\{x_n\}$ is a sequence in E converging strongly to $x \in E$, then $\langle f_n, x_n\rangle_{E^*,E} \longrightarrow \langle f, x\rangle_{E^*,E}$.

This first version is to be completed soon

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