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Fredholm Theory on Banach Spaces

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This work dedicated

to my parents.

Abstract

The primary goal of this work is to present the various concepts of the Fredholm theory and their perturbations in Banach spaces, as well as definitions of the essential spectrum found in the mathematical literature, beginning with Weyl's fundamental work. This theory has been investigated in relation to various classes of bounded operators defined by kernels and closed ranges. In order to construct these studies, we touched on a number of concepts and theories related to the algebraic properties of the kernel and range, and we had to present the majority of the concepts related to operators with closed range. All of this is covered in greater depth in the monograph work mentioned in the introduction. And we didn't forget to supplement the studies with some useful examples to help you understand more.

Résumé

L'objectif principal de ce travail est de présenter les différents concepts de la théorie de Fredholm et leurs perturbations dans les espaces de Banach, ainsi que les définitions du spectre essentiel trouvées dans la littérature mathématique, en commençant par les travaux fondamentaux de Weyl. Cette théorie a été étudiée en relation avec diverses classes d'opérateurs bornés définis par des noyaux et des images fermées.

Afin de construire ces études, nous avons abordé un certain nombre de concepts et de théories liés aux propriétés algébriques du noyau et du domaine, et nous avons dû présenter la majorité des concepts liés aux opérateurs à domaine fermé. Tout cela est traité plus en profondeur dans l'ouvrage monographique mentionné en introduction. Nous n'avons pas oublié de compléter les études avec quelques exemples utiles pour vous aider à mieux comprendre.

CONTENTS

INTRODUCTION

The purpose of this work is to introduce the Fredholm theory of bounded linear operators defined on Banach spaces. The main points of interest are Fredholm operators and semi-Fredholm operators, but we also look at some of their generalizations recently investigated.

The purpose of this thesis is to present a survey of results pertaining to various types of Fredholm theory that can be found in the form of research papers scattered throughout the literature.

To accomplish this goal, we tackled the following tasks: Kato and Kato type operators are essentially described by their properties and relationships to other known classes of operators. In addition, a definition of the generalised Kato decomposition for bounded operators on Banach spaces is introduced. This decomposition property results from the classical treatment of perturbation theory of Kato [36] has greatly benefited from the work of many authors in the last ten years, particularly Mbekhta [37], [38] and Muller [39].

The operators that satisfy this property belong to a class that includes semi-Fredholm operators. Definitions of essential spectra: Fredholm, Weyl, and Browder, as well as B-Fredholm and quasi-Fredholm classes, are provided, as are many other concepts that will be addressed in the monograph of this work. This monograph's architecture will be discussed in greater depth. This work is divided into three chapters: **The first chapter** The theory has been examined in connection with various classes of linear (resp: bounded) operators defined by means of kernels and ranges, that is why we presented the first chapter in the name of **Kernel and Range generalisation** , in which we introduce the various relationships between the kernel and the range, which led us to define important invariant subspaces called **hyper-kernel** and **hyper-range**, denoted by $\mathcal{N}(\mathbf{T})^{\infty}$ and $\mathcal{R}^{\infty}(\mathbf{T})$ respectively. The beginning of this chapter contains also an important proprieties of some classical algebraic quantities associated with an operator, such as **the ascent**, **the descent** , **the nullity** , and **deficiency** of an operator. These quantities are the basic bricks in the construction of the most important classes of linear operators.

The second chapter: The theory has been also examined in connection with various classes of bounded operators defined with closed ranges in Banach spaces. Therefore as a second chapter, we presented **operators with closed range and decomposition**. Therefore , we discussed theories about operators that have the same properties, and they were mentioned above (**Kato**, **essentially Kato** and **Kato type operators**). In addition to **the minimum modulus** theories that serve this. we also dealt with a light study on the **bounded below operators** . But the most important and important work was about **compact operators and Riesz-schauder theory**,because it is considered a watershed step for the study of the classical Fredholm theory deals with the solution of an equation of the type

$$x(s) = y(s) + \lambda \int_a^b k(s,t)\, x(t)dt \quad (a \leqslant s \leqslant b),$$

where k is given continuous scalar function of two real variables with domain $[a,b] \times [a,b]$, y is a given continuous scalar function of a single real variable with domain [a,b], λ is a scalar, and the continuous x scalar function of a single real variable with domain $[a,b]$ is to be determined (in much of the literature the scalars are taken to be real, but the extension to complex scalars is straightforward). Such an equation is known as a **Fredholm integral equation** . we call the function k the **Fredholm-kernel** of the equation (and of the associated **integral operator K** given by the equation

$$\mathbf{K}(s) = \int_a^b k(s,t)\, x(t)dt \ \ (a \leqslant s \leqslant b),$$

for every continuous scalar function x with domain [a,b].

We also introduced a new concept in the decomposition of operators called **Kato decomposition**, which allowed us to provide a short overview about a invariant subspaces (**analytic cor** and **quasi-nilpotant part**, these concepts have been studied by [**??**] ,[**??**] and [**??**]. At the end of this chapter, we presented general concepts about **closed operators** in order to generalize the concepts of Fredholm theory later.

The third chapter: The third chapter, called **Fredholm theory**, is the core of this work , its concepts based precisely on the concepts of the previous tow chapters. We give a survey of results concerning various types of **Fredholm**, **semi-Fredholm**, **Weyl**, **Browder**, **quasi-Fredholm** and **B-Fredholm** operators etc. A section of this chapter is also devoted to study some **perturbation** ideals which accur in Fredholm theory . In particular we study the class of **compact perturbations**, and some relationships between these classes and **Calkin algebra**. In the last chapter, we provided an overview of the concepts of the **essential spectra**,because the theory of the essential spectra of linear operators in Banach space is a modern section of spectral analysis widely used in the mathematical and physical sense when when resolving a number of applications that can be formulated in terms of linear operators. Within the spectral theory lie a vast number of essential spectra defined for an individual operators, that have been introduced and investigated extensively.

Key words:

Kernel, range, Hyper-kernel, Hyper-range, Ascent, Descent, Nullity, Deficiency, index, Minimum Modulus, Kato operators, Essentially Kato operators , Saphar operators , Essentially Saphar operators ,Kato-type operators, Kato decomposition, Analityc core, Quasi-nilpotent part , quasi-nilpotent operators, Fredholm operators, Semi-Fredholm operatord , Fredholm perturbations, weyl operators, Browder operators, B-Fredholm operators, Quasi-Fredholm operators, Essential spectrum.

CHAPTER 1

KERNEL AND RANGE GENERALISATION

In this chapter we recall the basic algebraic properties kernel and range of a linear operator in a vector spaces. Let us start by setting the stage, introducing the basic notions necessary to study linear operators. Through this monograph, an operator means a linear transformation defined on vector space. Although many of the results in these monograph are valid for real vector spaces. we always assume that all vector spaces are complex infinite-dimensional vector spaces. It can be said that these notions can also be generalized to the complex infinite-dimensional Banach space.First we study the most important properties of the kernel and rang of the power $\mathbf{T}^n$. Next that we present classical quantities associated with an operator. These quantities, such as the ascent, descent, the nullity and the deficiency of an operator are defined in the first and second section .

1.1 Algebraic properties of kernel and range

Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector spaces over the real or complex numbers (over filed $\mathbb{K} = \{\mathbb{R},\mathbb{C}\}$) and $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the set of linear operators from $\mathcal{X}$ into $\mathcal{Y}$, if $\mathcal{X} = \mathcal{Y}$ we put $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X},\mathcal{X})$. For $\mathbf{T} \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ we denote by $D(\mathbf{T}) \subseteq \mathcal{X}$ its domain, $Ker\mathbf{T} = \{x \in D(\mathbf{T}), \quad \mathbf{T}x = 0\}$ its kernel and $\mathcal{R}(\mathbf{T}) = \{\mathbf{T}x, \quad x \in D(\mathbf{T})\}$ its range. By induction we define the iterates $\mathbf{T}^2$, $\mathbf{T}^2 \cdots$. For $n \geqslant 1$, $\mathbf{T}^n$ is the linear operator with domain

$$D(\mathbf{T}^n) = \left\{x \in D(\mathbf{T}) \ : \ \mathbf{T}^k x \in D(\mathbf{T}), \ k = 1, \cdots, n-1\right\},$$

and such that for each x in $D(\mathbf{T}^n)$

$$\mathbf{T}^n x = \mathbf{T}(\mathbf{T}^{n-1} x).$$

Also we define $\mathbf{T}^0 = \mathbf{I}$ is the identity operator from $\mathcal{X}$ into $\mathcal{X}$, then we have $D(\mathbf{T}^0) = \mathcal{X}$ $Ker\mathbf{T}^0 = \{0\}$, and $\mathcal{R}(\mathbf{T}^0) = \mathcal{X}$. If $n \geqslant 1$:

$$Ker\mathbf{T}^n = \Big\{x \in D(\mathbf{T}^n), \quad \mathbf{T}^n x = 0\Big\},$$

and

$$\mathcal{R}(\mathbf{T}^n) = \Big\{\mathbf{T}^n x, \quad x \in D(\mathbf{T}^n)\Big\};$$

If $n = 0$:

$$\mathbf{T}^0 = \mathbf{I}\ ,\ D(\mathbf{T}^0) = \mathcal{X}\ ,\ Ker\mathbf{T}^0 = \{0\},\ and\ \mathcal{R}(\mathbf{T}^0) = \mathcal{X}.$$

Let n and m be non-negative integers. Then $x \in D(\mathbf{T}^{n+m})$ if and only if $\mathbf{T}^n \in D(\mathbf{T}^m)$, and in this case

$$\mathbf{T}^m(\mathbf{T}^n) = \mathbf{T}^{n+m}x.$$

1.1.1 The subspaces $Ker\mathbf{T}^n$ and $\mathcal{R}(\mathbf{T}^n)$

We will see that the kernels and the ranges of the iterates of a linear operator $\mathbf{T}$, defined on a vector space $\mathcal{X}$, form two increasing and decreasing chains, respectively. In this section we shall consider operators for which one, or both, of these chains becomes constant at some $n \in \mathbb{N}$.

The kernels and the ranges of the power $\mathbf{T}^n$ of a linear operator $\mathbf{T}$ on a vector space $\mathcal{X}$ form the following two sequences of subspaces:

$$Ker\mathbf{T}^0 = \{0\} \subseteq Ker\mathbf{T} \subseteq Ker\mathbf{T}^2 \subseteq \cdots$$

and

$$\mathcal{R}(\mathbf{T}^0) = \mathcal{X} \supseteq \mathcal{R}(\mathbf{T}) \supseteq \mathcal{R}(\mathbf{T}^2) \supseteq \cdots.$$

Generally all these inclusions are strict. For every $n \in \mathbb{N}_0$. Thus $\{Ker\mathbf{T}^n\}$ and $\{\mathcal{R}(\mathbf{T})\}$ are non-decreasing and non-increasing (in the inclusion ordering) sequences of subsets of $\mathcal{X}$, respectively.

Lemma 1.1. *:* $Ker\mathbf{T}^n$ *and* $\mathcal{R}(\mathbf{T}^n)$ *are* $\mathbf{T}$*-invariant subspaces of* $\mathcal{X}$ *for each* $n \in \mathbb{N}$.

▸**Proof:**

The kernel space of $\mathbf{T}^n$ is a T-invariant subspace of $\mathcal{X}$, that is, $\mathbf{T}(Ker\mathbf{T}^n) \subseteq Ker\mathbf{T}^n$ for every positive integer n. Indeed, if $x \in Ker\mathbf{T}^n$, then $\mathbf{T}^n x = 0$ and therefore, $\mathbf{T}^n(\mathbf{T}x) = \mathbf{T}(\mathbf{T}^n x) = 0$, i.e. $\mathbf{T}x \in Ker\mathbf{T}^n$.

$\mathcal{R}(\mathbf{T}^n)$ The rang space of $\mathbf{T}^n$ is a $\mathbf{T}$-invariant subspace of $\mathcal{X}$, that

$$\mathbf{T}(\mathcal{R}(\mathbf{T}^n)) = \mathcal{R}(\mathbf{T}^{n+1}) \subseteq \mathcal{R}(\mathbf{T}^n).$$

■

The following proposition shows the stability of kernel and range.

Proposition 1.1. *: Let p be an arbitrary integer in $\mathbb{N}_0$.*

1. *If $Ker\mathbf{T}^{p+1} = Ker\mathbf{T}^p$, then $Ker\mathbf{T}^{n+1} = Ker\mathbf{T}^n$, for every $n \geqslant p$;*
2. *If $\mathcal{R}(\mathbf{T}^{p+1}) = \mathcal{R}(\mathbf{T}^p)$, then $\mathcal{R}(\mathbf{T}^{n+1}) = \mathcal{R}(\mathbf{T}^n)$, for every $n \geqslant p$.*

▸**Proof:**
(1). Rewrite the statements in (1) as follows.

$$Ker\mathbf{T}^{p+1} = Ker\mathbf{T}^p \Longrightarrow Ker\mathbf{T}^{p+k+1} = Ker\mathbf{T}^{p+k} \ \ for\ every\ k \geqslant 0.$$

The claimed result holds trivially for $k = 0$. Suppose it holds for some $k \geqslant 0$. Take an arbitrary $x \in Ker\mathbf{T}^{p+k+2}$ so that $\mathbf{T}^{p+k+1}(\mathbf{T}x) = \mathbf{T}^{p+k+2}x = 0$. Thus $\mathbf{T}x \in Ker\mathbf{T}^{p+k+1} = Ker\mathbf{T}^{p+k}$ and so $\mathbf{T}^{p+k+1}x = \mathbf{T}^{p+k}(\mathbf{T}x) = 0$, which implies $x \in Ker\mathbf{T}^{p+k+1}$. Hence $Ker\mathbf{T}^{p+k+2} \subseteq Ker\mathbf{T}^{p+k+1}$. However, $Ker\mathbf{T}^{p+k+1} \subseteq Ker\mathbf{T}^{p+k+2}$ since $\{Ker\mathbf{T}^n\}$ is nondecreasing, and therefore $Ker\mathbf{T}^{p+k+2} = Ker\mathbf{T}^{p+k+1}$. Then the claimed result holds for $k+1$ whenever it holds for k, which completes the proof of (1) by induction.

(2). Rewrite the statements in (2) as follows.

$$\mathcal{R}(\mathbf{T}^{p+1}) = \mathcal{R}(\mathbf{T}^p) \Longrightarrow \mathcal{R}(\mathbf{T}^{p+k+1}) = \mathcal{R}(\mathbf{T}^{p+k}) \ \ for\ every\ k \geqslant 1.$$

The claimed result holds trivially for $k = 0$. Suppose it holds for some integer $k \geqslant 0$. Take an arbitrary $y \in \mathcal{R}(\mathbf{T}^{p+k+1})$ so that $y = \mathbf{T}^{p+k+1}x = \mathbf{T}(\mathbf{T}^{p+k}x)$ for some $x \in \mathbf{E}$, and hence $y = \mathbf{T}u$ for some $u \in \mathcal{R}(\mathbf{T}^{p+k})$. If $\mathcal{R}(\mathbf{T}^{p+k}) = \mathcal{R}(\mathbf{T}^{p+k+1})$, then $u \in \mathcal{R}(\mathbf{T}^{p+k+1})$, and so $y = \mathbf{T}(\mathbf{T}^{p+k+1}v)$ for some $v \in \mathbf{E}$. Thus $y \in \mathcal{R}(\mathbf{T}^{p+k+2})$. Therefore $\mathcal{R}(\mathbf{T}^{p+k+1}) \subseteq \mathcal{R}(\mathbf{T}^{p+k+2})$. Since the sequence $\{\mathcal{R}(\mathbf{T})\}$ is nonincreasing we get $\mathcal{R}(\mathbf{T}^{p+k+2}) \subseteq \mathcal{R}(\mathbf{T}^{p+k+1})$. Hence $\mathcal{R}(\mathbf{T}^{p+k+2}) = \mathcal{R}(\mathbf{T}^{p+k+1})$. Thus the claimed result holds for $k+1$ whenever it holds for k, which completes the proof of (2) by induction.

■

The next results exhibit some useful connections between the kernels and the ranges of the iterates $\mathbf{T}^n$ of an operator $\mathbf{T}$ on a vector space $\mathcal{X}$.

Corollary 1.1. *: For every $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, on a vector space $\mathcal{X}$. then*

1. *If $Ker\mathbf{T} = \{0\}$ then $Ker\mathbf{T}^n = \{0\}$, for all $n \geqslant 0$;*
2. *If $\mathcal{R}(\mathbf{T}) = \mathcal{X}$ then $\mathcal{R}(\mathbf{T}^n) = \mathcal{X}$, for all $n \geqslant 0$.*

Remark 1.1. *: We can observe that, if $\mathbf{T}$ is injective (resp: surjective) then $\mathbf{T}^n$ is also injective (resp: surjective). As a result if $\mathbf{T}$ is bijective , then $\mathbf{T}^n$ is also bijective.*

Lemma 1.2. *([4], **Lemma 1.4**): For every $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ on a vector space $\mathcal{X}$ we have .*

$$\mathbf{T}^m(Ker\mathbf{T}^{n+m}) = \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n \text{ for all } m,n \in \mathbb{N}.$$

▸**Proof:**

If we take $x \in Ker\mathbf{T}^{n+m}$ then we have $\mathbf{T}^m x \in \mathcal{R}(\mathbf{T}^m)$ and $\mathbf{T}^n(\mathbf{T}^m x) = 0$, so that $\mathbf{T}^m(Ker\mathbf{T}^{n+m}) \subseteq \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n$.

Cnversely, if $y \in \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n$,then $y = \mathbf{T}^m x$ and $x \in Ker\mathbf{T}^{n+m}$,so the opposite inclusion is verified.

■

Theorem 1.1. *([4], **Theorem 1.5**): Let $\mathcal{X}$ be a vector space and $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, the following statements are equivalent*

1. *$Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^m)$ for each $m \in \mathbb{N}$;*
2. *$Ker\mathbf{T}^n \subseteq \mathcal{R}(\mathbf{T})$ for each $n \in \mathbb{N}$;*
3. *$Ker\mathbf{T}^n \subset \mathcal{R}(\mathbf{T}^m)$ for each $(n,m) \in \mathbb{N}^2$;*
4. *$Ker\mathbf{T}^n = \mathbf{T}^m(Ker\mathbf{T}^{n+m})$ for each $(n,m) \in \mathbb{N}^2$.*

▸**Proof:**

4 $\Rightarrow$ **3** : If we apply **Lemma** 1.2 m we obtain that for each $n \in \mathbb{N}$

$$Ker\mathbf{T}^n = \mathbf{T}^m(Ker\mathbf{T}^{n+m}) = \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n \subseteq \mathcal{R}(\mathbf{T}^m).$$

3 $\Rightarrow$ **2** : We have $\mathcal{R}(\mathbf{T}^m) \subseteq \mathcal{R}(\mathbf{T})$, so we obtain that $Ker\mathbf{T}^n \subseteq \mathcal{R}(\mathbf{T})$ for each $n \in \mathbb{N}$.

2 ⇒ 1 : If we apply the inclusion (2) to the operator $\mathbf{T}^m$ we obtain $Ker(T^m)^n \subseteq \mathcal{R}(\mathbf{T}^m)$ and hence $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^m)$, since $Ker\mathbf{T} \subseteq Ker\mathbf{T}^{mn}$.

1 ⇒ 4 : If we apply the inclusion (1) to the operator $\mathbf{T}^n$ we obtain

$$Ker(\mathbf{T}^n) \subseteq \mathcal{R}((\mathbf{T}^n)^m) \subseteq \mathcal{R}(\mathbf{T}^n).$$

By **Lemma1.2** we then have

$$\mathbf{T}^m(Ker\mathbf{T}^{n+m}) = \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n = Ker\mathbf{T}^n,$$

so the proof is complete.

■

Lemma 1.3. *: Let $\mathcal{X}$ be a vector space and $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, for all $n \in \mathbb{N}$ and if $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^n)$ Then there exist $m \in \mathbb{N}$ such that :*

$$Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{m+k}), \quad for\ all\ k \in \mathbb{N}. \tag{1.1}$$

▸**Proof:**

Obviously, if $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^n)$ for all $n \in \mathbb{N}$, then

$$Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^n) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{n+k}) = Ker\mathbf{T},$$

for all integers $k \geqslant 0$.

■

Remark 1.2. *: If one of the **Theorem** 1.1 statements is valid , then **(**1.1**)** remains true.*

Definition 1.1. *: Let $\mathcal{X}$ be a vector space and $\mathbf{T} \in \mathcal{L}(\mathcal{X})$.Then*

the hyper-rang of $\mathbf{T}$ is the subspace

$$\mathcal{R}^\infty(\mathbf{T}) = \bigcap_{n\in\mathbb{N}} \mathcal{R}(\mathbf{T}^n).$$

The hyper-kernel of $\mathbf{T}$ is the subspace

$$\mathcal{N}^\infty(\mathbf{T}) = \bigcup_{n\in\mathbb{N}} Ker\mathbf{T}^n.$$

Corollary 1.2. *: $\mathcal{R}^{\infty}(\mathbf{T})$ and $\mathcal{N}^{\infty}(\mathbf{T})$ are $\mathbf{T}$-invariant subspaces of $\mathcal{X}$(i.e: $\mathbf{T}(\mathcal{R}^{\infty}(\mathbf{T})) \subseteq \mathcal{R}^{\infty}(\mathbf{T})$ and $\mathbf{T}(\mathcal{N}^{\infty}(\mathbf{T})) \subseteq \mathcal{N}^{\infty}(\mathbf{T})$), but generally are not closed.*

▸**Proof:**

The prove follows immediately from **Definition**1.1 and **Lemma**1.1.

