RECALL 1

Theorem.3.5. (Beppo-Levy monotone convergence Theorem)

Let (f_n) be an increasing sequence in $\mathcal{M}_+$, then:

$\lim\limits_n f_n = f \in \mathcal{M}_+$ and $\int\limits_X f.d\mu = \lim\limits_n \int\limits_X f_n\, d\mu$, in other words:

$$\lim_n \int\limits_X f_n\, d\mu = \int\limits_X \lim_n f_n\, d\mu$$

Proof. We know that $\lim\limits_n f_n = f \in \mathcal{M}_+$ and since (f_n) is increasing we have $\int\limits_X f_n\, d\mu \leq \int\limits_X f_{n+1}\, d\mu \leq \int\limits_X f.d\mu$, $\forall n$. So $a = \lim\limits_n \int\limits_X f_n\, d\mu$ exists and $a \leq \int\limits_X f.d\mu$. Let $s \in \mathcal{E}_+$ with $s \leq f$ and for $0 < c < 1$ put $E_n = \{f_n \geq c.s\}$.
We have $E_n \subset E_{n+1}$ since $f_n \leq f_{n+1}$ and $\underset{n}{\cup} E_n = X$ because $c.s < f = \sup\limits_n f_n$.
On the other hand $f_n \geq 0 \Longrightarrow f_n \geq c.s.I_{E_n}, \forall n$.
Now put $s = \sum\limits_i \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int\limits_X f_n.d\mu \geq \int\limits_X c.s.I_{E_n}.d\mu$ (since $f_n \geq c.s.I_{E_n}$ on X), then $\int\limits_X f_n.d\mu \geq c.\sum\limits_i \alpha_i.\mu\left(A_i \cap E_n\right), \forall n$. This implies $a = \lim\limits_n. \int\limits_X f_n.d\mu \geq \lim\limits_n. \left(c.\sum\limits_i \alpha_i.\mu\left(A_i \cap E_n\right)\right) = c.\sum\limits_i \alpha_i.\mu\left(A_i\right) = c.\int\limits_X s\, d\mu$, because $\mu\left(A_i \cap E_n\right)$ goes to $\mu\left(A_i\right)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get $a \geq \int\limits_X s\, d\mu$ for all $s \in \mathcal{E}_+$ with $s \leq f$, so $a \geq \sup\left\{\int\limits_X s\, d\mu, s \in \mathcal{E}_+, s \leq f\right\} = \int\limits_X f.d\mu$ by Theorem.5.3.4, then $a = \int\limits_X f.d\mu$.■

Remark. Theorem.**3.5.**is not valid in general for decreasing sequences (f_n) as is shown by the following example: let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ be the Borel measure space and $f_n = I_{]n,\infty[}$, then f_n decreases to 0 but $\lim\limits_n. \int\limits_X f_n.d\mu = \infty$.■

Lemma 3.6. (Fatou Lemma)

Let (f_n) be any sequence in $\mathcal{M}_+$, then:

$$\int\limits_X \liminf_n f_n\, d\mu \leq \liminf_n \int\limits_X f_n\, d\mu$$

Proof. Put $F_k = \inf\limits_{n \geq k} f_n$ then F_k is increasing in $\mathcal{M}_+$ to $\liminf\limits_n f_n$, so by Theorem..**3.5,** $\lim\limits_{k}.\int\limits_{X} F_k.d\mu = \int\limits_{X} \liminf\limits_{n} f_n \, d\mu.$

But $F_k \leq f_n, \forall n \geq k$, which implies $\int\limits_{X} F_k.d\mu \leq \inf\limits_{n\geq k} \int\limits_{X} f_n \, d\mu$ and then

making $k \longrightarrow \infty$ we get $\lim\limits_{k}.\int\limits_{X} F_k.d\mu = \int\limits_{X} \liminf\limits_{n} f_n \, d\mu \leq \lim\limits_{k} \inf\limits_{n\geq k} \int\limits_{X} f_n \, d\mu =$