■

Lemma 1.4. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. If there exist $m \in \mathbb{N}$ such that (1.1) holds , then*

$$Ker\mathbf{T}^n \cap \mathcal{R}(\mathbf{T}^m) \subseteq \mathcal{R}^{\infty}(\mathbf{T}), \quad for\ all \ \ n \geqslant 1. \tag{1.2}$$

▸**Proof:**

To prove (1.2), we proceed by indication , the hypotheses implies that

$$Ker\mathbf{T}^n \cap \mathcal{R}(\mathbf{T}^m) \subseteq \mathcal{R}(\mathbf{T}^{m+k}), \qquad for\ \ all\ \ k.$$

On other hand $Ker\mathbf{T}^n \cap \mathcal{R}(\mathbf{T}^m) \subseteq \cap_{i=0}^{m}\mathcal{R}(\mathbf{T}^i)$. hence $Ker\mathbf{T}^n \cap \mathcal{R}(\mathbf{T}^m) \subseteq \cap_{i=0}^{\infty}\mathcal{R}(\mathbf{T}^i)$, this proved the case n=1 . Now assume that the equality (1.2) is vitrified for n. Let $x \in Ker\mathbf{T}^{n+1} \cap \mathcal{R}(\mathbf{T}^m)$ and $k \geqslant m$ then

$$\begin{aligned} x \in Ker\mathbf{T}^{n+1} \cap \mathcal{R}(\mathbf{T}^m) \ &\Rightarrow x \in Ker\mathbf{T}^{n+1} \quad and \quad x \in \mathcal{R}(\mathbf{T}^m) \\ &\Rightarrow \mathbf{T}x \in Ker\mathbf{T}^n \ \ and \ \ \mathbf{T}^m y = x, \ \ y \in \mathcal{X} \\ &\Rightarrow \mathbf{T}x \in Ker\mathbf{T}^n \ and \ \mathbf{T}^{m+1}y = \mathbf{T}x, \ y \in \mathcal{X} \\ &\Rightarrow \mathbf{T}x \in Ker\mathbf{T}^n \cap \mathcal{R}(\mathbf{T}^m). \end{aligned}$$

And by the hypotheses of induction we have $Ker\mathbf{T}^n \cap \mathcal{R}(\mathbf{T}^m) \subseteq \mathcal{R}(\mathbf{T}^{m+k})$, hence $\mathbf{T}x = \mathbf{T}^{k+1}y$, $y \in \mathcal{X}$ and $x - \mathbf{T}^k y \in Ker\mathbf{T}$, so $x = \mathbf{T}^k y + u$, $u \in Ker\mathbf{T}$, since $k \geqslant m$ then $u \in \mathcal{R}(\mathbf{T}^m)$, so $x \in \mathcal{R}(\mathbf{T}^k) + (\mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}) \subset \mathcal{R}(\mathbf{T}^k)$. Hence $Ker\mathbf{T}^{n+1} \cap \mathcal{R}(\mathbf{T}^m) \subset \mathcal{R}(\mathbf{T}^k)$, for all $k \geqslant m$. This proves (1.2).

■

Proposition 1.2. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, $\mathcal{X}$ be a vector space and k is a fixed negative integer, then The sequence of subspaces $\{\mathcal{R}(\mathbf{T}^n) \cap Ker\mathbf{T}\}$ is constant for $n \geqslant k$, if and only if $\mathcal{R}(\mathbf{T}^k) \cap Ker\mathbf{T} = \mathcal{R}^{\infty}(\mathbf{T}) \cap Ker\mathbf{T}$.*

▸**Proof:**

It is clear that if $\mathcal{R}(\mathbf{T}^k) \cap Ker\mathbf{T} = \mathcal{R}^{\infty}(\mathbf{T}) \cap Ker\mathbf{T}$ then The sequence $\{\mathcal{R}(\mathbf{T}^n) \cap Ker\mathbf{T}\}$ is constant for $n \geqslant k$.

Conversely, if The sequence of subspaces $\{\mathcal{R}(\mathbf{T}^n) \cap Ker\mathbf{T}\}$ is constant for $n \geqslant k$, then $\mathcal{R}(\mathbf{T}^n) \supseteq \mathcal{R}(\mathbf{T}^k) \cap Ker\mathbf{T}$ for all $n \geqslant k$, so that $\mathcal{R}(\mathbf{T}^{\infty}) \supseteq \mathcal{R}(\mathbf{T}^k) \cap Ker\mathbf{T}$, which clearly implies that $\mathcal{R}(\mathbf{T}^k) \cap Ker\mathbf{T} = \mathcal{R}^{\infty}(\mathbf{T}) \cap Ker\mathbf{T}$.

■

Proposition 1.3. *: Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$*,* $\mathcal{X}$ *be a vector space and* k *is a fixed negative integer, then The sequence* $\{\mathcal{R}(\mathbf{T}^n) + Ker\mathbf{T}\}$ *is constant for* $n \geqslant k$*, if and only if* $\mathcal{R}(\mathbf{T}^k) + Ker\mathbf{T} = \mathcal{R}^\infty(\mathbf{T}) + Ker\mathbf{T}$.

▸**Proof:**

It is clear that if $\mathcal{R}(\mathbf{T}^k) + Ker\mathbf{T} = \mathcal{R}^\infty(\mathbf{T}) + Ker\mathbf{T}$. then The sequence $\{\mathcal{R}(\mathbf{T}^n) + Ker\mathbf{T}\}$ is constant for $n \geqslant k$.

Conversely, if sequence $\{\mathcal{R}(\mathbf{T}^n)+Ker\mathbf{T}\}$ is constant for $n \geqslant k$, then $\mathcal{R}(T^n) \supseteq \mathcal{R}(\mathbf{T}^k)+Ker\mathbf{T}$ for all $n \geqslant k$, so that $\mathcal{R}(T^\infty) \supseteq \mathcal{R}(\mathbf{T}^k) + Ker\mathbf{T}$, which clearly implies that $\mathcal{R}(\mathbf{T}^k) + Ker\mathbf{T} = \mathcal{R}^\infty(\mathbf{T}) + Ker\mathbf{T}$.

■

Remark 1.3. *: The statements of* ***Proposition*****1.2** *and* ***Proposition*****1.3** *are equivalent.*

Corollary 1.3. *([4],* ***Corollary 1.6****): Let* $\mathcal{X}$ *be a vector space and* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$*. Then the statements of* ***Theorem*****1.1** *are equivalent to each of the following inclusions:*

1. $Ker\mathbf{T} \subseteq \mathcal{R}^\infty(\mathbf{T})$;
2. $\mathcal{N}^\infty(\mathbf{T}) \subseteq \mathcal{R}(\mathbf{T})$;
3. $\mathcal{N}^\infty(\mathbf{T}) \subseteq \mathcal{R}^\infty(\mathbf{T})$.

Remark 1.4. *: We can observe that if one of the* ***Corollary*****1.3** *statements is valid, then (1.1) remains valid.*

Proposition 1.4. *: Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$*. If there exist* $m \in \mathbb{N}$ *such that (1.1) holds , then*

$$\mathbf{T}(\mathcal{R}^\infty(\mathbf{T})) = \mathcal{R}^\infty(\mathbf{T}).$$

▸**Proof:**

The fact that $\mathcal{R}^\infty(\mathbf{T})$ is invariant by $\mathbf{T}$ (see **Corollary**1.2), then the proof is done if we show that $\mathcal{R}^\infty(\mathbf{T}) \subseteq \mathbf{T}(\mathcal{R}^\infty(\mathbf{T}))$. Let $y \in \mathcal{R}^\infty(\mathbf{T})$ then $y \in \mathcal{R}(\mathbf{T}^n)$, for every $n \in \mathbb{N}$, so there exists $x_k \in \mathcal{X}$ such that $y = \mathbf{T}^{m+k}x_k$ for every $k \in \mathbb{N}$. If we set

$$z_k = \mathbf{T}^m x_1 - \mathbf{T}^{m+k-1}x, \quad k \in \mathbb{N}.$$

Then $z_k \in \mathcal{R}(\mathbf{T}^m)$ and since $\mathbf{T}z_k = \mathbf{T}^{m+1}x_1 - \mathbf{T}^{m+k}x = y - y = 0$, we also have $z_k \in Ker\mathbf{T}$, thus $z_k \in \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}$ and since $\mathcal{R}(\mathbf{T}^{m+k}) \cap Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^{m+k-1}) \cap Ker\mathbf{T}$ we deduce that $z_k \in \mathcal{R}(\mathbf{T}^{m+k-1})$. This implies that

$$y = \mathbf{T}^m x_1 = z_k + \mathbf{T}^{m+k-1}x_k \in \mathcal{R}(\mathbf{T}^{m+k-1}), \quad for\ all\ k \in \mathbb{N}$$

and therefore $y \in \mathcal{R}^\infty(\mathbf{T})$, we may conclude that $\mathcal{R}^\infty(\mathbf{T}) \subseteq \mathbf{T}(\mathcal{R}^\infty(\mathbf{T}))$, then $\mathcal{R}^\infty(\mathbf{T}) = \mathbf{T}(\mathcal{R}^\infty(\mathbf{T}))$.

■

The notion of hyper-rang and hyper-kernel holds for every bounded operator of a normed space (resp: Banach space) into itself . So when $\mathcal{X}$ is a normed space and $\mathbf{T}$ is bounded linear operator, then the subspaces $Ker\mathbf{T}^n$ are closed , but there is nothing to suggest that either $\mathcal{N}^\infty(\mathbf{T})$ or $\mathcal{R}^\infty(\mathbf{T})$ should be closed .

Theorem 1.2. *: if $\mathcal{X}$ is a normed space, and $\mathbf{T}$ is bounded linear operator . Then*

$$\mathbf{T}^{-1}(\mathcal{N}^\infty(\mathbf{T})) \subseteq \mathcal{N}^\infty(\mathbf{T}).$$

▸**Proof:**
Observe that if $n \in \mathbb{N}$ is arbitrary and $x \in \mathcal{X}$ then

$$\mathbf{T}x \in Ker\mathbf{T}^n \Longrightarrow x \in Ker\mathbf{T}^{n+1} \subseteq \mathcal{N}^\infty(\mathbf{T}).$$

Therefore $\mathbf{T}^{-1}(\mathcal{N}^\infty(\mathbf{T})) \subseteq \mathcal{N}^\infty(\mathbf{T})$.

■

Theorem 1.3. *([12], Theorem 7.8.3): if $\mathcal{X}$ is a normed space, and $\mathbf{T}$ is bounded linear operator . If $\mathbf{S}$ is bounded linear operator commutes with $\mathbf{T}$, then*

$$\mathbf{S}(\mathcal{R}^\infty(\mathbf{T})) \subseteq \mathcal{R}^\infty(\mathbf{T}) \quad and \quad \mathbf{S}(\mathcal{N}^\infty(\mathbf{T})) \subseteq \mathcal{N}^\infty(\mathbf{T}).$$

The following subspace, is important $\mathbf{T}$-invariant sub-spaces, is called the algebraic core $\mathrm{C}(\mathbf{T})$ of $\mathbf{T}$ is defined to be the largest linear subspace $\mathcal{M}$ such that $\mathbf{T}(\mathcal{M}) = \mathcal{M}$. Of course if $\mathbf{T}$ is surjective then $\mathrm{C}(\mathbf{T}) = \mathcal{X}$ and in general for every $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ we have $\mathrm{C}(\mathbf{T}) = \mathbf{T}^n(\mathrm{C}(\mathbf{T})) \subseteq \mathcal{R}(\mathbf{T}^n)$ for all $n \in \mathbb{N}$. From that it follows that $\mathrm{C}(\mathbf{T}) \subseteq \mathcal{R}^\infty(\mathbf{T})$. and we have:

$$\mathrm{C}(\mathbf{T}) = \left\{ x \in \mathcal{X} : \quad \exists (x_n)_{n \leqslant 0} \subset \mathcal{X}, \quad x_0 = x \text{ and } \mathbf{T}x_n = x_n - 1 \qquad \forall n \geqslant 1 \right\}.$$

we will prove this statement in the next theorem.

Theorem 1.4. *([4], **Theorem 1.8**): For a linear operator $\mathbf{T}$ on a vector space $\mathcal{X}$ the following statements are equivalent:*

1. $x \in C(\mathbf{T})$;
2. *There exists a sequence* $(u_n) \subset \mathcal{X}$ *such that* $x = u_0$ *and* $\mathbf{T}u_{n+1} = u_n$ *for every* $n \in \mathbb{Z}^+$.

▸**Proof:**

Let $\mathcal{M}$ denote the set of all $x \in \mathcal{X}$ for which there exists a sequence $(u_n) \subset \mathcal{X}$ such that $x = u_0$ and $Tu_{n+1} = u_n$ for all $n \in \mathbb{Z}^+$. We show first that $C(\mathbf{T}) \subseteq \mathcal{M}$.

Let $x \in C(\mathbf{T})$. From the equality $\mathbf{T}(C(\mathbf{T})) = C(\mathbf{T})$, we obtain that there is an element $u_1 \in C(\mathbf{T})$ such that $x = \mathbf{T}u_1$. Since $u_1 \in C(\mathbf{T})$, the same equality implies that there exists $u_2 \in C(\mathbf{T})$ such that $u_1 = \mathbf{T}u_2$. By repeating this process we can find the desired sequence (u_n), with $n \in \mathbb{Z}^+$, for which $x = u_0$ and $\mathbf{T}u_{n+1} = u_n$. Therefore $C(\mathbf{T}) \subseteq \mathcal{M}$. Conversely, to show the inclusion $\mathcal{M} \subseteq C(\mathbf{T})$ it suffices to prove, since $\mathcal{M}$ is a linear subspace of $\mathcal{X}$, that $\mathbf{T}(\mathcal{M}) = \mathcal{M}$.

Let $x \in \mathcal{M}$ and let (u_n), $n \in \mathbb{Z}^+$, be a sequence for which $x = u_0$ and $\mathbf{T}u_{n+1} = u_n$. Define (w_n) by

$$w_0 = \mathbf{T}x \text{ and } w_n := u_{n-1}, n \in \mathbb{Z}^+.$$

Then

$$w_n = u_{n-1} = \mathbf{T}u_n = \mathbf{T}w_{n+1},$$

and hence the sequence satisfies the definition of $\mathcal{M}$. Hence $w_0 = \mathbf{T}x \in \mathcal{M}$, and therefore $\mathbf{T}(M) \subseteq \mathcal{M}$. On the other hand, to prove the opposite inclusion, $\mathcal{M} \subseteq \mathbf{T}(M)$, let us consider an arbitrary element $x \in \mathcal{M}$ and let $(u_n)_{n \in \mathbb{Z}^+}$ be a sequence such that the equalities $x = u_0$ and $\mathbf{T}u_{n+1} = u_n$ hold for every $(n \in \mathbb{Z}^+)$ Since $x = u_0 = \mathbf{T}u_1$ it suffices to verify that $u_1 \in \mathcal{M}$. To see that let us consider the following sequence:

$$w_0 := u_1 \text{ and } w_n := u_{n+1}.$$

Then

$$w_n = u_{n+1} = \mathbf{T}u_{n+2} = \mathbf{T}w_{n+1} \text{ for all } n \in \mathbb{Z}^+,$$

and hence u_1 belongs to $\mathcal{M}$. Therefore $\mathcal{M} \subseteq \mathbf{T}(\mathcal{M})$, and hence $\mathcal{M} = \mathbf{T}(\mathcal{M})$.

■

Lemma 1.5. *: Let* $\mathbf{T}, \mathbf{S} \in \mathcal{L}(\mathcal{X})$ *such that* $\mathbf{TS} = \mathbf{ST}$, *then*

$$C(\mathbf{TS}) \subseteq C(\mathbf{T}) \cap C(\mathbf{S}).$$

▸**Proof:**

Let $x \in C(\mathbf{TS})$, then there exists $(x_n)_{n\geqslant 0} \subset \mathcal{X}$ such that $x_0 = x$, $\mathbf{TS}x_n = x_{n-1}$. Let $(y_n)_{n\geqslant 0}$ be defined by $y_n = \mathbf{S}^n x_n$, then $y_0 = x_0 = x$, hence $\mathbf{T}y_n = \mathbf{TS}^n x_n = \mathbf{TS}^{n-1}x_{n-1} = y_{n-1}$, consequently $x \in C(\mathbf{T})$. Similarly we have $x \in C(\mathbf{S})$.

■

Corollary 1.4. : *Let* $\mathbf{T}, \mathbf{S} \in \mathcal{L}(\mathcal{X})$, *then* $C(\mathbf{T}) = C(\mathbf{T}^n)$ *, for all* $n \geqslant \mathbb{N}$.

The next result shows that under certain purely algebraic conditions the algebraic core and the hyper-range of an operator coincide.

Lemma 1.6. *([4], **Lemma 1.9.**): Let* $\mathbf{T}$ *be a linear operator on vector space* $\mathcal{X}$. *Suppose that there exists an* $m \in \mathbb{N}$ *such that*

$$Ker(\mathbf{T}) \cap \mathcal{R}(\mathbf{T}^m) = Ker(\mathbf{T}) \cap \mathcal{R}(\mathbf{T}^{m+k}) \text{ for all integers } k \geqslant 0.$$

Then $C(\mathbf{T}) = \mathcal{R}^\infty(\mathbf{T})$

▸**Proof:**

We have only to prove that $\mathcal{R}^\infty(\mathbf{T}) \subseteq C(\mathbf{T})$. By **Proposition1.4** we show that $\mathbf{T}(\mathcal{R}^\infty(\mathbf{T})) = \mathcal{R}^\infty(\mathbf{T})$. Evidently the inclusion $\mathbf{T}(\mathcal{R}^\infty(\mathbf{T})) \subseteq \mathcal{R}^\infty(\mathbf{T})$ holds for every linear operator, so we need only to prove the opposite inclusion. Let

$$\mathbf{D} = Ker(\mathbf{T}) \cap \mathcal{R}(\mathbf{T}^m).$$

Obviously we have

$$\mathbf{D} = Ker(\mathbf{T}) \cap \mathcal{R}(\mathbf{T}^m) = Ker(\mathbf{T}) \cap \mathcal{R}^\infty(\mathbf{T}).$$

Let us now consider an element $y \in \mathcal{R}^\infty(\mathbf{T})$. Then $y \in \mathcal{R}(\mathbf{T}^n)$ for each $n \in \mathbb{N}$, so there exists an $x_k \in \mathcal{X}$ such that $y = \mathbf{T}^{m+k}x_k$ for every $k \in \mathbb{N}$. If we set

$$z_k = \mathbf{T}^m x_1 - \mathbf{T}^{m+k-1}x_k \qquad (k \in \mathbb{N}),$$

then $z_k \in \mathcal{R}(\mathbf{T}^m)$ and since

$$\mathbf{T}z_k = \mathbf{T}^{m+1}x_1 - \mathbf{T}^{m+k}x_k = y - y = 0,$$

we also have $z_k \in Ker(\mathbf{T})$. Thus $z_k \in \mathbf{D}$, and from the inclusion

$$\mathbf{D} = Ker(\mathbf{T}) \cap \mathcal{R}(\mathbf{T}^{m+k}) \subseteq Ker(\mathbf{T}) \cap \mathcal{R}(\mathbf{T}^{m+k-1}),$$

it follows that $z_k \in \mathcal{R}(\mathbf{T}^{m+k-1})$. This implies that

$$\mathbf{T}^m x_1 = z_k + \mathbf{T}^{m+k-1} x_k \in \mathcal{R}(\mathbf{T}^{m+k-1}),$$

for each $k \in \mathbb{N}$, and therefore $\mathbf{T}^m x_1 \in \mathcal{R}^\infty(\mathbf{T})$. Finally, from

$$\mathbf{T}(\mathbf{T}^m x_1) = \mathbf{T}^{m+1} x_1 = y,$$

we may conclude that $y \in \mathbf{T}(\mathcal{R}^\infty(\mathbf{T}))$. Therefore $\mathcal{R}^\infty(\mathbf{T}) \subseteq T(\mathcal{R}^\infty(\mathbf{T}))$, so the proof is complete.

■

Proposition 1.5. *: Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, *if* $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^n)$ *. Then*

$$\mathrm{C}(\mathbf{T}) = \mathcal{R}^\infty(\mathbf{T}).$$

▸**Proof:**

These results immediately follows from **Lemma 1.3** and **Lemma1.6** .

■

Remark 1.5. *: We can note that, if one of the **Theorem 1.1** or **Corollary 1.3** statements is valid, then* $\mathrm{C}(\mathbf{T}) = \mathcal{R}^\infty(\mathbf{T})$ *remains valid.*

1.1.2 Ascent and descent of an operator

We have already seen that the kernels and the ranges of the iterates of a linear operator $\mathbf{T}$, defined on a vector space $\mathcal{X}$, form two increasing and decreasing chains, respectively. In this section we shall consider operators for which one, or both, of these chains becomes constant at some $n \in \mathbb{N}$.

Now we will define very important classical algebraic quantities associated with an operator. They are the ascent and the descent.

Definition 1.2. *: Let* $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup +\{\infty\}$ *denote the set of all extended non-negative integers with its natural (extended) ordering. The ascent and descent of an operator* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ *are defined as follows.*

The ascent of $\mathbf{T}$*,* $asc(\mathbf{T})$*, is the least (extended) non-negative integer for which* $Ker\mathbf{T}^{n+1} = Ker\mathbf{T}^n$*:*

$$asc(\mathbf{T}) = \inf\Big\{n \in \overline{\mathbb{N}}_0 : Ker\mathbf{T}^{n+1} = Ker\mathbf{T}^n\Big\}.$$

and the descent of $\mathbf{T}$*,* $dsc(\mathbf{T})$*, is the least (extended) non-negative integer for which* $\mathcal{R}(\mathbf{T}^{n+1}) = \mathcal{R}(\mathbf{T}^n)$*:*

$$dsc(\mathbf{T}) = \inf\Big\{n \in \overline{\mathbb{N}}_0 : \mathcal{R}(\mathbf{T}^{n+1}) = \mathcal{R}(\mathbf{T}^n)\Big\}.$$

we can write also, $\mathbf{T}$ is said to have finite ascent if $\mathcal{N}^\infty(\mathbf{T}) = Ker\mathbf{T}^k$ for some positive integer k. Analogously, $\mathbf{T}$ is said to have finite descent if $\mathcal{R}^\infty(\mathbf{T}) = \mathcal{R}(\mathbf{T}^k)$ for some k.

Remark 1.6. *: Take any operator* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ *on a vector space* $\mathcal{X}$*. Then we have the following notions*

$$asc(\mathbf{T}) = n < \infty \;\Leftrightarrow Ker\mathbf{T}^{n+1} = Ker\mathbf{T}^n \text{ and } dsc(\mathbf{T}) = n < \infty \;\Leftrightarrow\; \mathcal{R}(\mathbf{T}^{n+1}) = \mathcal{R}(\mathbf{T}^n);$$

$$asc(\mathbf{T}) = \infty \Leftrightarrow\; Ker\mathbf{T}^n \subsetneq Ker\mathbf{T}^{n+1} \text{ and } dsc(\mathbf{T}) = \infty \;\Leftrightarrow\; \mathcal{R}(\mathbf{T}^n) \supsetneq \mathcal{R}(\mathbf{T}^{n+1}).$$

All following results are originally taken from ***Definition** 1.2*.

Now we give examples of descent and ascent of operators defined on ℓ^p $(1 \leqslant p \leqslant \infty)$, the Banach space of all p-summable sequences (bounded sequences for $p = \infty$) of complex numbers under the stander p-norm on it.

Example 1.1. *: Let* $\mathbf{T}$ *be defined by* $\mathbf{T}x = y$*, where* $x = (x_n)_{n\geqslant 0}$ *and* $y = (y_n)_{n\geqslant 0}$ *are related by*

$$y_n = \begin{cases} x_{n+2} & if \quad n \text{ is odd} \\ x_0 & if \quad n = 0, 2 \\ x_{n-2} & if \quad n \text{ is even and } n \geqslant 4 \end{cases}$$

Then for each $n \in \mathbb{N}$, $e_{n+2} \in Ker(\mathbf{T}^{n+1})$ *whilst* $e_{n+2} \notin Ker(\mathbf{T}^{n})$. *Hence* $Ker(\mathbf{T}^{n}) \neq Ker(\mathbf{T}^{n+1})$, $asc(\mathbf{T}) = \infty$. *Further for each* $n \in \mathbb{N}$, $\mathcal{R}(\mathbf{T}) = \Big\{(y_n) \in \ell^p(\mathbb{N}) \ : y_0 = y_2\Big\}$, *and* $\mathcal{R}(\mathbf{T}^2) = \Big\{(y_n) \in \ell^p(\mathbb{N}) \ : y_0 = y_2 = y_4\Big\}$ *and so on ... e.g. Thus* $\mathcal{R}(\mathbf{T}^n) \neq \mathcal{R}(\mathbf{T}^{n+1})$ *for each* $n \geqslant 1$. *Hence* $dsc(\mathbf{T}) = \infty$.

Proposition 1.6. *([16],* ***Lemma 1.4.1****): Take any operator* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ *on a vector space* $\mathcal{X}$ *. Then*

If $asc(\mathbf{T}) < \infty$ *and* $dsc(\mathbf{T}) = 0$, *then.* $asc(\mathbf{T}) = 0$.

▸**Proof:**

Suppose that the conclusion is false. Then there exists $x_1 \in \mathcal{X}$, $x \neq 0$ with $\mathbf{T}(x_1) = 0$. Inasmuch as $\mathcal{R}(\mathbf{T}) = \mathcal{X}$ there exists $x_2 \in \mathcal{X}$ with $\mathbf{T}(x_2) = x_1$ induction we define a sequence $\{x_n\} \in \mathcal{X}$ with $\mathbf{T}(x_{n+1}) = x_n$ for each $n \geqslant 1$. But then $\mathbf{T}^n(x_{n+1}) = x_1$ whereas $\mathbf{T}^{n+1}(x_{n+1}) = 0$ Thus $x_{n+1} \in Ker\mathbf{T}^{n+1}$ and $x_{n+1} \notin Ker\mathbf{T}^{n}$ for each n.

which is contrary to the hypothesis that $asc(\mathbf{T}) < \infty$.

■

Remark 1.7. : *The ascent of* $\mathbf{T}$ *is null if and only if* $\mathbf{T}$ *is injective, and the descent of* $\mathbf{T}$ *is null if and only if* $\mathbf{T}$ *is surjective. Indeed, since* $Ker\mathbf{T} = \{0\}$ *and* $\mathcal{R}(\mathbf{T}) = \mathcal{X}$,

$$asc(\mathbf{T}) = 0 \Longleftrightarrow Ker\mathbf{T} = \{0\}.$$

$$dsc(\mathbf{T}) = 0 \Longleftrightarrow \mathcal{R}(\mathbf{T}) = \mathcal{X}.$$

Example 1.2. : *Let* $\mathbf{T}$ *be defined by* $\mathbf{T}x = y$, *where* $x = (x_n)_{n\geqslant 0}$ *and* $y = (y_n)_{n\geqslant 0}$ *are related by*

$$y_n = \begin{cases} x_0 & if \quad n = 0,1 \\ x_n & if \quad n \geqslant 1 \end{cases}$$

$\mathbf{T}$ *is injective. Hence* $asc(\mathbf{T}) = 0$. *Further for each* $n \geqslant 1$, $\mathcal{R}(\mathbf{T}^n) = \Big\{(y_n)_{n\geqslant 0} \in \ell^p \ : y_0 = y_1 = y_2 = \cdots = y_n\Big\}$. *Thus* $\mathcal{R}(\mathbf{T}^n) \neq \mathcal{R}(\mathbf{T}^{n+1})$. *Therefore* $dsc(\mathbf{T}) = \infty$.

Example 1.3. : *Let* $\mathbf{T}_r$ *denote* ***the right shift operator*** *defined by* $\mathbf{T}_r(x) = y$, *where* $x = (x_n) \in \ell^2(\mathbb{N})$ *and* $y = (y_n) \in \ell^2(\mathbb{N})$ *are related by*

$$\mathbf{T}_r : \left| \begin{array}{ccc} \ell^2 & \longmapsto & \ell^2 \\ x & \longmapsto & \mathbf{T}_r(x) = (0, x_1, x_2, \ldots) \end{array} \right.$$

$\mathbf{T}_r$ *is injective. Hence* $asc(\mathbf{T}_r) = 0$. *Further for each* $n \in \mathbb{N}$, $\mathcal{R}(\mathbf{T}_r^n) = \Big\{(y_n) \in \ell^2(\mathbb{N}) : (y_1, y_2, y_3 \ldots) = (\underbrace{0,0,0}_{n.th}, x_1, x_2, \ldots)\Big\}$, *and* $\mathcal{R}(\mathbf{T}_r^{n+1}) = \Big\{(y_n) \in \ell^2(\mathbb{N}) : (y_1, y_2, y_3 \ldots) = (\underbrace{0,0,0}_{n+1.th}, x_1, x_2, \ldots)\Big\}$. *Thus* $\mathcal{R}(\mathbf{T}_r^n) \neq \mathcal{R}(\mathbf{T}_r^{n+1})$. *Therefore* $dsc(\mathbf{T}_r) = \infty$.

Example 1.4. *: Let* $\mathbf{T}_l$ *denote* ***the left shift operator*** *defined by* $\mathbf{T}_l(x) = y$, *where* $x = (x_n) \in \ell^2(\mathbb{N})$ *and* $y = (y_n) \in \ell^2(\mathbb{N})$ *are related by*

$$\mathbf{T}_l : \left| \begin{array}{ccc} \ell^2 & \longmapsto & \ell^2 \\ x & \longmapsto & \mathbf{T}_l(x) = (x_2, x_3, \ldots) \end{array} \right.$$

$\mathbf{T}_l$ *is surjective. Hence* $dsc(\mathbf{T}_l) = 0$. *Further it is easily seen that* $e_{n+1} \in Ker(\mathbf{T}_l^{n+1})$ *whilst* $e_{n+1} \notin Ker(\mathbf{T}_l^n)$ *for every* $n \in \mathbb{N}$, *thus* $Ker(\mathbf{T}_l^n) \neq Ker(\mathbf{T}_l^{n+1})$, *so* $asc(\mathbf{T}_l) = \infty$.

Since $\mathbf{T}_r$ and $\mathbf{T}_l$ are each one the adjoint of the other, then we have

$$dsc(\mathbf{T}_l) = asc(\mathbf{T}_r) = 0 \qquad \text{and} \qquad asc(\mathbf{T}_l) = dsc(\mathbf{T}_r) = \infty.$$

It is obvious that the sum $\mathcal{M} + \mathcal{N}$ of two linear subspaces $\mathcal{M}$ and $\mathcal{N}$ of a vector $\mathcal{X}$ space is again a linear subspace. If $\mathcal{M} \cap \mathcal{N} = \{0\}$ then this sum is called the direct sum of $\mathcal{M}$ and $\mathcal{N}$ and will be denoted by $\mathcal{M} \oplus \mathcal{N}$. In this case for every $z = x + y$ in $\mathcal{M} + \mathcal{N}$ the components x, y are uniquely determined. If $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ then $\mathcal{N}$ is called an algebraic complement of $\mathcal{M}$. In this case the (Hamel) basis of $\mathcal{X}$ is the union of the basis of $\mathcal{M}$ with the basis of $\mathcal{N}$. It is obvious that every subspace of a vector space admits at least one algebraic complement.

Lemma 1.7. *: Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ *and* $\mathcal{X}$ *be a vector space . For natural* $m \in \mathbb{N}$, *then we have:*

$$asc(\mathbf{T}) \leqslant m \leqslant \infty \text{ if and only if for every } n \in \mathbb{N} \text{ we have } \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n = \{0\}.$$

▸**Proof:**

Suppose $asc(\mathbf{T}) \leqslant m \leqslant \infty$ and $n \in \mathbb{N}$ any natural number. Consider an element $y \in \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n$. Then there exists $y \in \mathcal{X}$ such that $y = \mathbf{T}^m x$ and $\mathbf{T}^n y = 0$. From that we obtain $\mathbf{T}^{m+n} x = \mathbf{T}^n y = 0$ and therefore $x \in Ker\mathbf{T}^{m+n} = Ker\mathbf{T}^m$. Hence $y = \mathbf{T}^m x = 0$.

Conversely, suppose $\mathcal{R}(\mathbf{T}^m)\cap Ker\mathbf{T}^n = \{0\}$ for some natural m and let $x \in Ker\mathbf{T}^{m+1}$. Then $\mathbf{T}^m x \in Ker\mathbf{T}$ and therefore

$$\mathbf{T}^m x \in \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}^n = \{0\}.$$

Hence $x \in Ker\mathbf{T}^m$. We have shown that $Ker\mathbf{T}^{m+1} \subseteq Ker\mathbf{T}^m$. Since the opposite inclusion is verified for all operators we conclude that $Ker\mathbf{T}^{m+1} = Ker\mathbf{T}^m$, this will imply that $asc(\mathbf{T}) \leqslant m \leqslant \infty$.

■

Lemma 1.8. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ and $\mathcal{X}$ be a vector space. For natural $m \in \mathbb{N}$, then we have: $dsc(\mathbf{T}) \leqslant m \leqslant \infty$ if and only if for every $n \in \mathbb{N}$ there exists a subspace $Y_n \subseteq Ker\mathbf{T}^m$, $Y_n \cap \mathcal{R}(\mathbf{T}^n) = \{0\}$, such that $\mathcal{X} = Y_n \oplus \mathcal{R}(\mathbf{T}^n)$.*

▸**Proof:**

Let $k = dsc(\mathbf{T}) \leqslant m \leqslant \infty$ and Y be a complementary subspace to $\mathcal{R}(\mathbf{T}^m)$ in $\mathcal{X}$. Let $\{x_j : j \in J\}$ be a basis of Y. For every element x_j of the basis there exists, since $\mathbf{T}^k(Y) \subseteq \mathcal{R}(\mathbf{T}^k) = \mathcal{R}(\mathbf{T}^{k+n})$, an element $y_j \in \mathcal{X}$ such that $\mathbf{T}^k x_j = \mathbf{T}^{k+n} y_j$. Set $z_j = xj - \mathbf{T}y_j$. Then

$$\mathbf{T}^k z_j = \mathbf{T}^k x_j - \mathbf{T}^{k+n} y_j = 0.$$

From this it follows that the linear subspace Y_n generated by the elements z_j is contained in $Ker\mathbf{T}^k$ and a fortiori in $Ker\mathbf{T}^m$. From the decomposition $Y \oplus \mathcal{R}(\mathbf{T}^n) = \mathcal{X}$ we obtain for every $x \in \mathcal{X}$ a representation of the form

$$x = \sum_{j \in J} \lambda_j x_j + \mathbf{T}^n y = \sum_{j \in J} \lambda_j (z_j + \mathbf{T}^n y_j) + \mathbf{T}^n_y = \sum_{j \in J} \lambda_j z_j + \mathbf{T}^n z,$$

so $\mathcal{X} = Y_n + \mathcal{R}(\mathbf{T}^n)$. We show that this sum is direct. Indeed, suppose that $x \in Y_n \cap \mathcal{R}(\mathbf{T}^n)$. Then $x = \sum_{j \in J} \mu_j z_j = \mathbf{T}^n v$ for some $v \in \mathcal{X}$, and therefore

$$\sum_{j \in J} \mu_j x_j = \sum_{j \in J} \mu_j \mathbf{T}^n y_j + \mathbf{T}^n v \in \mathcal{R}(\mathbf{T}^n).$$

From the decomposition $\mathcal{X} = Y \oplus \mathcal{R}(\mathbf{T}^n)$ we then obtain that $\mu_j = 0$ for all $j \in J$ and hence $x = 0$. Therefore Y_n is a complement of $\mathcal{R}(\mathbf{T}^n)$ contained in $Ker\mathbf{T}^m$.

Conversely, It will suffice to prove that that $\mathcal{R}(\mathbf{T}^m) = \mathcal{R}(\mathbf{T}^{m+n})$ if for $n \in \mathbb{N}$ the subspace $\mathcal{R}(\mathbf{T}^n)$ has a complement $Y_n \subseteq Ker\mathbf{T}^m$ then

$$\mathcal{R}(\mathbf{T}^m) = \mathbf{T}^m(Y_n) + \mathcal{R}(\mathbf{T}^{m+n}) = \mathcal{R}(\mathbf{T}^{m+n}).$$

And therefore $dsc(\mathbf{T}) \leqslant m$.

■

Theorem 1.5. *([4], **Theorem 3.3**): Let $\mathcal{X}$ be a vector space and $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. Then*

$$\textit{If } asc(\mathbf{T}) < \infty \textit{ and } dsc(\mathbf{T}) < \infty. \textit{ then } asc(\mathbf{T}) = dsc(\mathbf{T}).$$

▸**Proof:**

Set $n = asc(\mathbf{T}) < \infty$ and $m = dsc(\mathbf{T}) < \infty$. Assume first that $n \leqslant m$, so that the inclusion $\mathcal{R}(\mathbf{T}^m) \subseteq \mathcal{R}(\mathbf{T}^n)$ holds. Obviously we may assume $m > 0$. From **Lemma 1.8** we have $\mathcal{X} = Ker\mathbf{T}^m + \mathcal{R}(\mathbf{T}^m)$, so every element $y = \mathbf{T}^n(x) \in \mathcal{R}(\mathbf{T}^n)$ admits the decomposition $y = z + \mathbf{T}^m w$, with $z \in Ker\mathbf{T}^m$. From $z = \mathbf{T}^n x - \mathbf{T}^m w \in \mathcal{R}(\mathbf{T}^m)$ we then obtain thatz $\in Ker\mathbf{T}^m \cap \mathcal{R}(\mathbf{T}^m)$ and hence the last intersection is $\{0\}$ by **Lemma1.7**. Therefore $y = \mathbf{T}^m w \in \mathcal{R}(\mathbf{T}^m)$ and this shows the equality $\mathcal{R}(\mathbf{T}^n) = \mathcal{R}(\mathbf{T}^m)$, from whence we obtain $m \geqslant n$, so that $n = m$.

Assume now that $m \leqslant n$ and $n > 0$, so that $Ker\mathbf{T}^m \subseteq Ker\mathbf{T}^n$. From **Lemma 1.8** we have $\mathcal{X} = Ker\mathbf{T}^m + \mathcal{R}(\mathbf{T}^n)$, so that an arbitrary element x of $Ker\mathbf{T}^n$ admits the representation $x = u + \mathbf{T}^n v$, with $u \in Ker\mathbf{T}^m$. From $\mathbf{T}^n x = \mathbf{T}^n u = 0$ it then follows that $\mathbf{T}^{2n} v = 0$, so that $v \in Ker\mathbf{T}^{2n} = Ker\mathbf{T}^n$. Hence $\mathbf{T}^n v = 0$ and consequently $x = u \in Ker\mathbf{T}^m$. This shows that $Ker\mathbf{T}^m = Ker\mathbf{T}^n$, hence $n \geqslant m$. Therefore $n = m$.

■

Example 1.5. *: Let $\mathbf{T}$ be defined by $\mathbf{T}x = y$, where $x = (x_n)_{n\geqslant 0}$ and $y = (y_n)_{n\geqslant 0}$ are related by*

$$y_n = \begin{cases} x_{n+1} & if \quad n \text{ is even} \\ x_n & if \quad n \text{ is odd.} \end{cases}$$

Then $Ker\mathbf{T} = Ker\mathbf{T}^2 = \Big\{(x_n)_{n\geqslant 0} \in \ell^p \ : \ x_{2n+1} = 0 \quad \textit{for each} \quad n \in \mathbb{N}\Big\}$. *Hence* $asc(\mathbf{T}) = 1$, *Also* $\mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{T}^2) = \Big\{(y_n)_{n\geqslant 0} \in \ell^p \ : \ y_{2n} = y_{2n+1} \quad \textit{for each} \quad n \in \mathbb{N}\Big\}$. *Therefore* $dsc(\mathbf{T}) = 1$.

Corollary 1.5. *: Let $\mathcal{X}$ be a vector space and $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. Then*

$$\textit{If } dsc(\mathbf{T}) < \infty \textit{ and } asc(\mathbf{T}) = 0, \textit{ then } dsc(\mathbf{T}) = 0.$$

Example 1.6. *: a simple example is bijective operators which is injective and surjective in same time this operators will have null ascent and descent.*

Given $n \in \mathbb{N}$, we denote by $\mathbf{T_n} = \mathbf{T}_{\backslash \mathcal{R}(\mathbf{T}^n)}$ the restriction of $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ on the subspace $\mathcal{R}(\mathbf{T}^n)$. Observe that

$$Ker\mathbf{T}_{n+1} = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{n+1}) \subseteq Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{n}) = Ker\mathbf{T}_n \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

and

$$\mathcal{R}(\mathbf{T}_n^m) = \mathcal{R}(\mathbf{T}^{n+m}) = \mathcal{R}(\mathbf{T}_m^n) \quad \text{for all } n, m \in \mathbb{N}. \tag{1.4}$$

Lemma 1.9. *: Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$*,* $\mathcal{X}$ *be a vector space. Then the following statements are equivalent:*

1. $asc(\mathbf{T}) < \infty$*;*
2. *there exists* $n \in \mathbb{N}$ *such that* $\mathbf{T}_n$ *is injective;*
3. *there exists* $n \in \mathbb{N}$ *such that* $asc(\mathbf{T}_n) = \infty$*.*

▸**Proof:**

(1)⇔(2) . If $m = asc(\mathbf{T}) < \infty$, by **Lemma** 1.7 then $Ker\mathbf{T_m} = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = \{0\}$. Conversely, suppose that $Ker\mathbf{T}_n = \{0\}$, for some $n \in \mathbb{N}$. If $x \in Ker\mathbf{T}^{n+1}$ then $\mathbf{T}(\mathbf{T}^n x) = 0$, so

$$\mathbf{T}^n x \in Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^n) = Ker\mathbf{T_n} = \{0\}.$$

Hence $x \in Ker\mathbf{T_n}$. This shows that $Ker\mathbf{T}^{n+1} \subseteq Ker\mathbf{T}^n$. The opposite inclusion is true for every operator, thus $Ker\mathbf{T}^{n+1} = Ker\mathbf{T}^n$ and consequently $asc(\mathbf{T}) \leqslant n$.