$\liminf\limits_{n} \int\limits_{X} f_n \, d\mu.$■

Theorem.3.7 (Lebesgue's dominated convergence theorem)
Let (f_n) be a sequence in $L_1(\mu)$ such that:
  (a) f_n converges $\mu - a.e$ to a function f
  (b) there is g in $L_1(\mu)$ such that $\forall n \geq 1 \quad |f_n| \leq |g| \quad \mu - a.e$
Then the function f is in $L_1(\mu)$ and $\lim\limits_{n} \int_X |f_n - f| \, d\mu = 0$
in particular $\lim\limits_{n} \int_X f_n \, d\mu = \int_X f \, d\mu$

SOLUTIONS TO SOME EXERCISES

24. (a) Prove that in any measure space the uniform convergence implies the convergence in measure.
(b) In the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ the uniform convergence is equivalent to the convergence in measure.

solution.
(a) Let $f_n, f : X \longrightarrow \mathbb{R}$ be measurable in the space $(X, \mathcal{F}, \mu)$ such that f_n converges uniformly to f
then we have $\forall \epsilon > 0, \exists N_\epsilon$ such that $\forall n \geq N_\epsilon, \ |f_n(x) - f(x)| < \epsilon$ for all $x \in X$
this implies $\{x : |f_n(x) - f(x)| > \epsilon\} = \phi, \forall n \geq N_\epsilon$
that is $\lim\limits_{n} \mu(|f_n - f| > \epsilon) = 0$ so f_n converges in measure to f . The result is true if f_n converges uniformly $\mu - a.e$ to f.
(b) use the fact that for the counting measure we have:
  $A \subset \mathbb{N}$ and $\mu(A) = 0 \Longrightarrow A = \phi$.

25. In the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ consider the sequence of indicator functions $f_n = I_{\{1,2,\dots,n\}}$; prove that f_n converges $\mu{-}a.e$ but does not converge in measure.

solution.
The sets $\{1, 2, ..., n\}$ increase to $\mathbb{N}$ as $n \longrightarrow \infty$ and so $I_{\{1,2,\dots,n\}}$ converges to 1 for any $x \in \mathbb{N}$.
On the other hand for $\epsilon > 0$ $\left\{\left|I_{\{1,2,\dots,n\}} - 1\right| > \epsilon\right\} = \{x \in \mathbb{N} : x > n\}$
$= \{n+1, n+2, n+3, ...\}$ which gives $\mu\left\{\left|I_{\{1,2,\dots,n\}} - 1\right| > \epsilon\right\} = \infty \quad \forall n.$■

26. Let $f_n, f \in \mathcal{M}(X, \mathbb{R})$ and suppose f_n converges pointwise to f and there is a positive measurable function g satisfying $\lim_n \mu\{g > \epsilon_n\} = 0$ for some sequence of positive numbers ϵ_n with $\lim_n \epsilon_n = 0$. Then if $|f_n| \leq g, \forall n$, prove that f_n converges in measure to f.

solution.
We have to prove that $n \longrightarrow \infty \Longrightarrow \mu(|f_n - f| > \epsilon) \longrightarrow 0, \forall \epsilon > 0$
Since $|f_n| \leq g$ and f_n converges pointwise to f we deduce that $|f| \leq g$
so $|f_n - f| \leq 2g$. Let $\epsilon > 0$, since $\lim_n \epsilon_n = 0$ there is $N \geq 1$ with $2\epsilon_n < \epsilon, \forall n \geq N$.
Now we have $(|f_n - f| > \epsilon) \subset (2g > \epsilon) \subset (2g > 2\epsilon_n) = (g > \epsilon_n), \forall n \geq N$
we deduce that $\lim_n \mu(|f_n - f| > \epsilon) \leq \lim_n \mu\{g > \epsilon_n\} = 0$. So f_n converges in measure to f.■

27. Let $f : X \longrightarrow \mathbb{R}$ be measurable in the space $(X, \mathcal{F}, \mu)$ and put:
$M(f) = \inf\{\alpha \geq 0 : \ \mu\{|f| > \alpha\} = 0\}$,Prove that $|f| \leq M(f) \ \ \mu - a.e.$
Prove that $\lim_n M(f_n - f) = 0$ iff $\lim_n f_n = f$ uniformly $\mu - a.e.$