(2)⇔(3) . The implication (2) ⇒ (3) is obvious. To show the opposite implication, suppose that $p = asc(\mathbf{T}_n) < \infty$. By **Lemma** 1.7 we then have:

$$\begin{aligned}\{0\} &= Ker\mathbf{T}_n \cap \mathcal{R}(\mathbf{T}_n^p) = (Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^n)) \cap \mathcal{R}(\mathbf{T}_n^p) \\ &= (Ker\mathbf{T}) \cap \mathcal{R}(\mathbf{T}_n^p) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{p+n}) = Ker\mathbf{T}_{p+n},\end{aligned}$$

so that the equivalence **(2)**⇔**(3)** is proved.

■

A similar result holds for the descent,

Lemma 1.10. *: Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$*,* $\mathcal{X}$ *be a vector space. Then the following statements are equivalent:*

1. $dsc(\mathbf{T}) < \infty$;
2. *there exists* $n \in \mathbb{N}$ *such that* $\mathbf{T}_n$ *is onto;*
3. *there exists* $n \in \mathbb{N}$ *such that* $dsc(\mathbf{T}_n) = \infty$

▸**Proof:**
(1)⇔(2) . Suppose that $m = dsc(\mathbf{T}) < \infty$. Then

$$\mathcal{R}(\mathbf{T}^m) = \mathcal{R}(\mathbf{T}^{m+1}) = \mathbf{T}(\mathcal{R}(\mathbf{T}^m)) = \mathcal{R}(\mathbf{T}_m),$$

hence $\mathbf{T}_m$ is onto. Conversely, if $\mathbf{T}_n$ is onto for some $n \in \mathbb{N}$ the

$$\mathcal{R}(\mathbf{T}^{n+1}) = \mathbf{T}(\mathcal{R}(\mathbf{T}^n)) = \mathcal{R}(\mathbf{T}_n) = \mathcal{R}(\mathbf{T}^n),$$

thus $dsc(\mathbf{T}) \leqslant n$.

The implication **(2)** ⇒ **(3)** is obvious. We show **(3)** ⇒ **(1)**. Suppose that $q = dsc(\mathbf{T}_n) < \infty$ for some $n \in \mathbb{N}$. Then $\mathcal{R}(\mathbf{T}_n^q) = \mathcal{R}(\mathbf{T}_n^{q+1})$, so $\mathcal{R}(\mathbf{T}^{n+q}) = \mathcal{R}(\mathbf{T}^{n+q+1})$, hence $dsc(\mathbf{T}) \leqslant n+q$.

■

Remark 1.8. *: As observed in the proof of* ***Lemma 1.9****, if* $m = asc(\mathbf{T}) < \infty$ *then* $Ker\mathbf{T_m} = \{0\}$ *and hence* $Ker\mathbf{T_i} = \{0\}$ *for all* $i \geqslant m$*. Conversely, if* $Ker\mathbf{T_n} = \{0\}$ *for some* $n \in \mathbb{N}$ *then* $asc(\mathbf{T}) < \infty$ *and* $asc(\mathbf{T}) < n$*. Hence, if* $asc(\mathbf{T}) < \infty$ *we have*

$$asc(\mathbf{T}) = \inf\Big\{n \in \mathbb{N} \;:\; \mathbf{T}_n \textit{ is injective } \Big\}.$$

Analogously, As observed in the proof of ***Lemma 1.10****, if* $m = dsc(\mathbf{T}) < \infty$ *then* $\mathbf{T}_i$ *is surjective for all* $i \geqslant m$*. Conversely, if* $\mathbf{T}_n$ *is onto for some* $n \in \mathbb{N}$ *then* $dsc(\mathbf{T}) \leqslant n$*, so that*

$$dsc(\mathbf{T}) = \inf\Big\{n \in \mathbb{N} \;:\; \mathbf{T}_n \textit{ is onto } \Big\}.$$

Proposition 1.7. *: Suppose that* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ *on a vector space* $\mathcal{X}$*. If* $n = asc(\mathbf{T}) = dsc(\mathbf{T}) < \infty$ *then we have the decomposition*

$$\mathcal{X} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n).$$

Conversely, if for a natural number n *we have the decomposition* $\mathcal{X} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n)$ *then* $asc(\mathbf{T}) = dsc(\mathbf{T}) < n$*. In this case* $\mathbf{T}_n$ *is bijective.*

▸**Proof:**

If $n < \infty$ and we may assume that $n > 0$, then the decomposition $\mathcal{X} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n)$ immediately follows from **Lemma**1.7, **Lemma**1.9 and **Lemma** 1.10.

If we denote by $\mathbf{T}_n$ the restriction of $\mathbf{T}$ to $\mathcal{R}(\mathbf{T}^n)$, then by **Lemma**1.9and **Lemma??** $\mathbf{T}_n$ is injective and onto itself, hence it is bijective. Conversely, if $\mathcal{X} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n)$ is valid, then $asc(\mathbf{T}), dsc(\mathbf{T}) < n$ by **Lemma**1.7 and **Lemma**1.8, and so $asc(\mathbf{T}) = dsc(\mathbf{T}) < n$ by **Theorem**1.5. If $asc(\mathbf{T}) = dsc(\mathbf{T}) = n < \infty$ (where we may assume $n > 0$), then decomposition $\mathcal{X} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n)$ immediately follows from **Lemma**1.7 and **Lemma**1.8.

■

The notion of ascent and descent holds for every bounded transformation of a Banach space into itself.

Theorem 1.6. : *Let* $\mathbf{E}$ *be a banach space over filed* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, *take any bounded operator* $\mathbf{T}$ *in* $\mathbf{E}$ *with closed rang. So* $\mathcal{R}(\mathbf{T}^n)$ *is closed then we have the* ***topological direct sum***:

$$\text{if } asc(\mathbf{T}) = dsc(\mathbf{T}) = n < \infty \text{ then } \mathbf{E} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n).$$

▸**Proof:**

If $asc(\mathbf{T}) < \infty$ and $dsc(\mathbf{T}) = n < \infty$, then $\mathbf{E} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n)$ is direct algebraic sum by **Theorem**1.7, $\mathbf{T}^n$ is a bounded operator. Therefore, $Ker\mathbf{T}^n$ is closed subspace of $\mathbf{E}$, which is a Banach space, then the topological direct sum $\mathbf{E} = Ker\mathbf{T}^n \oplus \mathcal{R}(\mathbf{T}^n)$ is verified.

■

1.1.3 The nullity and deficiency of an operator

The codimension of subspace $\mathcal{M}$ of a vector space $\mathcal{X}$ is the dimension of the quotient space $\mathcal{X}/\mathcal{M}$, it is denoted by $\mathbf{codim}\mathcal{M}$, therefore:

$$\mathbf{codim}(\mathcal{M}) = \mathbf{dim}(\mathcal{X}/\mathcal{M}).$$

In addition, let $\mathcal{X}$ be a vector space, when $\mathcal{M}$ is a subspace of $\mathcal{X}$ then, codimension of $\mathcal{M}$ is equal to the dimension of any algebraic complement of $\mathcal{M}$ in $\mathcal{X}$, which coincides with demention of the quotient space $\mathcal{X}/\mathcal{M}$.One has

$$\mathbf{dim}(\mathcal{M}) + \mathbf{dim}(\mathcal{X}/\mathcal{M}) = \mathbf{dim}(\mathcal{M}) + \mathbf{codim}(\mathcal{M}) = \mathbf{dim}\mathcal{X}.$$

Definition 1.3. *: Let* $\mathbf{T}$ *an operator on a vector space* $\mathcal{X}$*. Then*

The nullity of $\mathbf{T}$ *is the positive integer defined by*

$$\alpha(\mathbf{T}) = \boldsymbol{dim} Ker \mathbf{T}.$$

The deficiency of $\mathbf{T}$ *is the positive defined integer by*

$$\beta(\mathbf{T}) = \boldsymbol{codim} \mathcal{R}(\mathbf{T}).$$

Let $\Delta(\mathcal{X})$ denote the set of all linear operators on vector space $\mathcal{X}$, for which $\alpha(\mathbf{T})$ and $\beta(\mathbf{T})$ are both finite. With every operator $\mathbf{T} \in \Delta(\mathcal{X})$ of finite nullity and deficiency one associates its index, which is the integer

$$ind(\mathbf{T}) = \alpha(\mathbf{T}) - \beta(\mathbf{T}).$$

Example 1.7. *: Let* $\mathcal{X} = \ell^p$ *, write each* $u \in \mathcal{X}$ *in the form of a bilateral sequence* $u = (....\xi_{-1}, \xi_0, \xi_1, ...)$*. Let* $\{x_i\}$ *be the canonical basis of* $\mathcal{X}$ *and let* $\mathbf{T}$ *be bounded operator in* $\mathcal{X}$*, be such that* $\mathbf{T}x_0 = 0, \mathbf{T}x_j = x_{j-1}$ $(j = \pm 1, \pm 2, ...)$*.* $Ker(\mathbf{T})$ *is the one-dimensional subspace spanned by* x_0 *and* $\mathcal{R}(\mathbf{T})$ *is the subspace spanned by all the* x_1 *except* x_1*. Hence* $\alpha(\mathbf{T}) = 1$*,* $\beta(\mathbf{T}) = 1$*,* $ind(\mathbf{T}) = 0$*.*

The basic assertion concerning the index is made by the following index theorem,

Theorem 1.7. ***(index theorem)****: Let* $\mathcal{X}$ *be a vector space,* $\mathbf{T}, \mathbf{S} \in \Delta(\mathcal{X})$*. Then*

$$ind(\mathbf{TS}) = ind(\mathbf{T}) + ind(\mathbf{S}).$$

▸**Proof:**
See [25] Theorem 23.1. p 108.

■

Example 1.8. *: Let's take as a practical example the* ***right shift operator*** *mentioned in* ***Example 1.3****, and* $\mathcal{X} = \ell^p$ *. As is easily verified,* $Ker(\mathbf{T}_r)$ *is the one-dimensional subspace spanned by* x_1*, and* $\mathcal{R}(\mathbf{T}_r) = \mathcal{X}$*. Thus* $\alpha(\mathbf{T}_r) = 1$*,* $\beta(\mathbf{T}_r) = 0$*,* $ind(\mathbf{T}_r) = 1$*.*

Example 1.9. *: Let's take as a practical example the* ***left shift operator*** *mentioned in* ***Example 1.4****, and* $\mathcal{X} = \ell^p$ *It is easily verified that* $Ker(\mathbf{T}_l) = 0$*, and that* $\mathcal{R}(\mathbf{T}_l)$ *is the subspace spanned by* $x_2, x_3, \ldots\ldots$ *Hence* $\alpha(\mathbf{T}_l) = 0$*,* $\beta(\mathbf{T}_l) = 1$*,* $ind(\mathbf{T}_l) = -1$*.*

We can observe that $\mathbf{T}_l, \mathbf{T}_r \in \Delta(\mathcal{X})$. Then

$$ind(\mathbf{T}_l\mathbf{T}_r) = ind(\mathbf{T}_r\mathbf{T}_l) = 0. \tag{1.5}$$

Remark 1.9. *: If for an operator* $\mathbf{T} \in \Delta(\mathcal{X})$ *an equation of the form*

$$\mathbf{TS} = \mathbf{C} \;\; or \;\; \mathbf{ST} = \mathbf{C} \;\; with \;\; ind(\mathbf{C}) = 0$$

holds, then also $\mathbf{S} \in \Delta(\mathcal{X})$ *and* $ind(\mathbf{S}) = -ind(\mathbf{T})$*.*

Example 1.10. *: Like an example see (1.5). We can observe that* $ind(\mathbf{T}_l) = -ind(\mathbf{T}_r)$*.*

Proposition 1.8. *: Let* $\mathcal{X}$ *be a vector space,* $\mathbf{T} \in \Delta(\mathcal{X})$*, and* $\mathbf{S}$ *a finite-dimensional one. Then*

$$ind(\mathbf{T} + \mathbf{S}) = ind(\mathbf{T}).$$

▸**Proof:**

See [25] Proposition 23.3. p 109.

■

Lemma 1.11. *([5],* ***Lemma 1.21****): Let* $\mathbf{T}$ *be a linear operator on a linear vector space* $\mathcal{X}$*. If* $\alpha(\mathbf{T}) < \infty$ *then* $\alpha(\mathbf{T}^n) < \infty$ *for all* $n \in \mathbb{N}$*. Analogously, if* $\beta(\mathbf{T}) < \infty$ *then* $\beta(\mathbf{T}^n) < \infty$ *for all* $n \in \mathbb{N}$

▸**Proof:**

We use an inductive argument. Suppose that $\alpha(\mathbf{T}^n) < \infty$. Since $\mathbf{T}(Ker\mathbf{T}^{n+1}) \subseteq Ker\mathbf{T}^n$ then the restriction

$$\mathbf{T}_0 = \mathbf{T}_{/Ker\mathbf{T}^{n+1}} : Ker\mathbf{T}^{n+1} \longrightarrow Ker\mathbf{T}^n$$

has kernel equal to $Ker\mathbf{T}$, so the canonical mapping $\hat{\mathbf{T}} : Ker\mathbf{T}^{n+1}/Ker\mathbf{T} \longrightarrow Ker\mathbf{T}^n/Ker\mathbf{T}$ is injective. Therefore we have

$$\mathbf{dim}(Ker\mathbf{T}^{n+1}/Ker\mathbf{T}) \leqslant \mathbf{dim}(Ker\mathbf{T}^n/Ker\mathbf{T}) \leqslant \mathbf{dim}(Ker\mathbf{T}^n) = \alpha(\mathbf{T}^n) < \infty.$$

and since $\alpha(\mathbf{T}) < \infty$ we then conclude that $\alpha(\mathbf{T}^{n+1}) < \infty$.

Suppose now that $\beta(\mathbf{T}^n) < \infty$. Since the map

$$\tilde{\mathbf{T}} : \mathcal{R}(\mathbf{T}^n)/\mathcal{R}(\mathbf{T}^{n+1}) \longrightarrow \mathbf{T}^{n+1}(\mathcal{R}(\mathbf{T}^n)/\mathcal{R}(\mathbf{T}^{n+2}),$$

defined by

$$\tilde{\mathbf{T}}(z + \mathcal{R}(\mathbf{T}^{n+1})) = \tilde{\mathbf{T}}z + \mathcal{R}(\mathbf{T}^{n+2}), \quad z \in \mathcal{R}(\mathbf{T}^n),$$

is onto, $\mathbf{dim}\mathcal{R}(\mathbf{T}^{n+1})/\mathcal{R}(\mathbf{T}^{n+2}) \leqslant \mathbf{dim}\ \mathcal{R}(\mathbf{T}^n)/\mathcal{R}(\mathbf{T}^{n+1})$.

This easily implies that $\beta(\mathbf{T}^{n+1}) < \infty$.

■

In the next theorem we establish the basic relationships between the quantities $\alpha(\mathbf{T})$, $\beta(\mathbf{T})$, $asc(\mathbf{T})$ and $dsc(\mathbf{T})$.

Theorem 1.8. *([4], **Theorem 3.4**): If $\mathbf{T}$ is is a linear operator on a vector space $\mathcal{X}$, then the following properties hold:*

1. *If $asc(\mathbf{T}) < \infty$ then $\alpha(\mathbf{T}) \leqslant \beta(\mathbf{T})$;*
2. *If $dsc(\mathbf{T}) < \infty$ then $\beta(\mathbf{T}) \leqslant \alpha(\mathbf{T})$;*
3. *If $asc(\mathbf{T}) = dsc(\mathbf{T}) < \infty$ then $\alpha(\mathbf{T}) = \beta(\mathbf{T})$;*
4. *If $\alpha(\mathbf{T}) = \beta(\mathbf{T}) < \infty$ and if either $asc(\mathbf{T}) < \infty$ or $dsc(\mathbf{T}) < \infty$ then $asc(\mathbf{T}) = dsc(\mathbf{T})$.*

▸**Proof:**

(1). Let $p = asc(\mathbf{T}) < \infty$, i.e., $Ker\mathbf{T}^p = Ker\mathbf{T}^{p+n}$ for all $n \in \mathbb{N}$. Obviously if $\beta(\mathbf{T}) = \infty$ there is nothing to prove. Assume that $\beta(\mathbf{T}) < \infty$. By **Lemma1.7**, we have $Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^p) = \{0\}$. Since $\beta(\mathbf{T}^n) < \infty$, by **Lemma 1.11**, this implies that $\alpha(\mathbf{T}) < \infty$, so $\mathbf{T}$ has finite deficiency. According to the index theorem we then obtain for all $n \geqslant p$ the following equality:

$$n\,.\,ind(\mathbf{T}) = ind(\mathbf{T}^n) = \alpha(\mathbf{T}^n) - \beta(\mathbf{T}^n) = \alpha(\mathbf{T}^p) - \beta(\mathbf{T}^n).$$

Now suppose that $q = dsc(\mathbf{T}) < \infty$. For all integers $n \geqslant \max\{p, q\}$ the quantity $n\,.\,ind(\mathbf{T}) = \alpha(\mathbf{T}^p) - \beta(\mathbf{T}^p)$ is then constant, so that $ind(\mathbf{T}) = 0$, i.e., $\alpha(\mathbf{T}) = \beta(\mathbf{T})$. Consider the other case $q = \infty$. Then $\beta(\mathbf{T}^n) \longrightarrow 0$ as $n \longrightarrow \infty$, so $n\,.\,ind(\mathbf{T})$ becomes eventually negative, and hence $ind(\mathbf{T}) < 0$. Therefore, in this case we have $\alpha(\mathbf{T}) < \beta(\mathbf{T})$.

(2). Let $q = dsc(\mathbf{T}) < \infty$. Also here we can assume that $\alpha(\mathbf{T}) < \infty$, otherwise there is nothing to prove. Consequently, as is easy to check, also $\beta(\mathbf{T}^n) < \infty$ and by of **Lemma 1.8** $\mathcal{X} = Y \oplus \mathcal{R}(\mathbf{T})$ with $Y \subseteq Ker\mathbf{T}^q$. From this it follows that

$$\beta(\mathbf{T}) = \mathbf{dim}Y \leqslant \alpha(\mathbf{T}^q) < \infty.$$

If we use, with appropriate changes, the index argument used in the proof of part (1) then we obtain that $\beta(\mathbf{T}) = \alpha(\mathbf{T})$ if $asc(\mathbf{T}) < \infty$, and $\alpha(\mathbf{T}) < \beta(\mathbf{T})$ if $dsc(\mathbf{T}) = \infty$.

(3). This is clear from part (1) and part (2).

(4). This is an immediate consequence of the equality $\alpha(\mathbf{T}^n) - \beta(\mathbf{T}^n) = ind(\mathbf{T}^n) = n\,.\,ind(\mathbf{T}) = 0$, valid for every $n \in \mathbb{N}$.

∎

Remark 1.10. : *We can observe that if* $\mathbf{T} \in \Delta(\mathcal{X})$, *and ascent and descent are both finite then* $ind(\mathbf{T}) = 0$.

Lemma 1.12. *([15],* ***Lemma 5.4****): Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, *and let* $k \in \mathbb{N}$. *Then*

1. *If* $\alpha(\mathbf{T}) < \infty$, *then* $\alpha(\mathbf{T}^k) \leqslant k\alpha(\mathbf{T})$;
2. *If* $\beta(\mathbf{T}) < \infty$, *then* $\beta(\mathbf{T}^k) \leqslant k\beta(\mathbf{T})$.

▸**Proof:**

(1). Let $n \geqslant 0$. Since $Ker\mathbf{T}^n \subset Ker\mathbf{T}^{n+1}$ it follows that there exists a complementary subspace $\mathbf{N}$ (relative to $Ker\mathbf{T}^{k+1}$) such that $Ker\mathbf{T}^{k+1} = Ker\mathbf{T}^k \oplus \mathbf{N}$. It will be shown that $\mathbf{dimN} \leqslant \alpha(\mathbf{T})$. The case $\mathbf{dimN} = 0$ is trivial, hence assume that $\mathbf{dimN} > 0$. Let $x_1, x_2, ..., x_p \in \mathbf{N}$ be linearly independent, $1 \leqslant p \leqslant \mathbf{dimN}$. Then (because $\mathbf{N} \subset Ker\mathbf{T}^{n+1}$) there exist $y_1, y_2, ..., y_p \in Ker\mathbf{T}$ such that

$$\{x_1, y_1\}, \{x_2, y_2\}, ..., \{x_p, y_p\} \in \mathbf{T}^n.$$

Assume that $\sum_{i=1}^{p} c_i y_i = 0$ for certain $c_i \in \mathbb{K}, 1 \leqslant i \leqslant p$. Then

$$\sum_{i=1}^{p} c_i\{x_i, y_i\} = \left\{\sum_{i=1}^{p} c_i x_i, \sum_{i=1}^{p} c_i y_i\right\} = \left\{\sum_{i=1}^{p} c_i x_i, 0\right\} \in \mathbf{T}^n$$

so that $\sum_{i=1}^{p} c_i X_i \in Ker\mathbf{T}^n \cap \mathbf{N}$. Since $\mathbf{N}$ and $Ker\mathbf{T}^n$ are complementary spaces it follows that $\sum_{i=1}^{p} c_i X_i = 0$ which implies that $c_i = 0, 1 \leqslant i \leqslant p$. This means that for any p linearly independent vectors in $\mathbf{N}$ there exist p linearly independent vectors in $Ker\mathbf{T}$. Hence $\mathbf{dimN} \leqslant \mathbf{dim}Ker\mathbf{T} = \alpha(\mathbf{T})$. Thus $\alpha(\mathbf{T}^{n+1}) \leqslant \alpha(\mathbf{T}^n) + \alpha(\mathbf{T})$, so that the statement follows by induction; recall that $\alpha(\mathbf{T}^0) = 0$.

(2) . $\beta(\mathbf{T}^0) = 0$, the case $k = 0$ is trivial. Assume $k \in \mathbb{N}$ and define

$$\mathbf{M}_k = \mathcal{R}(\mathbf{T}^{k-1})/\mathcal{R}(\mathbf{T}^k).$$

we have that $\beta(\mathbf{T}^k) = \beta(\mathbf{T}^{k-1}) + \mathbf{dimM}_k$ and a repeated application gives

$$\beta(\mathbf{T}^k) = \mathbf{dimM}_1 + \mathbf{dimM}_2 + ... + \mathbf{dimM}_k, \ \ k \in \mathbb{N}. \tag{1.6}$$

Note that $\mathbf{dimM} = \mathbf{dim}(\mathcal{R}(\mathbf{T}^0)/\mathcal{R}(\mathbf{T}^1)) = \mathbf{dim}(\mathcal{X}/\mathcal{R}(\mathbf{T})) = \mathbf{codim}\mathcal{R}(\mathbf{T}) = \beta(\mathbf{T})$. Now the inequality

$$\mathbf{dimM}_{n+1} \leqslant \mathbf{dimM}_n, \ \ n \in \mathbb{N}, \tag{1.7}$$

will be shown. Let $[y_1], [y_2], ..., [y_p] \in \mathbf{dimM}_{n+1}$ be linearly independent cosets. Then $y_i \in \mathcal{R}(\mathbf{T}^n)$ for $1 \leqslant i \leqslant p$, so there exist $x_1, x_2, ..., x_p \in \mathcal{R}(\mathbf{T}^{n-1})$ such that $\{x_1, y_1\}, \{x_2, y_2\}, ..., \{x_p, y_p\} \in \mathbf{T}$ (even if $n = 1$ because $\mathcal{D}(\mathbf{T}) \subset \mathcal{X} = \mathcal{R}(\mathbf{T}^0)$). Now if $\sum_{i=1}^p c_i[x_i] = [\sum_{i=1}^p c_i x_i] = [0]$ in $\mathbf{M}_n$ for certain $c_i \in \mathbb{K}$, then $\sum_{i=1}^p c_i x_i \in \mathcal{R}(\mathbf{T}^n)$, and hence $\sum_{i=1}^p c_i y_i \in \mathcal{R}(\mathbf{T}^{n+1})$. It follows that

$$\sum_{i=1}^{p} c_i[y_i] = \left[\sum_{i=1}^{p} c_i y_i\right] = [0]$$

which implies $c_i = 0, \ \ 1 \leqslant i \leqslant p$. Hence for any p linearly independent vectors in $\mathbf{M}_{n+1}$ there exist p linearly independent vectors in $\mathbf{M}_n$. Therefore (1.7) has been established. The statement now follows from (1.6) and (1.7), since $\mathbf{dimM}_1 = \beta(\mathbf{T})$.

■

Proposition 1.9. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. Assume there exists some $n \in N$ such that $\alpha(\mathbf{T}^k) \leqslant n$ for $k \in \mathbb{N}$. Then $asc(\mathbf{T}) \leqslant n$.*

▸**Proof:**

If $asc(\mathbf{T}) = \infty$ then $Ker\mathbf{T}^{k+1} \supsetneqq Ker\mathbf{T}^k, k \in \mathbb{N}$ Ḣence $\alpha(\mathbf{T}^k) < \alpha(\mathbf{T}^k + 1), \ \ k \in \mathbb{N},$ which implies that the sequence $\alpha(\mathbf{T}^k)$ is unbounded. This contradiction implies that $asc(\mathbf{T}) < \infty$. Assume that $asc(\mathbf{T}) = p$ for some $p \in \mathbb{N}$. In the case $p = 0$ the statement is trivial, so what remains to be shown is that $p \leqslant n$ if $p > 0$. Cleary

$$\{0\} = Ker\mathbf{T}^0 \subsetneqq Ker\mathbf{T} \subsetneqq \ \subsetneqq Ker\mathbf{T}^{p-1} \subsetneqq Ker\mathbf{T}^p,$$

and thus

$$0 = \alpha(\mathbf{T}^0) < \alpha(\mathbf{T}) < \ < \alpha(\mathbf{T}^{p-1}) < \alpha(\mathbf{T}^p).$$

Therefore, $p - 1 < \alpha(\mathbf{T}^p) \leqslant n$, leading to $p \leqslant M$.

■

Proposition 1.10. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. Assume there exists some $n \in N$ such that $\beta(\mathbf{T}^k) \leqslant n$ for $k \in \mathbb{N}$. Then $dsc(\mathbf{T}) \leqslant n$.*

▸Proof:

If $dsc(\mathbf{T}) = \infty$ then $\mathcal{R}(\mathbf{T}^{k+1}) \subsetneq \mathcal{R}(\mathbf{T}^{k})$, $k \in \mathbb{N}$. Hence $\beta(\mathbf{T}^{k+1}) > \beta(\mathbf{T}^{k})$, $k \in \mathbb{N}$, which implies that the sequence $\beta(\mathbf{T}^{k})$ is unbounded. This contradiction implies that $dsc(\mathbf{T}) < \infty$. Assume that $dsc(\mathbf{T}) = q$ for some $q \in \mathbb{N}$. The case $q = 0$ is obvious, so let $q > 0$. Since $\mathbf{dim}(\mathcal{R}(\mathbf{T}^{k})/\mathcal{R}(\mathbf{T}^{k+1})) > 0$ for $k < q$, we have that

$$0 = \beta(\mathbf{T}^{0}) < \beta(\mathbf{T}) < ... < \beta(\mathbf{T}^{q-1}) < \beta(\mathbf{T}^{q}).$$

Therefore, $q - 1 < \beta(\mathbf{T}^{q}) \leqslant n$, leading to $q \leqslant n$.

■

Corollary 1.6. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$, and let $k \in \mathbb{N}$. Then*

1. $\alpha(\mathbf{T}^{k}) \leqslant asc(\mathbf{T})\alpha(\mathbf{T})$;
2. $\beta(\mathbf{T}^{k}) \leqslant dsc(\mathbf{T})\beta(\mathbf{T})$.

▸Proof:

(1) . We firstly observe that $asc(\mathbf{T}) = 0$ if and only if $\alpha(\mathbf{T}) = 0$. Hence the product $asc(\mathbf{T})\alpha(\mathbf{T})$ is well defined. We need only consider the case where both $asc(\mathbf{T}) and \alpha(\mathbf{T})$ are finite. Let $asc(\mathbf{T}) = p$. Then $\alpha(\mathbf{T}^{k}) \leqslant \alpha(\mathbf{T}^{p})$ for any k and if we show $\alpha(\mathbf{T}^{k}) \leqslant k\alpha(\mathbf{T})$ for every non-negative integer k, the result will follow. Therefore the result follows immediately by **Lemma** 1.12.

(2) . Again, since $dsc(\mathbf{T})$ is zero if and only if $\beta(\mathbf{T})$ is zero, the product $dsc(\mathbf{T})\beta(\mathbf{T})$ is well defined and we need only consider the case when $dsc(\mathbf{T})$ and $\beta(\mathbf{T})$ are finite. Again it suffices to prove that for each positive integer k, $\beta(\mathbf{T}^{k}) \leqslant k\beta(\mathbf{T})$. So by **Lemma** 1.12, this completes the proof of **(2)**.

■

Theorem 1.9. *: Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. If one of the following conditions holds:*

1. $\alpha(\mathbf{T}) < \infty$;
2. $\beta(\mathbf{T}) < \infty$;

   Then there exist $m \in \mathbb{N}$ such that :

$$Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{m}) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{m+k}), \quad for\ all\ \ n \in \mathbb{N}. \tag{1.8}$$

▸Proof:

(1). If $Ker\mathbf{T}$ is finite-dimensional then there exists a positive integer m such that

$$Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{m+k})$$

(2) . Suppose that $\mathcal{X} = Y \oplus \mathcal{R}(\mathbf{T})$ with $\mathbf{dim}(Y) < \infty$. Clearly, if we let $D_n = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^n)$ then we have $D_n \supseteq D_{n+1}$ for all $n \in \mathbb{N}$. Suppose that there exist k distinct subspaces D_n. There is no loss of generality in assuming $D_j \neq Dj+1$ for $j = 1, 2, \cdots k$. Then for every one of these j we can find an element $w_j \in \mathcal{X}$ such that $\mathbf{T}^j w_j \in D_j$ and $\mathbf{T}^j w_j \in D_{j+1}$. By means of the decomposition $\mathcal{X} = Y \oplus \mathcal{R}(\mathbf{T})$ we also find $u_j \in Y$ and $v_j \in \mathcal{R}(\mathbf{T})$ such that $w_j = u_j + v_j$. We claim that the vectors $u_1, \cdots, u_k$ are linearly independent.

To see this let us suppose $\sum_{j=1}^{k} \lambda_j u_j = 0$. Then

$$\sum_{j=1}^{k} \lambda_j w_j = \sum_{j=1}^{k} \lambda_j v_j$$

and therefore from the equalities $\mathbf{T}^k w_1 = \ldots = \mathbf{T}^k w_{k-1} = 0$ we deduce that

$$\mathbf{T}^k(\sum_{j=1}^{k} \lambda_j w_j) = \lambda_k \mathbf{T}^k w_k = \mathbf{T}^k(\sum_{j=1}^{k} \lambda_j v_j) \in \mathbf{T}^k(\mathcal{R}(\mathbf{T})) = \mathcal{R}(\mathbf{T}^{k+1})$$

From $\mathbf{T}^k w_k \in Ker\mathbf{T}$ we obtain $\lambda_k \mathbf{T}^k w_k \in D_{k+1}$, and since $\mathbf{T}^k w_k \notin D_{k+1}$ this is possible only if $\lambda_k = 0$. Analogously we have $\lambda_{k-1} = \ldots = \lambda_1 = 0$, so the vectors $u_1, \ldots, u_k$ are linearly independent. From this it follows that k is smaller than or equal to the dimension of Y. But then for a sufficiently large m we obtain that

$$Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{m+j}).$$

■

Lemma 1.13. *([25], **Lemma 38.1**): Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ maps the linear space $\mathcal{R}^\infty(\mathbf{T})$ into itself, and in the case of $\alpha(\mathbf{T}) < \infty$ even onto itself.*

▸Proof:

It is trivial that $\mathcal{R}^\infty(\mathbf{T})$ is mapped into itself by $\mathbf{T}$. We now assume that $\alpha(\mathbf{T}) < \infty$ and show that every element of $\mathcal{R}^\infty(\mathbf{T})$ is the image of an element of $\mathcal{R}^\infty(\mathbf{T})$ under $\mathbf{T}$. From $Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^n) \supset Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{n+1})$ for $n = 0, 1, 2, \ldots$ it follows, because of $\alpha(\mathbf{T}) < \infty$ and, that there exists a natural number m with

$$\mathbf{D} := Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^{m+k}) \quad for \quad k = 0, 1, 2, \ldots. \tag{1.9}$$

Obviously also $\mathbf{D} := Ker\mathbf{T} \cap \mathcal{R}^\infty(\mathbf{T})$. Let now y be an arbitrary element of $\mathcal{R}^\infty(\mathbf{T})$. Then for every $k = 0, 1, 2, \ldots$ there exists an $x_k \in \mathcal{X}$ so that $y = \mathbf{T}^{m+k} x_k$. If we set

$$z_k = \mathbf{T}^m x_1 - \mathbf{T}^{m+k-1} x_k \;\; for \quad k = 1, 2, \ldots, \tag{1.10}$$

then z_k lies in $\mathcal{R}(\mathbf{T}^m)$ and, because of $\mathbf{T}z_k = \mathbf{T}^{m+1}x_1 - \mathbf{T}^{m+k}x_k = y - y = 0$, also in $Ker\mathbf{T}$, hence $z_k \in Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = \mathbf{D}$. From (1.9) it follows that z_k lies also in $\mathcal{R}(\mathbf{T}^{m+k-1})$ and with the aid of (1.10) this implies

$$\mathbf{T}^m x_1 = z_k + \mathbf{T}^{m+k-1}x_k \in \mathcal{R}(\mathbf{T}^{m+k-1}) \;\; for \quad k = 1, 2, \ldots,$$

hence $\mathbf{T}^m x_1 \in \mathcal{R}^\infty(\mathbf{T})$. Because of $\mathbf{T}(\mathbf{T}^m x_1) = \mathbf{T}^{m+1}x_1 = y$, we see that y is indeed the image of an element of $\mathcal{R}^\infty(\mathbf{T})$ under $\mathbf{T}$.

■

Proposition 1.11. *([25], **Proposition 28.2**): Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ with $\alpha(\mathbf{T}) < \infty$ the following assertions are equivalent:*

1. *$asc(\mathbf{T}) < \infty$;*

2. *On every subspace $\mathbf{D}$ of $\mathcal{X}$, which is mapped by $\mathbf{T}$ onto itself, $\mathbf{T}$ is injective;*

3. *$\mathbf{T}$ is injective on the subspace $\mathcal{R}^\infty(\mathbf{T})$.*

▸**Proof:**

1 $\Rightarrow$ 2: If $\mathbf{T}(\mathbf{D}) = \mathbf{D}$ and $\tilde{\mathbf{T}}$ is the restriction of $\mathbf{T} to \mathbf{D}$, then $dsc(\tilde{\mathbf{T}}) = 0$. From $Ker\mathbf{T}^n = Ker\mathbf{T}^n \cap \mathbf{D}$ it follows because of (1) that $asc(\tilde{\mathbf{T}}) < \infty$. Because of **Theorem1.5** we thus have $asc(\tilde{\mathbf{T}}) = asc(\tilde{\mathbf{T}}) = 0$, and so $\tilde{\mathbf{T}}$ is injective.

2 $\Rightarrow$ **3**: This implication is trivial because of **Lemma 1.13**.

3 $\Rightarrow$ **1**: From (3) it follows in the first place that $\mathbf{D} = Ker\mathbf{T} \cap \mathcal{R}^\infty(\mathbf{T}) = \{0\}$. Because of (1.9) we have thus also $Ker\mathbf{T} \cap \mathcal{R}(\mathbf{T}^m) = \{0\}$ for some natural number m. Assertion (1) is now a consequence of **Lemma1.7**.

■

Lemma 1.14. *([4], Lemma 1.2): Let $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ and $\mathcal{X}$ be a vector space. then we have, if p_1 and p_2 are relatively prime polynomials then there exist polynomials q_1 and q_2 such that*

$$p_1(\mathbf{T})q_1(\mathbf{T}) + p_2(\mathbf{T})q_2(\mathbf{T}) = \mathbf{T}.$$

▸**Proof:**

If p_1 and q_1 are relatively prime polynomials then there are polynomials such that $p_1(\lambda)q_1(\lambda) + p_2(\lambda)q_2(\lambda) = 1$ for every $\lambda \in \mathbb{C}$.

■

Lemma 1.15. *: Let $\mathcal{X}$ be a vector space, $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ and $\lambda, \mu \in \mathbb{C}$. Then we have*

1. $(\lambda\mathbf{I} - \mathbf{T})(Ker\mathbf{T}^n) = Ker\mathbf{T}^n$ *for all* $n \in \mathbb{N}$ *and* $\lambda \neq 0$;
2. $Ker((\lambda\mathbf{I} - \mathbf{T})^n) \subseteq \mathcal{R}((\mu\mathbf{I} - \mathbf{T})^n)$ *for all* $n \in \mathbb{N}$ *and* $\lambda \neq \mu$.

▸**Proof:**

(**1**). We prove that $(\lambda\mathbf{I} - \mathbf{T})(Ker\mathbf{T}^n) = Ker\mathbf{T}^n$ for every $n \in \mathbb{N}$ and $\lambda \neq 0$. Clearly, $(\lambda\mathbf{I} - \mathbf{T})(Ker\mathbf{T}^n) \subseteq Ker\mathbf{T}^n$ holds for all $n \in \mathbb{N}$. By **Lemma 1.14** there exist polynomials p and q such that

$$(\lambda\mathbf{I} - \mathbf{T})p(\mathbf{T}) + q(\mathbf{T})\mathbf{T}^n = \mathbf{I}.$$

If $x \in Ker\mathbf{T}^n$ then $(\lambda\mathbf{I} - \mathbf{T})p(\mathbf{T})x = x$ and since $p(\mathbf{T})x \in Ker\mathbf{T}^n$ this implies $Ker\mathbf{T}^n \subseteq (\lambda\mathbf{I} - \mathbf{T})(Ker\mathbf{T}^n)$. Then we have

$$(\lambda\mathbf{I} - \mathbf{T})(Ker\mathbf{T}^n) = Ker\mathbf{T}^n.$$

(2). Put $\mathbf{S} = \lambda\mathbf{I} - \mathbf{T}$ and write $\mu\mathbf{I} - \mathbf{T} = (\mu - \lambda)\mathbf{I} + \lambda\mathbf{I} - \mathbf{T} = (\mu - \lambda)\mathbf{I} + \mathbf{S}$. By assumption $\mu - \lambda \neq 0$, so by part (1) we obtain that $(\mu\mathbf{I} - \mathbf{T})(Ker((\lambda\mathbf{I} - \mathbf{T})^n)) = ((\mu - \lambda)\mathbf{I} + \mathbf{S})(Ker\mathbf{S}^n) = Ker((\lambda\mathbf{I} - \mathbf{T})^n)$. From this it easily follows that $(\mu\mathbf{I} - \mathbf{T})^n(Ker\mathbf{S}^n) = Ker\mathbf{S}^n)$ for all $n \in \mathbb{N}$, and consequently $Ker((\lambda\mathbf{I} - \mathbf{T})^n) \subseteq \mathcal{R}((\lambda\mathbf{I} - \mathbf{T})^n)$.

■

Corollary 1.7. *:*
Let $\mathcal{X}$ be a vector space and $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ (a linear operator on $\mathcal{X}$). Then we have:

1. $(\lambda\mathbf{I} - \mathbf{T})(\mathcal{N}^\infty(\mathbf{T})) = \mathcal{N}^\infty(\mathbf{T})$ *for every* $\lambda \neq 0$;
2. $\mathcal{N}^\infty(\lambda\mathbf{I} - \mathbf{T}) \subseteq \mathcal{R}^\infty(\mu\mathbf{I} - \mathbf{T})$ *for every* $\lambda \neq \mu$.

▸**Proof:**

(**1**). It suffices to prove that $(\lambda\mathbf{I} - \mathbf{T})(Ker\mathbf{T}^n) = Ker\mathbf{T}^n$ for every $n \in \mathbb{N}$ and $\lambda \neq 0$, so by part (1) of **Lemma 1.15**. Then we have

$$(\lambda\mathbf{I} - \mathbf{T})(\mathcal{N}^\infty(\mathbf{T})) = \mathcal{N}^\infty(\mathbf{T}).$$

(2). It suffices to prove that $Ker((\lambda\mathbf{I} - \mathbf{T})^n) \subseteq \mathcal{R}((\mu\mathbf{I} - \mathbf{T})^n)$ for all $n \in \mathbb{N}$ and $\lambda \neq \mu$, so by part (2) of **Lemma 1.15**, consequently we have

$$\mathcal{N}^\infty(\lambda\mathbf{I} - \mathbf{T}) \subseteq \mathcal{R}^\infty(\lambda\mathbf{I} - \mathbf{T}).$$

■

We have the following statement

$$\mathcal{R}((\lambda\mathbf{I}-\mathbf{T})^n)+\mathcal{R}(\mathbf{T}^m)=\mathcal{X}\ for\ \ all\ n\in\mathbb{N}\ and\ \lambda\neq 0 \tag{1.11}$$

we will use it in the next lemma .

Lemma 1.16. *: Let* $\mathbf{T}\in\mathcal{L}(\mathcal{X})$ *and* $\mathbf{T}_m=\mathbf{T}\backslash\mathcal{R}(\mathbf{T}^m)$. *For all* $\lambda\in\mathbb{C}$ *and* $\lambda\neq 0$, *we have*

1. $\beta((\lambda\mathbf{I}-\mathbf{T})^n)=\beta((\lambda\mathbf{I}-\mathbf{T}_m)^n)$, *for all* $n\in\mathbb{N}$;
2. $\alpha((\lambda\mathbf{I}-\mathbf{T}_m)^n)=\alpha((\lambda\mathbf{I}-\mathbf{T})^n)$, *for all* $n\in\mathbb{N}$.