solution.
We have to prove that $\mu\{|f| > M(f)\} = 0$
If $M(f) = \infty$ the result is true.
Suppose $M(f)$ finite then we have $\{|f| > M(f)\} = \underset{n}{\cup}\left\{|f| > M(f) + \frac{1}{n}\right\}$
but $M(f) < M(f) + \frac{1}{n} \Longrightarrow \exists \alpha_n \in \{\alpha \geq 0 : \ \mu\{|f| > \alpha\} = 0\}$
with $M(f) < \alpha_n < M(f) + \frac{1}{n}$ so $\left\{|f| > M(f) + \frac{1}{n}\right\} \subset \{|f| > \alpha_n\}$ and then $\mu\left\{|f| > M(f) + \frac{1}{n}\right\} \leq \mu\{|f| > \alpha_n\} = 0, \forall n$, we deduce $\mu\{|f| > M(f)\} = \mu\left(\underset{n}{\cup}\left\{|f| > M(f) + \frac{1}{n}\right\}\right) \leq \sum_n \mu\left\{|f| > M(f) + \frac{1}{n}\right\} = 0.$■

28 Let $f_n, f : X \longrightarrow \mathbb{R}$ be measurable functions in the space $(X, \mathcal{F}, \mu)$ and suppose that f_n converges in measure to f ; if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a uniformly continuous function prove that the sequence $g \circ f_n$ converges in measure to $g \circ f$

solution.
We have to prove that $n \longrightarrow \infty \Longrightarrow \mu(|g \circ f_n - g \circ f| > \epsilon) \longrightarrow 0, \forall \epsilon > 0$
g uniformly continuous implies:
$(*) \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R} \qquad |x - y| < \delta \Longrightarrow |g(x) - g(y)| < \epsilon$
$(**) \quad f_n$ converges in measure to $f \Longrightarrow \mu(|f_n - f| > \alpha) \longrightarrow 0, \forall \alpha > 0$
$(*) \Longrightarrow \forall \epsilon > 0 \ \ \exists \delta > 0$ such that $(|g \circ f_n - g \circ f| > \epsilon) \subset (|f_n - f| > \delta)$
then applying μ we get $\mu(|g \circ f_n - g \circ f| > \epsilon) \leq \mu(|f_n - f| > \delta)$
$(**) \Longrightarrow \lim_n \mu(|f_n - f| > \delta) = 0$ so we deduce
$\lim_n \mu(|g \circ f_n - g \circ f| > \epsilon) = 0, \forall \epsilon > 0.$■

29.(a) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be the counting measure on $\mathbb{N}$.
If $f : \mathbb{N} \longrightarrow [0, \infty[$ is given by $f(i) = a_i$ $i \in \mathbb{N}$ prove that:

$$\int_{\mathbb{N}} f.d\mu = \sum_i a_i$$

(b) Let $\mu = \delta_{x_0}$ be the Dirac measure on the power set $\mathcal{P}(X)$ of X.
then for any $f : X \longrightarrow [0, \infty[$, $\int_X f.d\mu = f(x_0)$.

solution. $f : \mathbb{N} \longrightarrow [0, \infty[$

(a) Suppose f simple function of the form $\sum_1^n a_i . I_{\{i\}}$ then $\int_{\mathbb{N}} f.d\mu = \sum_1^n a_i . \mu\{i\}$

but $\mu\{i\} = 1$ since μ is the counting measure so $\int_{\mathbb{N}} f.d\mu = \sum_1^n a_i$

now take f of the form $f = .\sum_i a_i . I_{\{i\}}$ which is the limit pointwise of the increasing sequence $\varphi_n = \sum_1^n a_i . I_{\{i\}}$, by Beppo-Levy theorem we get

$$\int_{\mathbb{N}} f.d\mu = \lim_n \int_{\mathbb{N}} \varphi_n . d\mu = \lim_n \sum_1^n a_i = \sum_i a_i.$$

(b) Recall that Dirac measure is defined on $\mathcal{P}(X)$ by

$$\delta_{x_0}(A) = I_A(x_0) = \left\{ \begin{array}{l} 1 \text{ if } x_0 \in A \\ 0 \text{ if } x_0 \notin A \end{array} \right\}$$

so we have $\delta_{x_0}(A) = \int_X I_A . d\delta_{x_0}$ and generalize this formula by usual procedure

to get for any $f : X \longrightarrow [0, \infty[$, $\int_X f.d\delta_{x_0} = f(x_0)$.■

30. Let (f_n) be any sequence in $\mathcal{M}_+$, prove that $\sum_n f_n \in \mathcal{M}_+$ and:

$$\int_X \sum_n f_n \, d\mu = \sum_n \int_X f_n . d\mu$$

solution.
$\sum_1^n f_i$ increases to $\sum_n f_n$ and use Beppo-Levy Theorem, see the recall.■

31. Let $f \in \mathcal{M}_+$

(a) Prove that the set function $\nu : A \longrightarrow \int_A f.d\mu$, defined on $\mathcal{F}$ is a positive measure

(b) If $g \in \mathcal{M}_+$ prove that $\int_X g.d\nu = \int_X f.g.d\mu$

solution.

(a) Let (A_n) be a pairwise disjoint sequence of sets in $\mathcal{F}$

we have to prove that $\int\limits_{\underset{n}{\cup} A_n} f.d\mu = \sum\limits_n \int\limits_{A_n} f.d\mu$

since the sets A_n are pairwise disjoint we have $I_{\underset{n}{\cup} A_n} = \sum\limits_n I_{A_n}$ and $f \geq 0$ then

$f.I_{\underset{n}{\cup} A_n} = \sum\limits_n f.I_{A_n}$, so we get $\int\limits_X f.I_{\underset{n}{\cup} A_n}.d\mu = \int\limits_X \sum\limits_n f.I_{A_n}.d\mu = \sum\limits_n \int\limits_{A_n} f.d\mu$

where the last equality comes from **Beppo-Levy** Theorem **3.5** (see recall 1)

(b) check (b) for $g \in \mathcal{E}_+$ and apply Beppo-Levy Theorem for $g \in \mathcal{M}_+$.■

32. Let (f_n) be a sequence in $\mathcal{M}_+$ with $\lim\limits_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$.Suppose $\sup\limits_n \int\limits_X f_n.d\mu < \infty$, and prove that $\int\limits_X f.d\mu < \infty$

solution.

(Apply **Fatou Lemma 3.6** see recall 1)

$\int\limits_X \liminf\limits_n f_n\, d\mu \leq \liminf\limits_n \int\limits_X f_n\, d\mu$ with $\liminf\limits_n f_n = \lim\limits_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$ so $\int\limits_X f.d\mu \leq \liminf\limits_n \int\limits_X f_n\, d\mu \leq \sup\limits_n \int\limits_X f_n.d\mu < \infty$.■

33. Let (f_n) be a decreasing sequence in $\mathcal{M}_+$ such that

$\int\limits_X f_{n_0}.d\mu < \infty$, for some $n_0 \geq 1$

Prove that $\lim\limits_n \int\limits_X f_n\, d\mu = \int\limits_X \lim\limits_n f_n\, d\mu$

solution.

apply Theorem **3.5** (Recall 1) to the increasing positive sequence $(f_{n_0} - f_n)$ $n \geq n_0$

indeed we have $f_{n+1} \leq f_n \implies f_{n_0} - f_n \leq f_{n_0} - f_{n+1}, \forall n \geq n_0$ and so $\lim\limits_n (f_{n_0} - f_n) = f_{n_0} - f$

by Theorem **3.5** we deduce $\lim\limits_n \int\limits_X (f_{n_0} - f_n).d\mu = \int\limits_X f_{n_0}.d\mu - \lim\limits_n \int\limits_X f_n.d\mu =$ $\int\limits_X f_{n_0}.d\mu - \int\limits_X f.d\mu$ since $f \in \mathcal{M}_+$

by the fact $\int\limits_X f_{n_0}.d\mu < \infty$, we get $\lim\limits_n \int\limits_X f_n.d\mu = \int\limits_X f.d\mu$ ■

34. Let the interval $]0,1[$ of real numbers be endowed with Lebesgue measure. (Apply **Fatou Lemma 3.6** see recall 1) to the following sequence:

$f_n(x) = n, 0 \leq x \leq \frac{1}{n}$ and $f_n(x) = 0, \frac{1}{n} < x < 1$.

solution.