▸**Proof:**
(1). By (1.11), we have :

$$\begin{aligned}\beta((\lambda\mathbf{I}-\mathbf{T})^n) &= \mathbf{dim}(\mathcal{X}/\mathcal{R}(\lambda\mathbf{I}-\mathbf{T})^n))\\ &= \mathbf{dim}(\mathcal{R}(\lambda\mathbf{I}-\mathbf{T})^n)+\mathcal{R}(\mathbf{T}^m)/\mathcal{R}(\lambda\mathbf{I}-\mathbf{T})^n)\\ &= \mathbf{dim}(\mathcal{R}(\mathbf{T}^m)/\mathcal{R}(\mathbf{T}^m)\cap\mathcal{R}(\lambda\mathbf{I}-\mathbf{T})^n))\\ &= \beta((\lambda\mathbf{I}-\mathbf{T}_m)^n)\end{aligned}$$

.

(2). It will suffice to show that

$$Ker(\lambda-\mathbf{T})^n=Ker(\lambda-\mathbf{T}_m)^n \tag{1.12}$$

for $n=0,1....$. When $n=0$, (1.12) holds trivially, so fix $n\geqslant 1$. Let $x\in Ker(\lambda\mathbf{I}-\mathbf{T})^n$. Then

$$0=(\lambda-\mathbf{T})^n x=\lambda^n x+p(\mathbf{T})x,$$

where $p(\mathbf{T})$ is a linear combination of the iterates $\mathbf{T},\mathbf{T}^2.....\mathbf{T}^n$ of $\mathbf{T}$. Then

$$x=[-\lambda^{-n}p(\mathbf{T})]^i x$$

for $i=1$, and hence for $i=2,3.....$ Thus $x\in\mathcal{R}(\mathbf{T}^i)$ for $i=1,2,....$ In particular $x\in\mathcal{R}(\mathbf{T}^m)$, so $0=(\lambda-\mathbf{T}_m)^n x$ and $x\in Ker(\lambda-\mathbf{T})^n$. Thus $Ker(\lambda-\mathbf{T})^n\subset Ker(\lambda-\mathbf{T}_m)^n$. Containment in the other direction is obvious.

■

Corollary 1.8. *: Let* $\mathbf{T}\in\mathcal{L}(\mathcal{X})$ *and for some fixed* $m\leqslant 0$, $\mathbf{T}_m=\mathbf{T}\backslash\mathcal{R}(\mathbf{T}^m)$. *be the restriction of* $\mathbf{T}$. *For all* $\lambda\in\mathbb{C}$ *and* $\lambda\neq 0$, *we have:*

1. $\beta(\lambda\mathbf{I}-\mathbf{T})=\beta(\lambda\mathbf{I}-\mathbf{T}_m)$;
2. $\alpha(\lambda\mathbf{I}-\mathbf{T})=\alpha(\lambda\mathbf{I}-\mathbf{T}_m)$.

▸**Proof:**

The result follows frome **Lemma 1.16**, when $n = 1$.

■

Corollary 1.9. : *Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$ *and* $\mathbf{T}_{\infty} = \mathbf{T}_{\backslash \mathcal{R}^{\infty}(\mathbf{T})}$. *For all* $\lambda \in \mathbb{C}$ *and* $\lambda \neq 0$, *we have*

1. $\beta(\lambda \mathbf{I} - \mathbf{T}) = \beta(\lambda \mathbf{I} - \mathbf{T}_{\infty})$;
2. $\alpha(\lambda \mathbf{I} - \mathbf{T}) = \alpha(\lambda \mathbf{I} - \mathbf{T}_{\infty})$.

▸**Proof:**

The proof of this lemma also follows immediately from that of **Lemma1.16**.

■

Proposition 1.12. : *Let* $\mathbf{T} \in \mathcal{L}(\mathcal{X})$. *If one of the following conditions holds:*

1. $\alpha(\mathbf{T}) < \infty$;
2. $\beta(\mathbf{T}) < \infty$;

Then $\mathrm{C}(\mathbf{T}) = \mathcal{R}^{\infty}(\mathbf{T})$.

▸**Proof:**

These results immediately follows from **Theorem 1.9** and **lemma 1.6**.

■

CHAPTER 2

OPERATORS WITH CLOSED RANGE AND DECOMPOSITION

Let $\mathbf{E}$, $\mathbf{F}$ are Banach spaces, we says that an operator $\mathbf{T}$ is bounded (or continuous) if there is a constant $c \geqslant 0$ such that

$$\|\mathbf{T}x\| \leqslant c\|x\| \qquad \forall x \in \mathbf{E}$$

We denote the Banach space of all bounded linear operators from $\mathbf{E}$ into $\mathbf{F}$ by $\mathcal{B}(\mathbf{E},\mathbf{F})$, $\mathcal{B}(\mathbf{E},\mathbf{E})$ is also denoted $\mathcal{B}(\mathbf{E})$. Recall that if $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, the norm of $\mathbf{T}$ is defined by :

$$\|\mathbf{T}\| = \sup_{x \neq 0} \frac{\|\mathbf{T}x\|}{\|x\|}.$$

Recall that when $\mathbf{E}$ is Banach spaces, the dual space $\mathbf{E}' := \mathcal{B}(\mathbf{E},\mathbb{C})$, consists of the bounded linear functionals f on $\mathbf{E}$; it is Banach space with norm

$$\|f\|_{\mathbf{E}'} = \inf\Big\{ \ |f(x)| \ : \quad x \in \mathbf{E}, \ \|x\| = 1\Big\}.$$

2.1 Minimum modulus and Kato operator theory

Let $\mathcal{M}$ and $\mathcal{N}$ be tow nonzero subsets of $\mathbf{E}$ and $\mathbf{E}'$, respectively (i.e $\mathcal{M} \subset \mathbf{E}$, and $\mathcal{N} \subset \mathbf{E}'$):

1. Let $\mathcal{M}$ be a subset of a Banach space $\mathbf{E}$ The annihilator of $\mathcal{M}$ is the closed subspace of $\mathbf{E}'$ defined by

$$\mathcal{M}^{\perp} = \Big\{ f \in \mathbf{E}' \quad : \forall x \in \mathcal{M}, \;\; f(x) = 0 \Big\}.$$

2. Let $\mathcal{N}$ be a subset of $\mathbf{E}'$. The pre-annihilator of $\mathcal{N}$ is the closed subspace of a Banach space $\mathbf{E}$ defined by

$$^{\perp}\mathcal{N} = \Big\{ x \in \mathbf{E} \quad : \forall f \in \mathcal{N}, \;\; f(x) = 0 \Big\}.$$

Even if $\mathcal{M}$ and $\mathcal{N}$ are not subspaces, and $\mathcal{M}^{\perp}$ and $^{\perp}\mathcal{N}$ are closed subspaces of $\mathbf{E}'$ and $\mathbf{E}'$ respectively. We have $\mathcal{M}^{\perp} = \mathbf{E}'$ (resp. $^{\perp}\mathcal{N}$) if and only if $\mathcal{M} = \{0\}$ (resp. $\mathcal{N} = \{0\}$).

Clearly $^{\perp}(\mathcal{M}^{\perp}) = \mathcal{M}^{\perp}$ if $\mathcal{M}$ is closed. Moreover, if $\mathcal{M}$ and $\mathcal{W}$ are closed linear subspaces of $\mathbf{E}$ then $(\mathcal{M} + \mathcal{N})^{\perp} = \mathcal{M}^{\perp} \cap \mathcal{W}^{\perp}$. The dual relation $\mathcal{M}^{\perp} + \mathcal{W}^{\perp} = (\mathcal{M} \cap \mathcal{W})^{\perp}$ is not always true, since $(\mathcal{M} \cap \mathcal{W})^{\perp}$ is always closed but $\mathcal{M}^{\perp} + \mathcal{W}^{\perp}$ need not be closed. However, a classical theorem establishes that

$$\mathcal{M}^{\perp} + \mathcal{W}^{\perp} \text{ is closed in } \mathbf{E}' \iff \mathcal{M} + \mathcal{W} \text{ is closed in } \mathbf{E}.$$

Definition 2.1. *: If* $\mathbf{T} \in \mathcal{B}(\mathbf{E}, \mathbf{F})$, *then the dual map(adjoint) of* $\mathbf{T}$ *is the map* $\mathbf{T}' \in \mathcal{B}(\mathbf{F}', \mathbf{E}')$ *defined by :*

$$\mathbf{T}'(g) = g \circ \mathbf{T} \quad for \;\; g \in \mathbf{F}'.$$

Example 2.1. *: Let define* $\mathbf{T} : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ *by* $\mathbf{T}p = p'$.

Suppose φ *is the linear functional on* $\mathcal{P}(\mathbb{R})$ *defined by* $\varphi(p) = p(3)$. *Then* $\mathbf{T}'(\varphi)$ *is the linear functional on* $\mathcal{P}(\mathbb{R})$ *given by*

$$(\mathbf{T}'(\varphi))(p) = (\varphi \circ \mathbf{T})(p) = \varphi(\mathbf{T}p) = \varphi(p') = p'(3).$$

Thus $\mathbf{T}'(\varphi)$ *is the linear functional on* $\mathcal{P}(\mathbb{R})$ *that takes* p *to* $p'(3)$ *.*

Example 2.2. *: Suppose* φ *is the linear functional on* $\mathcal{P}(\mathbb{R})$ *defined by* $\varphi(p) = \int_0^1 p$. *Then* $\mathbf{T}'(\varphi)$ *is the linear functional on* $\mathcal{P}(\mathbb{R})$ *given by*

$$(\mathbf{T}'(\varphi))(p) = (\varphi \circ \mathbf{T})(p) = \varphi(\mathbf{T}p) = \varphi(p') = \int_0^1 p' = p(1) - p(0).$$

Thus $\mathbf{T}'(\varphi)$ *is the linear functional on* $\mathcal{P}(\mathbb{R})$ *that takes* p *to* $p(1) - p(0)$ *.*

Remark 2.1. : *All adjoint operators in Hilbert spaces are dual map but the opposite is not true.*

Example 2.3. : *Consider the Banach space $L^2[a,b]$ of all integrable complex-valued functions on a bounded closed interval $[a,b]$ with the sup-norm.*

A continuous function $k(s,t)$ defined on $[a,b]\times[a,b]$ defines an operator $\mathbf{T}\in\mathcal{B}(L^2[a,b])$ by

$$(\mathbf{T}x)(s)=\int_a^b k(s,t)\,x(t)\,dt,\quad x\in L^2[a,b].$$

Then $\mathbf{T}'$ is given y

$$(\mathbf{T}'x)(s)=\int_a^b \overline{k(t,s)}\,x(t)\,dt,\quad x\in L^2[a,b].$$

Example 2.4. : *If $(\mathbf{E},\Omega,\mu)$ is a a-finite measure space and $\phi\in L^\infty(\mathbf{E},\Omega,\mu)$, define $\mathbf{M}_\phi :\in L^p(\mathbf{E},\Omega,\mu)\to L^p(\mathbf{E},\Omega,\mu)$, $1\leqslant p\leqslant\infty$ by $\mathbf{M}_\phi f=\phi f$ for all f in $L^p(\mathbf{E},\Omega,\mu)$. Then $\mathbf{M}_\phi\in\mathcal{B}(L^p(\mathbf{E},\Omega,\mu))$ and $\|\mathbf{M}_\phi\|=\|\phi\|_\infty$ İf $1/p+1/q=1$, then $\mathbf{M}'_\phi :\in L^p(\mu)\to L^p(\mu)$ is given by $\mathbf{M}'_\phi f=\phi f$. That is, $\mathbf{M}'_\phi f=\mathbf{M}_\phi f$.*

Remark 2.2. : *If $\mathbf{T}$ is bounded operator from $\mathbf{E}$ into $\mathbf{F}$ then $\mathbf{T}'$ is also a bounded operator from $\mathbf{F}'$ into $\mathbf{E}'$ and, moreover, $\|\mathbf{T}\|=\|\mathbf{T}'\|$.*

And we have:

$$Ker(\mathbf{T})={}^{\perp}\overline{\mathcal{R}(\mathbf{T}')}\ \text{ and }\ \overline{\mathcal{R}(\mathbf{T})}={}^{\perp}Ker(\mathbf{T}');$$

and

$$Ker(\mathbf{T}')=\overline{\mathcal{R}(\mathbf{T})}^{\perp}\ \text{ and }\ \overline{\mathcal{R}(\mathbf{T}')}\subseteq Ker(\mathbf{T})^{\perp}.$$

Lemma 2.1. : *Let $\mathbf{T}\in\mathcal{B}(\mathbf{E},\mathbf{F})$. If $\mathcal{R}(\mathbf{T})$ is closed, then :*

$$\mathcal{R}(\mathbf{T}')=Ker(\mathbf{T})^{\perp}\ \text{ and }\ \mathcal{R}(\mathbf{T})={}^{\perp}Ker(\mathbf{T}').$$

▸**Proof:**

See [27] **Lemma 7.1.** p 18.

■

Let $\mathbf{M}$ be a closed subspace of a Banach space $\mathbf{E}$. Then $\mathbf{M}'$ is isometrically isomorphic to the quotient $\mathbf{E}'/\mathbf{M}^{\perp}$, while $(\mathbf{E}/\mathbf{M})'$ is isometrically isomorphic to $\mathbf{M}^{\perp}$.

Proposition 2.1. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *with closed rang,* $\mathbf{E}$ *and* $\mathbf{F}$ *are Banach spaces. Then:*

1. $\alpha(\mathbf{T}) = \beta(\mathbf{T}')$;
2. $\beta(\mathbf{T}) = \alpha(\mathbf{T}')$;
3. $\alpha(\mathbf{T}) = \alpha(\mathbf{T}'')$;
4. $\beta(\mathbf{T}) = \beta(\mathbf{T}'')$.

▸**Proof:**

By **Lemma** 2.1 asserts that $\mathcal{R}(\mathbf{T}') = Ker\mathbf{T}^{\perp}$, so we have that $[Ker\mathbf{T}]'$ is isomorphic to $\mathbf{E}'/\mathcal{R}(\mathbf{T}')$. Therefore

$$\begin{aligned}\alpha(\mathbf{T}) &= \mathbf{dim} Ker\mathbf{T} = \mathbf{dim}(Ker\mathbf{T})' \\ &= \mathbf{dim}(\mathbf{E}'/(Ker\mathbf{T})^{\perp}) \\ &= \mathbf{dim}(\mathbf{E}'/\mathcal{R}(\mathbf{T}')) \\ &= \beta(\mathbf{T}').\end{aligned}$$

As noted earlier also we have that $[\mathbf{F}/\mathcal{R}(\mathbf{T})]'$ is isomorphic to $\mathcal{R}(\mathbf{T})^{\perp}$. Therefore

$$\begin{aligned}\beta(\mathbf{T}) &= \mathbf{dim}(\mathbf{F}/\mathcal{R}(\mathbf{T})) \\ &= \mathbf{dim}(\mathbf{F}/\mathcal{R}(\mathbf{T}))' \\ &= \mathbf{dim}\mathcal{R}(\mathbf{T})^{\perp} \\ &= \mathbf{dim} Ker\mathbf{T}' \\ &= \alpha(\mathbf{T}').\end{aligned}$$

Then immediately we have $\alpha(\mathbf{T}) = \alpha(\mathbf{T}'')$ and $\beta(\mathbf{T}) = \beta(\mathbf{T}'')$.

■

Example 2.5. *: Let* ***right shift*** *operator* $\mathbf{T}_r$ *and the* ***left shift*** *operator* $\mathbf{T}_l$*, which defined in* ***Example*** *1.3, and* ***Example*** *1.4 respectively, we have the following results*

$$\mathbf{T}_r' = \mathbf{T}_l.$$

and

$$\alpha(\mathbf{T}_r) = \beta(\mathbf{T}_l) = 0 \quad \text{and} \quad \alpha(\mathbf{T}_l) = \beta(\mathbf{T}_r) = 1.$$

Definition 2.2. *: Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces ,then* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *is said to be* ***Kato*** *if* $\mathcal{R}(\mathbf{T})$ *is closed and* $\mathbf{T}$ *verifies one of the equivalent conditions of* ***Theorem 1.1****.*

Example 2.6. *: Trivial examples of Kato*[1] *operators are surjective operators as well as injective operators with closed range.*

Example 2.7. *:. Let* $\mathcal{H}$ *be a Hilbert space with an orthonormal basis* $(e_{i,j})$ *where* i, j *are integers and* $i \geqslant 1$*. Let* $\mathbf{T}$ *be defined by:*

$$\mathbf{T}e_{i,j} := \begin{cases} e_{i,j+1} & if \quad j \neq 0, \\ 0 & if \quad j = 0. \end{cases}$$

Clearly $\mathcal{R}(\mathbf{T})$ *is closed and*

$$Ker\mathbf{T} = \underset{j\geqslant 1}{span}\{e_{0,j}\} \subset \mathcal{R}^{\infty}(\mathbf{T}),$$

so that $\mathbf{T}$ *is Kato.*

Definition 2.3. *: Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces, then* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *is said to be* ***essentially Kato*** *if* $\mathcal{R}(\mathbf{T})$ *is closed and* d *there exists a finite dimensional subspace* $\mathbf{F}$*, such that* $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^n) + \mathbf{F}$*, for all* $n \in \mathbb{N}$ *.*

Let (M,N) be a pair of closed subspaces of $\mathbf{E}$. $\mathbf{T}$ is said to be decomposed according to $\mathbf{E} = M \oplus N$, if $\mathbf{T}(M) \subset M$, and $\mathbf{T}(N) \subset N$. When $\mathbf{T}$ is decomposed as above, the pair $\mathbf{T}_M$, $\mathbf{T}_N$ of $\mathbf{T}$ in M, N, respectively, can be defined: $\mathbf{T}_M$ is an operator in the Banach space M with $\mathcal{D}(\mathbf{T}_M) = M$ such that $\mathbf{T}_M x = \mathbf{T}x \in M$, and $\mathbf{T}_N$ is similarly defined. In this case, we write $\mathbf{T} = \mathbf{T}_M \oplus \mathbf{T}_N$.

Definition 2.4. *: Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces ,then* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *is said to be of* ***Kato- type*** *of order* $d \in N$ *if, there exist a pair of closed subspaces* (M,N) *of* $\mathbf{E}$ *such that* $\mathbf{T} = \mathbf{T}_M \oplus \mathbf{T}_N$*, where* $\mathbf{T}_M$ *is Kato operator and* $\mathbf{T}_N$ *is nilpotent of order* d *(i.e.,* $\mathbf{T}_N^d = 0$*).*

An operator $\mathbf{T}$ *is said to be of Kato type if, there exists* $d \in \mathbb{N}$ *such that* $\mathbf{T}$ *is a Kato type of order* d *.*

Example 2.8. *:*

- *Clearly, every Kato operator is of Kato type with* $M = \mathbf{E}$ *and* $N = \{0\}$ *and a nilpotent operator has a decomposition with* $M = \{0\}$ *and* $N = \mathbf{E}$*.*

[1]Tosio Kato, August 25, 1917 - October 2, 1999. Japanese mathematician.

- *Every essentially Kato operator admits a decomposition (M,N) such that N is a finite-dimensional vector space.*

Definition 2.5. *: If $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, where $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces* ***the reduced minimum modulus*** *of a non-zero operator $\mathbf{T}$ is defined to be*

$$\gamma(\mathbf{T}) := \inf_{x \notin Ker\mathbf{T}} \frac{\|\mathbf{T}x\|}{\operatorname{dist}(x, Ker\mathbf{T})}.$$

Remark 2.3. *: If $\mathbf{T} = 0$ then we take $\gamma(\mathbf{T}) = \infty$.*

Therefore. It easily seen that if $\mathbf{T}$ is bijective then $\gamma(\mathbf{T}) = \frac{1}{\|\mathbf{T}^{-1}\|}$. In fact, if $\mathbf{T}$ is bijective then $dist(x, Ker\mathbf{T}) = dist(x, \{0\}) = \|x\|$, thus if $\mathbf{T}x = y$,

$$\gamma(\mathbf{T}) = \inf_{x\neq 0} \frac{\|\mathbf{T}x\|}{\|x\|} = (\sup_{x\neq 0} \frac{\|x\|}{\|\mathbf{T}x\|})^{-1}$$
$$= (\sup_{y\neq 0} \frac{\|\mathbf{T}^{-1}y\|}{\|y\|})^{-1} = \frac{1}{\|\mathbf{T}^{-1}\|}.$$

Example 2.9. *: Trivial example is the operator which defined in* ***Example1.7****. For any $u = (\xi_j)$ we have $\|\tilde{u}\| = dist(u, Ker(\mathbf{T})) = (\sum_{j=2}^{\infty} |\ \xi_j\ |)^{\frac{1}{p}}) = \|\mathbf{T}u\|$. Hence $\|\mathbf{T}u\|\ \|\tilde{u}\| = 1$ for every $u \in \ell^p$, so that $\gamma(\mathbf{T}) = 1$.*

Theorem 2.1. *([6],* ***Theorem 1.2****): Let $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces. Then*

1. *$\gamma(\mathbf{T}) > 0$ if and only if $\mathcal{R}(\mathbf{T})$ is closed.*
2. *$\gamma(\mathbf{T}) = \gamma(\mathbf{T}')$.*

▸**Proof:**

(1) . Let $\tilde{\mathbf{E}} = \mathbf{E} \backslash Ker\mathbf{T}$ and let $\tilde{\mathbf{T}} : \tilde{\mathbf{E}} \longmapsto \mathbf{F}$ denote the continuous injection corresponding to $\mathbf{T}$, defined by

$$\tilde{\mathbf{T}}\tilde{x} = \mathbf{T}x \text{ for every } x \in \tilde{\mathbf{E}}.$$

Clearly $\mathcal{R}(\tilde{\mathbf{T}}) = \mathcal{R}(\mathbf{T})$. From the open mapping theorem it follows that $\mathcal{R}(\tilde{\mathbf{T}})$ is closed if and only if $\tilde{\mathbf{T}}$ admits a continuous inverse, there exists a constant $\delta > 0$ such that $\|\tilde{\mathbf{T}}\tilde{x}\| \geqslant \delta\|\tilde{x}\|$ for every $x \in \mathbf{E}$. From the equality

$$\gamma(\mathbf{T}) = inf_{\tilde{x}\neq 0} \frac{\|\tilde{\mathbf{T}}\tilde{x}\|}{\|\tilde{x}\|}$$

we then conclude that $\mathcal{R}(\tilde{\mathbf{T}}) = \mathcal{R}(\mathbf{T})$ is closed if and only if $\gamma(\mathbf{T}) > 0$.