$\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$ with $\liminf_n f_n = \sup_n . \inf_{k \geq n} f_k = 0$ and $\int_X f_n \, d\mu = 1, \forall n$

whence $0 \leq \liminf_n \int_X f_n \, d\mu \leq 1$.■

35 (continuity of integrals depending on a parameter)

Let T be an interval of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:

(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$

(b) there is g in $L_1(\mu)$such that$|f(x,t)| \leq |g(x)|$ $\mu - a.e$ for all $t \in T$

if $\lim_{t \to t_0} .f(x,t) = f(x,t_0)$ then we have $\lim_{t \to t_0} \int_X f(x,t) \, d\mu = \int_X f(x,t_0) \, d\mu$

solution.

Consider the function $h : T \longrightarrow \mathbb{R}$ given by $h(t) = \int_X f(x,t) \, d\mu$

we have to prove that $\lim_{t \to t_0} h(t) = h(t_0)$

that is the function h is continuous on T which is equivalent to: for any sequence (t_n) with $\lim_n t_n = t_0$ we have $\lim_n h(t_n) = h(t_0)$

let us observe that the functions u_n defined by $u_n(x) = f(x,t_n)$

satisfies **Theorem.3.7** by (b) and $\lim_n .u_n(x) = f(x,t_0)$, so $\int_X u_n.d\mu = h(t_n)$

converges to $\int_X \lim_n .u_n(x).d\mu = \int_X f(x,t_0).d\mu = h(t_0)$.■

36 (Derivative of integrals depending on a parameter)

Let T be an open set of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:

(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$

(b) the function $t \longrightarrow f(x,t)$ derivable on T for each $x \in X$

(c) there is $g \in L_1(\mu)$ $\left| \frac{d}{dt} f(x,t) \right| \leq |g(x)|$ $\mu - a.e$ for all $t \in T$

Then the function $t \longrightarrow \int_X f(x,t) \, d\mu$ is differentiable on T

and $\frac{d}{dt} \int_X f(x,t) \, d\mu = \int_X \frac{d}{dt} f(x,t) \, d\mu$

solution.

Let (t_n) be a sequence with $\lim_n t_n = t$ and define the sequence (g_n) of functions by

$g_n(x) = \dfrac{f(x,t_n) - f(x,t)}{t_n - t}$ then $\lim_n g_n(x) = \dfrac{d}{dt} f(x,t)$. By the Mean Value Theorem

there is $\theta_n(x)$ between t_n and t such that $g_n(x) = \dfrac{d}{dt} f(x, \theta_n(x))$.

Now we have $\lim_n t_n = t$ so $\lim_n \theta_n(x) = t$ and $\lim_n g_n(x) = \dfrac{d}{dt} f(x,t)$. But $|g_n(x)| \leq |g(x)|$ by (c) then

we can apply **Theorem.3.7** to $g_n(x)$ with

$$\int_X g_n(x).d\mu = \frac{\int_X f(x,t_n).d\mu - \int_X f(x,t).d\mu}{t_n - t}$$

to get $\lim_n \int_X g_n(x).d\mu = \dfrac{d}{dt}\int_X f(x,t)\ d\mu = \int_X \lim_n g_n(x)\ d\mu = \int_X \dfrac{d}{dt} f(x,t)\ d\mu.$ ■

37 (Change of variable formula)

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $(Y, \mathcal{G})$ be a measurable space:

If $\varphi : X \longrightarrow Y$ is a measurable mapping from $(X, \mathcal{F})$ into $(Y, \mathcal{G})$ then:

(1) the set function $\nu : \mathcal{G} \longrightarrow [0, \infty]$ given by $G \in \mathcal{G}, \nu(G) = \mu\left(\varphi^{-1}(G)\right)$ is a measure on $(Y, \mathcal{G})$

(2) for every function $g : Y \longrightarrow \mathbb{C}$, $\nu-$integrable the function $g \circ \varphi$ is $\mu-$integrable and

$$(*) \int_Y g.d\nu = \int_X g \circ \varphi.d\mu$$

$$(**) \int_E g.d\nu = \int_{\varphi^{-1}(E)} g \circ \varphi.d\mu \ \forall E \in \mathcal{G}.$$

solution.