(2). The assertion is obvious if $\gamma(\mathbf{T}) = 0$. Suppose that $\gamma(\mathbf{T}) > 0$. Then $\mathcal{R}(\mathbf{T})$ is closed. If $\tilde{\mathbf{T}}_0 : \tilde{\mathbf{E}} \longrightarrow \mathcal{R}(\mathbf{T})$ is defined by $\tilde{\mathbf{T}}_0\tilde{x} = \mathbf{T}$ for every $x \in \tilde{\mathbf{E}}$, then $\gamma(\mathbf{T}) = \gamma(\mathbf{T})$ and $\mathbf{T} = \mathbf{J}\tilde{\mathbf{T}}_0\mathbf{Q}$, where $\mathbf{J} : \mathcal{R}(\mathbf{T}) \longrightarrow \mathbf{F}$ denotes the natural embedding and $\mathbf{Q} : \mathbf{E} \longrightarrow \tilde{\mathbf{E}}$ is the canonical projection defined by $\mathbf{Q}x = \tilde{x}$. Clearly, $\tilde{\mathbf{T}}_0$ is bijective, and from $\mathbf{T} = \mathbf{J}\tilde{\mathbf{T}}_0\mathbf{Q}$ it then follows that $\mathbf{T}' = \mathbf{Q}'\,(\tilde{\mathbf{T}}_0)'\,\mathbf{J}'$. From this we easily obtain that

$$\gamma(\mathbf{T}) = \frac{1}{\|\,(\tilde{\mathbf{T}}_0)^{-1}\,\|} = \frac{1}{\|\,(\tilde{\mathbf{T}}'_0)^{-1}\,\|} = \gamma(\mathbf{T}').$$

■

Corollary 2.1. : *Let $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$. Then $\mathcal{R}(\mathbf{T})$ is closed if and only if $\mathcal{R}(\mathbf{T}')$ is closed*

▸**Proof:**

It is obvious from the equality $\gamma(\mathbf{T}) = \gamma(\mathbf{T}')$ (by **Theorem** 2.1) observed above. We have that $\gamma(\mathbf{T}) = \gamma(\mathbf{T}') > 0$. Therefore $\mathcal{R}(\mathbf{T})$ is closed if and only if $\mathcal{R}(\mathbf{T}')$ is closed.

■

Example 2.10. : *a trivial example is also **right shift operator** and **left shift operator**, which defined in **Example**1.3 and **Example**1.1respectively. A consideration similar to that in **Example** 2.9 shows that $\gamma(\mathbf{T}_r) = 1$. suppose that $\mathbf{E} = \ell^p$. In this case $\tilde{\mathbf{E}} = \mathbf{E}$, $\tilde{\mathbf{T}}_l = \mathbf{T}_l$, $\|\mathbf{T}_l u\| = \|u\|$ so that $\gamma(\mathbf{T}_l) = 1$.*

Corollary 2.2. : *The function $\gamma : \mathcal{B}(\mathbf{E},\mathbf{F}) \to \langle 0,\infty\rangle$ is upper sem-icontinuous.*

Example 2.11. : *In general, the function γ is not continuous. Let $\mathbf{T}_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\gamma(\mathbf{T}_n) = 1/n$, $\mathbf{T}_n \to \mathbf{T}$ and $\gamma(\mathbf{T}) = 1$.*

Theorem 2.2. *([4], **Theorem 1.14**): Let $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ a Banach space, and suppose that there exists a closed subspace $\mathbf{Y}$ of $\mathbf{E}$ such that $\mathcal{R}(\mathbf{T}) \cap \mathbf{Y} = \{0\}$ and $\mathcal{R}(\mathbf{T}) \oplus \mathbf{Y}$ is closed. Then $\mathcal{R}(\mathbf{T})$ is also closed.*

▸**Proof:**

Consider the product space $\mathbf{E} \times \mathbf{Y}$ under the norm

$$\|(x,y)\| = \|x\| + \|y\| \quad (x \in \mathbf{E},\ y \in \mathbf{Y}).$$

Then $\mathbf{E} \times \mathbf{Y}$ is a Banach space, and the continuous map $\mathbf{S} : \mathbf{E} \times \mathbf{Y} \longmapsto \mathbf{E}$ defined by $\mathbf{S}(x,y) = \mathbf{T}x + y$ has range $\mathbf{S}(\mathbf{E}\times\mathbf{Y}) = \mathcal{R}(\mathbf{T}) \oplus \mathbf{Y}$, which is closed by assumption. Consequently by **Theorem 2.1** we have

$$\gamma(\mathbf{S}) = \inf_{(x,y)\notin Ker\mathbf{S}} \frac{\|\mathbf{S}(x,y)\|}{dist((x,y),Ker\mathbf{S})} > 0.$$

Clearly, $Ker\mathbf{S} = Ker\mathbf{T} \times \{0\}$, so that if $x \notin Ker\mathbf{T}$ then $(x,0) \notin Ker\mathbf{S}$. Moreover

$$dist((x,0),Ker\mathbf{S}) = dist(x,Ker\mathbf{T}),$$

and therefore

$$\|\mathbf{T}x\| = \|\mathbf{S}(x,0)\| \geqslant \gamma(\mathbf{S})dist((x,0),Ker\mathbf{S}) = \gamma(\mathbf{S})dist(x,Ker\mathbf{T}).$$

This implies that $\gamma(\mathbf{T}) \geqslant \gamma(\mathbf{S}) > 0$, and therefore $\mathbf{T}$ has closed range.

■

Corollary 2.3. : *Let $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ a Banach space, and $\mathbf{Y}$ a finite-dimensional subspace of $\mathbf{E}$ such that $\mathcal{R}(\mathbf{T}) + \mathbf{Y}$ is closed. Then $\mathcal{R}(\mathbf{T})$ is closed.*
In particular,

$$\textit{if} \quad \beta(\mathbf{T}) < \infty \textit{ then } \mathcal{R}(\mathbf{T}) \textit{ is closed.}$$

▸**Proof:**
Let $\mathbf{Y_1}$ be any subspace of $\mathbf{Y}$ for which $\mathbf{Y_1} \cap \mathcal{R}(\mathbf{T}) = \{0\}$ and $\mathcal{R}(\mathbf{T}) + \mathbf{Y_1} = \mathcal{R}(\mathbf{T}) + \mathbf{Y}$. From the assumption we infer that $\mathcal{R}(\mathbf{T}) \oplus \mathbf{Y_1}$ is closed, so $\mathcal{R}(\mathbf{T})$ is closed by **Theorem** 2.2.

The second statements is clear, since every finite-dimensional subspace of a Banach space **E** is always closed, we know that $\mathbf{dimY} = \mathbf{codim}\mathcal{R}(\mathbf{T})$.

■

Theorem 2.3. *([4], **Theorem 1.16**): Suppose that $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ is a Banach space. Then we have*

$$\textit{If} \quad \mathbf{T} \textit{ is Kato. Then } \gamma(\mathbf{T}^n) \geqslant \gamma(\mathbf{T})^n.$$

▸**Proof:**
We proceed by induction. The case $n = 1$ is trivial.
Suppose that $\gamma(\mathbf{T}^n) \geqslant \gamma(\mathbf{T})^n$. For every element $x \in \mathbf{E}$, and $u \in Ker\mathbf{T}^{n+1}$ we have

$$\begin{aligned} dist(x, Ker\mathbf{T}^{n+1}) &= dist(x-u, Ker\mathbf{T}^{n+1}) \\ &\leqslant dist(x-u, Ker\mathbf{T}). \end{aligned}$$

By assumption $\mathbf{T}$ is Kato, so by **Theorem 1.2** $Ker\mathbf{T} = \mathbf{T}^n(Ker\mathbf{T}^{n+1})$ and therefore

$$\begin{aligned} dist(\mathbf{T}^n x, Ker\mathbf{T}) &= dist(\mathbf{T}^n x, \mathbf{T}^n(Ker\mathbf{T}^{n+1})) \\ &= \inf_{u \notin Ker\mathbf{T}^{n+1}} \|\mathbf{T}^n(x-u)\| \\ &\geqslant \gamma(\mathbf{T}^n). \inf_{u \notin Ker\mathbf{T}^{n+1}} dist(x-u, Ker\mathbf{T}^n) \\ &\geqslant \gamma(\mathbf{T}^n) dist(x, Ker\mathbf{T}^{n+1}). \end{aligned}$$

From this estimate it follows that

$$\|\mathbf{T}^{n+1}x\| \geqslant \gamma(\mathbf{T}) dist(\mathbf{T}^n x, Ker\mathbf{T}) \geqslant \gamma(\mathbf{T})\gamma(\mathbf{T}^n).dist(x, Ker\mathbf{T}^{n+1});$$

Consequently from our inductive assumption we obtain that

$$\gamma(\mathbf{T}^{n+1}) \geqslant \gamma(\mathbf{T})\gamma(\mathbf{T})^n \geqslant \gamma(\mathbf{T})^{n+1}.$$

which completes the proof.

■

Proposition 2.2. *([19], **Proposition 6**): Let $\mathbf{T}, \mathbf{S} \in \mathcal{B}(\mathbf{E})$, $\mathbf{TS} = \mathbf{ST}$. If $\mathbf{TS}$ is Kato, then both $\mathbf{T}$ and $\mathbf{S}$ are Kato.*

▸**Proof:**

It is sufficient to show that $\mathbf{T}$ is Kato. We have $Ker\mathbf{T}^n \subset Ker(\mathbf{TS})^n \subset \mathcal{R}(\mathbf{TS}) \subset \mathcal{R}(\mathbf{T})$ for all n, and so $\mathcal{N}^\infty(\mathbf{T}) \subset \mathcal{R}(\mathbf{T})$.

It remains to show that $\mathcal{R}(\mathbf{T})$ is closed. Let $x_k \in \mathbf{E}$ and $\mathbf{T}x_k \to v$ for some $v \in \mathbf{E}$. Then $\mathbf{ST}x_k \to \mathbf{S}v$, and so $\mathbf{S}v = \mathbf{ST}u$ for some $u \in \mathbf{E}$. Thus $v - \mathbf{T}u \in Ker\mathbf{S} \subset Ker(\mathbf{TS}) \subset \mathcal{R}(\mathbf{TS}) \subset \mathcal{R}(\mathbf{T})$, and so $v \in \mathcal{R}(\mathbf{T})$.

■

Remark 2.4. *: the product of two Kato operators, also commuting Kato operators, need not be Kato.*

Example 2.12. *: . Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis (e_i, j) where i, j are integers for which $ij \leqslant 0$. Let $\mathbf{T} \in \mathcal{B}(\mathcal{H})$, and $\mathbf{S} \in \mathcal{B}(\mathcal{H})$ are defined by the assignment:*

$$\mathbf{T}e_i, j = \begin{cases} 0 & if \quad i = 0, j > 0 \\ e_{i+1,j} & \quad otherwise, \end{cases}$$

and

$$\mathbf{S}e_i, j = \begin{cases} 0 & if \quad j = 0, i > 0 \\ e_{i,j+1} & otherwise, \end{cases}$$

Then

$$\mathbf{T}e_i, j = \mathbf{S}e_i, j = \begin{cases} 0 & if \quad i = 0, j \geqslant 0, i \geqslant 0 \\ e_{i+1,j+1} & otherwise, \end{cases}$$

Hence $\mathbf{TS} = \mathbf{ST}$ *and, as it is easy to verify*

$$Ker\mathbf{T} = \underset{i>1}{span}\{e_i, 0\} \subset \mathcal{R}^{\infty}(\mathbf{T}).$$

where $\underset{j>1}{span}\{e_0, j\}$ *denotes the linear subspace of* $\mathcal{H}$ *generated by the set* $\{e_j \ : j > 0\}$. *Analogously we have*

$$Ker\mathbf{S} = \underset{i>1}{span}\{e_i, 0\} \subset \mathcal{R}^{\infty}(\mathbf{S}).$$

Corollary 2.4. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, *where* $\mathbf{E}$ *is a Banach space. Then*

If $\mathbf{T}$ *is Kato then* $\mathbf{T}^n$ *is also Kato for all* $n \in \mathbb{N}$.

▸**Proof:**

If $\mathbf{T}$ is Kato then by **Theorem** 2.3 we have

$$\gamma(\mathbf{T}^n) \geqslant \gamma(\mathbf{T})^n \geqslant 0.$$

So $\mathbf{S} = \mathbf{T}^n$ has closed range .

Furthermore, $\mathcal{R}(\mathbf{S}^{\infty}) = \mathcal{R}(\mathbf{T}^{\infty})$ and, by **Theorem** 1.1, $Ker\mathbf{S} \subseteq \mathcal{R}(\mathbf{T}^{\infty}) = \mathcal{R}(\mathbf{S}^{\infty})$. From **Corollary** 1.3, which equivalent to the statements of **Theorem**1.1 we conclude that $\mathbf{T}^n$ is Kato.

■

Corollary 2.5. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, *where* $\mathbf{E}$ *is a Banach space. Then*

$\mathbf{T}$ *is Kato if and only if* $\mathcal{R}(\mathbf{T}^n)$ *is closed for all* $n \in \mathbb{N}$ *and* $\mathbf{T}$ *verifies one of the equivalent conditions of* ***Theorem*** *1.1.*

Theorem 2.4. *([4], **Theorem 1.19**): Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space. Then:*

$\mathbf{T}$ *is Kato if and only if* $\mathbf{T}'$ *is Kato.*

▸**Proof:**

Suppose that $\mathbf{T}$ is Kato. Then $\mathcal{R}(\mathbf{T})$ is closed so that $\gamma(\mathbf{T}) > 0$ by **Theorem** 2.1. From **Theorem** 2.3 we then obtain that $\gamma(\mathbf{T}^n) \geqslant \gamma(\mathbf{T})^n \geqslant 0$ and this implies, again by **Theorem**2.1, that $\mathcal{R}(\mathbf{T^n})$ is closed for every $n \in \mathbb{N}$. The same argument also shows that $\mathcal{R}((\mathbf{T}^n)') = \mathcal{R}((\mathbf{T}')^n)$ is closed for every $n \in \mathbb{N}$ by part (1) of **Theorem** 2.1. Therefore by part (2) of **Theorem**2.1 it follows that the equalities

$$Ker(\mathbf{T}^n)^{\perp} = \mathcal{R}(\mathbf{T}^{n\prime}) \;\; and \;\; {}^{\perp}Ker(\mathbf{T}^{n\prime}) = \mathcal{R}(\mathbf{T}^n) \tag{2.1}$$

hold for all $n \in \mathbb{N}$.

Now, since $\mathbf{T}$ is Kato then $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^n)$ for every $n \in \mathbb{N}$ and therefore $\mathcal{R}(\mathbf{T}^n)^{\perp} \subseteq Ker(\mathbf{T})^{\perp} = \mathcal{R}(\mathbf{T}')$. Moreover, from the second equality of (2.1) we obtain $Ker(\mathbf{T}'^n) = \mathcal{R}(\mathbf{T}^n)^{\perp}$, so that $Ker(\mathbf{T}'^n) \subseteq \mathcal{R}(\mathbf{T}')$ holds for every $n \in \mathbb{N}$. This shows, since $\mathcal{R}(\mathbf{T}')$ is closed, that $\mathbf{T}'$ is Kato.

A similar argument shows that if $\mathbf{T}'$ is Kato then also $\mathbf{T}$ is Kato .

■

Let $\mathbf{E}$, $\mathbf{F}$ be Banach spaces and $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ an operator. An operator $\mathbf{S} : \mathbf{F} \longrightarrow \mathbf{E}$ is called a generalized inverse of $\mathbf{T}$ if $\mathbf{TST} = \mathbf{T}$ and $\mathbf{STS} = \mathbf{S}$. It is easy to see that if $\mathbf{S} : \mathbf{F} \longrightarrow \mathbf{E}$ is a one-sided inverse of $\mathbf{T}$ (i.e., either $\mathbf{TS} = \mathbf{I_F}$ or $\mathbf{ST} = \mathbf{I_E}$), then $\mathbf{S}$ is a generalized inverse of $\mathbf{S}$.

Definition 2.6. *: An operator* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is called* ***Saphar*** *if* $\mathbf{T}$ *is Kato and has a generalized inverse. Equivalently,* $\mathbf{T}$ *is Saphar if and only if* $\mathbf{T}$ *has a generalized inverse and* $Ker\mathbf{T} \subseteq \mathcal{R}^{\infty}(\mathbf{T})$.

Remark 2.5. *Obviously, in Hilbert spaces the Saphar operators coincide with the* ***Kato operators****.*

Definition 2.7. *: A closed operator* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is called* ***essentially Saphar****, if* $\mathbf{T}$ *has a generalized inverse and* $Ker\mathbf{T} \subseteq_e \mathcal{R}^{\infty}(\mathbf{T})$.

Remark 2.6. *Obviously, in Hilbert spaces the Saphar operators coincide with the* ***essentially Kato*** *operators.*

Our aim is to show that the set of all **Saphar** operators is a regularity. This will be an immediate consequence of the following tow propositions, which are of independent interest.

Proposition 2.3. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *be a Saphar operator, let* $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ *satisfy* $\mathbf{TST} = \mathbf{T}$ *and let* $n \in \mathbb{N}$*. Then* $\mathbf{T}^n\mathbf{S}^n\mathbf{T}^n = \mathbf{T}^n$*. In particular,* $\mathbf{T}^n$ *is a Saphar operator.*

▸**Proof:**

Let $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ satisfy $\mathbf{TST} = \mathbf{T}$. We prove by induction on n that $\mathbf{T}^n\mathbf{S}^n\mathbf{T}^n = \mathbf{T}^n$

Suppose that $n \geqslant 1$ and $\mathbf{T}^n\mathbf{S}^n\mathbf{T}^n = \mathbf{T}^n$. Then

$$\mathbf{T}^{n+1}\mathbf{S}^{n+1}\mathbf{T}^{n+1} = \mathbf{T}(\mathbf{T}^n\mathbf{S}^n(\mathbf{ST}-\mathbf{I}) + \mathbf{T}^n\mathbf{S}^n)\mathbf{T}^n.$$

Since $\mathbf{T}^n\mathbf{S}^n\mathbf{T}^n = \mathbf{T}^n$ and $\mathbf{TST} = \mathbf{T}$, we can check easily that $\mathbf{T}^n\mathbf{S}^n$ is a projection onto $\mathcal{R}(\mathbf{T}^n)$ and $\mathbf{I}-\mathbf{ST}$ is a projection onto $Ker\mathbf{T} \subset \mathcal{R}(\mathbf{T}^n)$. Thus

$$\mathbf{T}^{n+1}\mathbf{S}^{n+1}\mathbf{T}^{n+1} = \mathbf{T}((\mathbf{ST}-\mathbf{I}) + \mathbf{T}^n\mathbf{S}^n)\mathbf{T}^n = \mathbf{T}\mathbf{T}^n\mathbf{S}^n\mathbf{T}^n = \mathbf{T}^{n+1}.$$

■

Proposition 2.4. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *be a Saphar operator. Then there exists* $\varepsilon > 0$ *such that* $\mathbf{T}-\mathbf{U}$ *has a generalized inverse for every operator* $\mathbf{U} \in \mathcal{B}(\mathbf{E})$ *commuting with* $\mathbf{T}$ *such that* $\|\mathbf{U}\| < \varepsilon$*. More precisely, if* $\mathbf{T}$ *is Kato,* $\mathbf{TST} = \mathbf{T}$*,* $\mathbf{UT} = \mathbf{TU}$ *and* $\|\mathbf{U}\| < \|\mathbf{S}\|^{-1}$*, then* $(\mathbf{T}-\mathbf{U})\mathbf{S}(\mathbf{I}-\mathbf{US})^{-1}(\mathbf{T}-\mathbf{U}) = \mathbf{T}-\mathbf{U}$ *.*

▸**Proof:**

Let $\mathbf{TST} = \mathbf{T}$, $\mathbf{UT} = \mathbf{TU}$ and $\|\mathbf{U}\| < \|\mathbf{S}\|^{-1}$.