Apply usual procedure:

start with g simple then g in $\mathcal{M}_+$ and finally g integrable for ν. ■

38 Measure defined by an integral. (see exercise **31 for the proof)**

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{M}_+$ then

(a) the set function $\nu : \mathcal{F} \longrightarrow [0, \infty]$ given by: $A \in \mathcal{F}, \nu(A) = \int_A f.d\mu$

is a positive measure on $\mathcal{F}$ and we have:

(b) $\int_X g.d\nu = \int_X f.g.d\mu$, for every $g \in \mathcal{M}_+$.

RECALL 2
INTEGRATION IN PRODUCT SPACES
Product Measure and Fubini Theorem

In this part we give without proofs the most important results on product spaces useful in applications.Proofs are classical and in general simple.

1. Preliminaries and Notations

1.1 In all what follows, $(X, \mathcal{F}, \mu)$, $(Y, \mathcal{G}, \nu)$ will be fixed measure spaces.
1.2 Let us recall that the product $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ on $X \times Y$ is generated by the family $\{A \times B, \text{ with } A \in \mathcal{F}, B \in \mathcal{G}\}$, (Definition **3.4** Chapter **1**)
1.3 The set $\mathbb{R}$ will be endowed with its Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$. The set $\mathbb{R}^2$ endowed with the σ-field $\mathcal{B}_{\mathbb{R}^2}$=$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$(Theorem**2.9**Chap.**3**)

2. Product Measure

2.1 Definition
For any subset $E \subset X \times Y$ and any $(x, y) \in X \times Y$, define:
the section of E at x, $E_x = \{y \in Y, \quad (x, y) \in X \times Y\}$
the section of E at y, $E_y = \{x \in X, \quad (x, y) \in X \times Y\}$

2.2 Proposition
For every $E \in \mathcal{F} \otimes \mathcal{G}$, $E_x \in \mathcal{G}$ and $E_y \in \mathcal{F}$.

2.3 Theorem
Suppose that the measure μ and ν are $\sigma-$finite
then for every $E \in \mathcal{F} \otimes \mathcal{G}$, we have:
the function $x \longrightarrow \nu(E_x)$ is $\mathcal{F}$ measurable
the function $y \longrightarrow \mu(E_y)$ is $\mathcal{G}$ measurable

Moreover we have $\displaystyle\int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$

Corollary.(Product measure)
Under the conditions of Theorem **1.6** the set function $\mu \otimes \nu$ defined on $\mathcal{F} \otimes \mathcal{G}$ by:

$$\mu \otimes \nu(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu,\ E \in \mathcal{F} \otimes \mathcal{G}$$

is a $\sigma-$finite measure on $\mathcal{F} \otimes \mathcal{G}$. Moreover $\mu \otimes \nu$ is the unique $\sigma-$finite measure on $\mathcal{F} \otimes \mathcal{G}$ satisfying $\mu \otimes \nu(A \times B) = \mu(A).\nu(B)$ for every $A \in \mathcal{F}, B \in \mathcal{G}$.

3 Integration in Product Spaces

3.1 Definition Let $f : X \times Y \longrightarrow \mathbb{R}$ be any function and $(x, y) \in X \times Y$, define:
$f_x : Y \longrightarrow \mathbb{R}$ by $f_x(y) = f(x, y)$ (section of f at x)
$f_y : X \longrightarrow \mathbb{R}$ by $f_y(x) = f(x, y)$ (section of f at y)

3.2 Proposition

Let $f : X \times Y \longrightarrow \mathbb{R}$ be $\mathcal{F} \otimes \mathcal{G}$-measurable then

f_x is $\mathcal{G}$-measurable and f_y is $\mathcal{F}$-measurable

3.3 Theorem (Fubini)

Suppose that the measure μ and ν are $\sigma-$finite and $f : X \times Y \longrightarrow \mathbb{R}$ is $\mathcal{F} \otimes \mathcal{G}$-measurable positive then:

the function $x \longrightarrow \int\limits_Y f(x, y)\ d\nu$ is $\mathcal{F}$-measurable

the function $y \longrightarrow \int\limits_X f(x, y)\ d\mu$ is $\mathcal{G}$-measurable

and we have:

$$\int\limits_{X \times Y} f(x, y)\ d\mu \otimes \nu = \int\limits_X d\mu \int\limits_Y f(x, y)\ d\nu = \int\limits_Y d\nu \int\limits_X f(x, y)\ d\mu$$

3.4 Theorem (Fubini)

For every $f \in L_1(\mu \otimes \nu)$ we have:

(a) $\int\limits_Y f(x, y)\ d\nu \in L_1(\mu)$ and $\int\limits_X f(x, y)\ d\mu \in L_1(\nu)$

(b) $\int\limits_{X \times Y} f(x, y)\ d\mu \otimes \nu = \int\limits_X d\mu \int\limits_Y f(x, y)\ d\nu = \int\limits_Y d\nu \int\limits_X f(x, y)\ d\mu$

3.5 Application. (Convolution of functions)

Let μ be the Lebesgue measure on $\mathbb{R}$,$\mathcal{B}_{\mathbb{R}}$ and $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions in $L_1(\mu)$, then:

$$\int\limits_{\mathbb{R}} |f(x-y)| . |g(y)| . d\mu(y) < \infty \text{ for each } x$$

Let us define the convolution of f and g by the function $h : \mathbb{R} \longrightarrow \mathbb{R}$:

$$h(x) = \int\limits_{\mathbb{R}} f(x-y) . g(y) . d\mu(y)$$

we denote h by $h = f * g$

Since $\left| \int\limits_{\mathbb{R}} f(x-y) . g(y) . d\mu(y) \right| \leq \int\limits_{\mathbb{R}} |f(x-y)| . |g(y)| . d\mu(y) < \infty$ we deduce that $h \in L_1(\mu)$

3.6 Lemma

Under the definition above we have $\|f * g\| \leq \|f\| . \|g\|$.■

4 Convolution of Measures

4.1 Definition
Let us consider on the set $\mathbb{R}^2$ endowed with the σ-field $\mathcal{B}_{\mathbb{R}^2}$=$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $T(x, y) = x + y$ which is measurable because continuous. Let $\mu_1 \otimes \mu_2$ be the product of two finite measures μ_1, μ_2 defined on $\mathbb{R}$, $\mathcal{B}_{\mathbb{R}}$. The convolution $\mu_1 * \mu_2$ of the measures μ_1, μ_2 is the measure on $\mathcal{B}_{\mathbb{R}}$ given by: $B \in \mathcal{B}_{\mathbb{R}}$, $(\mu_1 * \mu_2)(B) = (\mu_1 \otimes \mu_2)\left(T^{-1}(B)\right)$. Then we have:

4.2 Proposition Let $B \in \mathcal{B}_{\mathbb{R}}$ and define:

$\left[T^{-1}(B)\right]_x = \{y \in \mathbb{R}, x + y \in B\} = B - x$

$\left[T^{-1}(B)\right]_y = \{x \in \mathbb{R}, x + y \in B\} = B - y$

then we get: $(\mu_1 * \mu_2)(B) = \int_{\mathbb{R}} .\mu_2(B - x) .\mu_1(dx) = \int_{\mathbb{R}} .\mu_1(B - y) .\mu_2(dy)$

by applying Fubini Theorem and the relation $(\mu_1 * \mu_2)(B) = (\mu_1 \otimes \mu_2)\left(T^{-1}(B)\right) = \int_{X \times Y} .I_{T^{-1}(B)}(x, y) . (\mu_1 \otimes \mu_2)(dx, dy)$.

Moreover if we take a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ integrable with respect to $\mu_1 * \mu_2$ we obtain the following nice relation:

$\int_{\mathbb{R}} f(t) . (\mu_1 * \mu_2)(dt) = \int_{\mathbb{R}} \mu_2(dy) \int_{\mathbb{R}} f(x + y) .\mu_1(dx) = \int_{\mathbb{R}} \mu_1(dx) \int_{\mathbb{R}} f(x + y) .\mu_2(dy)$

4.3 Proposition With the definitions above we have:

(1) $\mu_1 * \mu_2 = \mu_2 * \mu_1$

(2) $(\mu_1 * \mu_2)(\mathbb{R}) = (\mu_1 \otimes \mu_2)\left(T^{-1}(\mathbb{R})\right) = (\mu_1 \otimes \mu_2)\left(\mathbb{R}^2\right) = \mu_1(\mathbb{R}) .\mu_2(\mathbb{R})$

(3) $\mu_1 * \delta_0 = \delta_0 * \mu_1 = \mu_1$, δ_0 is the Dirac measure at 0.■