We first prove by induction on n that $\mathbf{U}(\mathbf{SU})^n Ker\mathbf{T} \subset Ker\mathbf{T}^{n+1}$. This is clear for $n = 0$. Suppose that $\mathbf{U}(\mathbf{SU})^{n-1}Ker\mathbf{T} \subset Ker\mathbf{T}^n \subset \mathcal{R}(\mathbf{T})$ and let $z \in Ker\mathbf{T}$. Then $\mathbf{U}(\mathbf{SU})^{n-1}z = \mathbf{T}v$ for some $v \in \mathbf{E}$, and

$$\mathbf{T}^{n+1}\mathbf{U}(\mathbf{SU})^n z = \mathbf{T}^{n+1}\mathbf{UST}v = \mathbf{T}^n\mathbf{UTST}v = \mathbf{T}^n\mathbf{UT}v = \mathbf{UT}^n\mathbf{U}(\mathbf{SU})^{n-1}z = 0,$$

by the induction assumption. Thus $\mathbf{U}(\mathbf{SU})^n Ker\mathbf{T} \subset Ker\mathbf{T}^{n+1}$ for all n. Since $\mathbf{I}-\mathbf{ST}$ is a projection onto $Ker\mathbf{T}$, we have

$$\mathbf{U}(\mathbf{SU})^n(\mathbf{I}-\mathbf{ST})\mathbf{E} \subset Ker\mathbf{T}^{n+1} \subset \mathcal{R}(\mathbf{T}) \quad (n \geqslant 0),$$

and so

$$(\mathbf{I}-\mathbf{TS})\mathbf{U}(\mathbf{SU})^n(\mathbf{I}-\mathbf{ST}) = 0 \quad (n \geqslant 0).$$

Then

$$
\begin{aligned}
(\mathbf{T}-\mathbf{U})\mathbf{S}(\mathbf{I}-\mathbf{US})^{-1}(\mathbf{T}-\mathbf{U}) &= (\mathbf{T}-\mathbf{U})\mathbf{S}\sum_{i=0}^{\infty}(\mathbf{US})^{i}(\mathbf{T}-\mathbf{U}) \\
&= \mathbf{TST}-\mathbf{UST}-\mathbf{TSU}+\mathbf{TSUST} \\
&+\sum_{i=0}^{\infty}(\mathbf{TS}(\mathbf{US})^{i+2}\mathbf{T}-\mathbf{US}(\mathbf{US})^{i+1}\mathbf{T}-\mathbf{TS}(\mathbf{US})^{i+1}\mathbf{U}+\mathbf{US}(\mathbf{US})^{i}\mathbf{U}) \\
&= \mathbf{T}-\mathbf{UST}-TS\mathbf{U}+\mathbf{TSUST}+\sum_{i=0}^{\infty}(\mathbf{I}-\mathbf{TS})(US)^{i+1}\mathbf{U}(\mathbf{I}-\mathbf{ST}) \\
&= \mathbf{T}-\mathbf{U}+(\mathbf{I}-\mathbf{TS})\mathbf{U}(\mathbf{I}-\mathbf{ST}) \\
&= \mathbf{T}-\mathbf{U}.
\end{aligned}
$$

Hence $\mathbf{T}-\mathbf{U}$ has a generalized inverse.

■

2.2 Bounded below operators theory

A very important class of operators is the class of injective operators having closed range.

Definition 2.8. *: An operator $\mathbf{T}\in\mathcal{B}(\mathbf{E},\mathbf{F})$ is said to be bounded below if $\mathbf{T}$ is injective and has closed rang.*

Theorem 2.5. *([6], **Theorem 1.5**): Let $\mathbf{T}\in\mathcal{B}(\mathbf{E},\mathbf{F})$ is bounded below if and only if there exists $c>0$ such that*

$$\|\mathbf{T}x\| \geqslant c\|x\| \quad \forall x\in\mathbf{E}. \tag{2.2}$$

▸**Proof:**

Indeed, if $\|\mathbf{T}x\|\geqslant c\|x\|$ for some $c>0$ and all $x\in\mathbf{E}$ then $\mathbf{T}$ is injective. Moreover, if (x_n) is a sequence in $\mathbf{E}$ for which $(\mathbf{T}x_n)$ converges to $y\in\mathbf{E}$ then (x_n) is a Cauchy sequence and hence convergent to some $x\in\mathbf{E}$. Since $\mathbf{T}$ is continuous then $\mathbf{T}x=y$ and therefore $\mathcal{R}(\mathbf{T})$ is closed. Conversely, if $\mathbf{T}$ is injective and $\mathcal{R}(\mathbf{T})$ is closed then, from the open mapping theorem, it easily follows that there exists a $c>0$ for which the inequality (2.2) holds.

■

Example 2.13. *: Let $\mathbf{E}=C^1[0,1]$ and $\mathbf{F}=C[0,1]$, both with $\|\,.\,\|_\infty$ and $\mathbf{T}x=x'$, $\quad x\in C^1[0,1]$.*

Taking $x_n(t)=t \quad n$ for $t\in[a,b]$, $n\in\mathbb{N}$, we have

$$x_n\in\mathbf{E},\quad \|x_n\|_\infty=1,\quad \|\mathbf{T}x_n\|_\infty=n$$

for every $n\in\mathbb{N}$. Hence, $\mathbf{T}$ is not a bounded operator.

If $\mathbf{E}=\{x\in C^1[0,1]:\quad x(0)=0\}$, then $\mathbf{T}$ is bounded below, for in this case, we have

$$x(t)=\int_0^1 x(s)ds \quad \forall x\in\mathbf{E},\quad t\in[0,1],$$

so that

$$\|x\|_\infty \leqslant \|x_0\|_\infty = \|\mathbf{T}x\|_\infty \quad \forall x \in \mathbf{E}.$$

Theorem 2.6. *([7], **Theorem 1.2**) (Bounded Inverse Theorem): Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces and take any*
$\mathbf{T} \in \mathcal{B}(\mathbf{E}, \mathbf{F})$ *. The following assertions are equivalent:*

1. $\mathbf{T}$ *has a bounded inverse on its range;*

2. $\mathbf{T}$ *is bounded below;*

3. $\mathbf{T}$ *is injective and has a closed range .*

▸**Proof:**

Part (i). The equivalence between (1) and (2) still holds if $\mathbf{E}$ and $\mathbf{F}$ are just normed spaces. Indeed, if there exists $\mathbf{T}^{-1} \in \mathcal{B}(\mathcal{R}(\mathbf{T}), \mathbf{E})$, then there exists a constant $\beta > 0$ for which $\|\mathbf{T}\mathbf{T}^{-1}y\| \leqslant \beta\|y\| foreveryy \in \mathcal{R}(\mathbf{T})$. Take an arbitrary $x \in \mathbf{E}$ so that $\mathbf{T}x \in \mathcal{R}(\mathbf{T})$. Thus $\|x\| = \|\mathbf{T}^{-1}\mathbf{T}x\| \leqslant \beta\|\mathbf{T}x\|$, and so $\frac{1}{\beta}\|x\| \leqslant \|\mathbf{T}x\|$. Hence (1) implies (2). Conversely, if (2) holds true, then $0 < \|\mathbf{T}x\|$ for every nonzero $x \in \mathbf{E}$, and so $Ker(\mathbf{T}) = \{0\}$. Then $\mathbf{T}$ has a (linear) inverse on its range a linear transformation is injective if and only if it has a null kernel. Take an arbitrary $y \in \mathcal{R}(\mathbf{T})$ so that $y = \mathbf{T}x$ for some $x \in \mathbf{E}$. Thus $\|\mathbf{T}^{-1}y\| = \|\mathbf{T}^{-1}\mathbf{T}x\| = \|\|x\| \leqslant \frac{1}{\alpha}\|\mathbf{T}x\| = \frac{1}{\alpha}\|y\|$ for some constant $\alpha > 0$. Hence $\mathbf{T}^{-1}$ is bounded. Thus (2) implies (1).

Part (ii). Take an arbitrary $\mathcal{R}(\mathbf{T})$-valued convergent sequence $\{y_n\}$. Since each y_n lies in $\mathcal{R}(\mathbf{T})$, then there exists an $\mathbf{E}$ -valued sequence $\{x_n\}$ for which $y_n = \mathbf{T}x_n$ for each n . Since $\{\mathbf{T}x_n\}$ converges in $\mathbf{F}$, then it is a Cauchy sequence in $\mathbf{E}$. Thus if $\mathbf{T}$ is bounded below, then there exists $\alpha > 0$ such that

$$0 \leqslant \alpha\|x_m - x_n\| \leqslant \|\mathbf{T}(x_m - x_n)\| = \|\mathbf{T}x_m - \mathbf{T}x_n\|.$$

for every m, n. Hence $\{x_n\}$ is a Cauchy sequence in $\mathbf{E}$, and so it converges in $\mathbf{E}$ to, say, $x \in \mathbf{E}$ if $\mathbf{E}$ is a Banach space. Since $\mathbf{T}$ is continuous, it preserves convergence and hence $y_n = \mathbf{T}x_n \longrightarrow \mathbf{T}x$. Then the (unique) limit of $\{y_n\}$ lies in $\mathcal{R}(\mathbf{T})$. Conclusion: $\mathcal{R}(\mathbf{T})$ is closed in $\mathbf{F}$ by the classical closed set theorem. That is, $\overline{\mathcal{R}(\mathbf{T})} = \mathcal{R}(\mathbf{T})$ whenever $\mathbf{E}$ is a Banach space, where $\overline{\mathcal{R}(\mathbf{T})}$ stands for the closure of $\mathcal{R}(\mathbf{T})$. Moreover, since (2) trivially implies $Ker(\mathbf{T}) = \{0\}$, it follows that (2) implies (3). On the other hand, if $Ker(\mathbf{T}) = \{0\}$, then $\mathbf{T}$ is injective. If in addition the linear manifold : $\mathcal{R}(\mathbf{T})$ is closed in the Banach space $\mathbf{F}$, then it is itself a Banach space and so $\mathbf{T} : \mathbf{E} \longrightarrow \mathcal{R}(\mathbf{T})$ is an injective and surjective bounded linear transformation of the Banach space $\mathbf{E}$ onto the Banach space $\mathcal{R}(\mathbf{T})$. Hence its inverse $\mathbf{T}^{-1}$

lies in $\mathcal{B}(\mathcal{R}(\mathbf{T}), \mathbf{E})$ by the Inverse Mapping Theorem. Thus (3) implies (1).

∎

Example 2.14. *: Let* $\mathbf{E} = L^2[0,1] = \mathbf{F}$,

$$\mathbf{E}_0 = \left\{ x \in L^2[0,1] : \ x \ \textit{absolutely continuous} \ x(0) = 0, \ x_0 \in L^2[0,1] \right\}$$

and

$$\mathbf{T}x = x', \ \ x \in \mathbf{E}_0.$$

By the fundamental theorem of Lebesgue integration[2]*, we have*

$$x(t) = \int_0^t x'(s)\, ds,$$

for every $x \in \mathbf{E}_0$ *and* $t \in [0,1]$ *so that*

$$| x(t) | \leqslant \int_0^t | x'(s) | \leqslant \|x'\|_2.$$

Hence

$$\|x\|_2 \leqslant \|x'\|_2 \quad \forall x \in \mathbf{E}_0,$$

that is, $\mathbf{T}$ *is bounded below. Therefore,* $\mathbf{T}$ *has a bounded inverse from its range. Again, by fundamental theorem of Lebesgue integration, for every* $y \in L^2[0,1]$*, the function* x *defined by*

$$x(t) = \int_0^t y(s)\, ds, \quad t \in [0,1],$$

belongs to $\mathbf{E}_0$ *and* $x \quad 0 = y$ *so that* $\mathcal{R}(\mathbf{T}) = L^2[0,1]$*. Therefore,* $\mathbf{T}^{-1}$ *is a bounded operator with closed domain.*

Now we will define tow important quantities .

Definition 2.9. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E}, \mathbf{F})$*. The quantity*

$$j(\mathbf{T}) = \inf_{\|x\|=1} \|\mathbf{T}x\| = \inf_{x \neq 0} \frac{\|\mathbf{T}x\|}{\|x\|} = \inf \left\{ \ \|\mathbf{T}x\| \ : x \in \mathbf{E}, \ \|x\| = 1 \right\},$$

is called the injectivity modulus of $\mathbf{T}$.

[2] Fundamental theorem of Lebesgue integration: If $y \in L^1[0,1]$, then x defined by $x(t) = \int_0^t y(s)ds$ is absolutely continuous, differentiable a.e., and $x' = y$. Conversely, if $x : [0,1] \to \mathbb{K}$ is absolutely continuous, then it is differentiable a.e., $x' \in L^1[,b] and x(t) = \int_0^t y(s)ds$.

Definition 2.10. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$. *The quantity*

$$k(\mathbf{T}) = \inf\Big\{ \ r \geqslant 0 : \mathbf{T}B_{\mathbf{E}} \supset r.B_{\mathbf{F}} \Big\},$$

is called the surjectivity modulus of $\mathbf{T}$.

Remark 2.7. *:* $\mathbf{T}$ *is onto if and only if* $k(\mathbf{T}) > 0$. *Furthermore, if* $\mathbf{T}$ *is onto, then* $k(\mathbf{T}) > 0$ *by the open mapping theorem. If* $\mathbf{T}$ *is not onto, then* $k(\mathbf{T}) = 0$ *by definition.*

We have also the following results

$$\mathbf{T} \text{ is bounded below if and only if } j(\mathbf{T}) > 0,$$

and in this case $j(\mathbf{T}) = \gamma(\mathbf{T})$.

The next theorem shows that some properties of injectivity and surjectivity modulus .

Theorem 2.7. *: Let* $\mathbf{E},\mathbf{F},\mathbf{G}$ *be Banach spaces,* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$. *Then:*

1. $j(\mathbf{ST}) \leqslant \|\mathbf{S}\| j(\mathbf{T})$;
2. $j(\mathbf{ST}) \geqslant j(\mathbf{S}) j(\mathbf{T})$;
3. $k(\mathbf{ST}) \leqslant k(\mathbf{S}) \|\mathbf{T}\|$;
4. $k(\mathbf{ST}) \geqslant k(\mathbf{S}) k(\mathbf{T})$.

▸**Proof:**

See [19] .Theorem 6. p 87.

■

Remark 2.8. *: If* $\mathbf{T}$ *is bijective, then* $j(\mathbf{T}) = \|\mathbf{T}^{-1}\|^{-1} = k(\mathbf{T})$.

The relation between Kato operator and bounded below operators in the next example .

Example 2.15. *: Any operator that is either onto or bounded below is Kato. In particular, the isometrical shift* $\mathbf{S}$ *on a Hilbert space* $\mathcal{H}$ *is Kato. Note that in this case* $\mathcal{R}^{\infty}(\mathbf{S}) = \{0\} = \mathcal{N}\infty(\mathbf{S})$.

Similarly, $\mathbf{S}'$ *is also Kato and* $\mathcal{R}^{\infty}(\mathbf{S}) = \mathcal{H} = \mathcal{N}\infty(\mathbf{S})$.

The direct sum $\mathbf{S} \oplus \mathbf{S}'$ *is an example of a Kato operator that is neither onto nor bounded below.*

The next result shows that the properties of being bounded below or being surjective are dual each other.

Theorem 2.8. *([6], **Theorem 1.6**) : Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, *then*

1. $\mathbf{T}$ *is bounded below (respectively, surjective) if and only if* $\mathbf{T}'$ *is surjective (respectively, bounded below);*
2. *If* $\mathbf{T}$ *is bounded below (respectively, surjective) then* $\lambda\mathbf{I}-\mathbf{T}$ *is surjective (respectively, bounded below) for all* $|\lambda| < \gamma(\mathbf{T})$.

▸**Proof:**

(1). Suppose that $\mathbf{T}$ is surjective. Trivially $\mathbf{T}$ has closed range and therefore also $\mathbf{T}'$ has closed range. From the equality $Ker\mathbf{T}' = \mathcal{R}(\mathbf{T})^{\perp} = ^{\perp}\mathbf{E} = \{0\}$, we conclude that $\mathbf{T}'$ is injective.

Conversely, suppose that $\mathbf{T}'$ is bounded below. Then $\mathbf{T}'$ has closed range and hence by **Theorem** 2.1 the operator $\mathbf{T}$ has also closed range. From the equality $\mathcal{R}(\mathbf{T}) = ^{\perp} Ker\mathbf{T}' = ^{\perp} \{0\} = \mathbf{E}$ we then conclude that $\mathbf{T}$ is surjective. The proof of $\mathbf{T}$ being bounded below if and only if $\mathbf{T}'$ is surjective is analogous.

(2). Suppose that $\mathbf{T}$ is injective with closed range. Then $\gamma(\mathbf{T}) > 0$ and from definition of $\gamma(\mathbf{T})$ we obtain

$$\gamma(\mathbf{T})dist(x, Ker\mathbf{T}) = \gamma(\mathbf{T})\|x\| \leqslant \|\mathbf{T}x\| \quad for \;\; all \;\; x \in \mathbf{E}.$$

From that we obtain

$$\|(\lambda\mathbf{I}-\mathbf{T})x\| \geqslant \|\mathbf{T}x\| - |\lambda| \, \|x\| \geqslant (\gamma(\mathbf{T}) - |\lambda|)\|x\|,$$

thus for all $|\lambda| < \gamma(\mathbf{T})$, the operator $\lambda\mathbf{I}-\mathbf{T}$ is bounded below. The case that $\mathbf{T}$ is surjective follows now easily by considering the adjoint $\mathbf{T}'$.

■

Remark 2.9. *For all bounded operators in Banach space we have the following results*

$$j(\mathbf{T}) = k(\mathbf{T}') \qquad and \qquad k(\mathbf{T}) = j(\mathbf{T}').$$

2.3 Compact operators and Riesz-Schauder theory

There is a class of bounded operators, called compact (or completely continuous) operators, which are in many respect analogous to operators in finite-dimensional spaces. So in this section we reassume some of the basic properties of compact linear operators.

Definition 2.11. *: A bounded operator $\mathbf{T}$ from a Banach space $\mathbf{E}$ into a Banach space $\mathbf{F}$ is said to be compact if for every bounded sequence (x_n) of elements of $\mathbf{E}$ the corresponding sequence $(\mathbf{T}x_n)$ contains a convergent subsequence. This is equivalent to saying that the closure of $\mathbf{T}(B_{\mathbf{E}})$, $B_{\mathbf{E}}$ the closed unit ball of $\mathbf{E}$, is a compact subset of $\mathbf{F}$.*

Now we give a supplementary examples of compact operator

Example 2.16. *: An important example of compact operators are integral operators.*

Consider the Banach space $C[a,b]$ of all continuous complex-valued functions on a bounded closed interval $[a,b]$ with the sup-norm

A continuous function $k(s,t)$ defined on $[a,b]\times[a,b]$ defines an operator $\mathbf{T}$ on $C[a,b]$ by

$$(\mathbf{T}f)(s)=\int_a^b k(s,t)f(t)dt.$$

It follows from classical results of analysis that $\mathbf{T}$ is a compact operator.

The classical Fredholm[3] integral equation is

$$\lambda f(s)-\int_a^b k(s,t)f(t)dt=g(s) \qquad (a\leqslant s\leqslant b),$$

where $g\in C[a,b]$ is given, λ is a parameter and f is unknown. Clearly, we can write the equation as $(\lambda\mathbf{I}-\mathbf{T})f=g$.

This was the original motivation that led to the study of operators of the form $\lambda\mathbf{I}-\mathbf{T}$ where $\mathbf{T}$ is compact, . The theory of these operators is sometimes referred to as the Riesz-Schauder theory.

Example 2.17. *: Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $(e_i)_{i\geqslant 1}$. Operators $\mathbf{T}\in\mathcal{B}(\mathcal{H})$ defined by*

$$\mathbf{T}e_i=\sum_{j=1}^{\infty}\alpha_{i,j}e_j \quad (j\geqslant 1),$$

*where $\alpha_{i,j}\in\mathbb{C}$ satisfy $\sum_{i,j}|\alpha_{i,j}|^2<\infty$, are called **Hilbert-Schmidt**. Clearly, $\sum_{i,j}|\alpha_{i,j}|^2=\sum_j\|\mathbf{T}e_j\|^2$; this number does not depend on the choice of an orthonormal basis (e_j). Hilbert-Schmidt operators are an important example of compact operators.*

Example 2.18. *: The **Volterra** operator $\mathbf{V}:L^2[0,1]\to L^2[0,1]$ is defined by*

[3] It is named in honor of Erik Ivar Fredholm, Swedish mathematician, April 7, 1866- August 17, 1927.

$$(\mathbf{V}f)(x) = \int_0^x f(y)dy.$$

$\mathbf{V}$ *is an example of a compact operator.*

Definition 2.12. *: We say that* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *is of finite rank if* $\boldsymbol{dim}\mathcal{R}(\mathbf{T}) < \infty$.

The set of all compact (resp finite-rank) operators from $\mathbf{E}$ to $\mathbf{F}$ will be denoted by $\mathcal{K}(\mathbf{E},\mathbf{F})$ and (resp. $\mathcal{F}(\mathbf{E},\mathbf{F})$) , respectively. If $\mathbf{E} = \mathbf{F}$, then we write $\mathcal{K}(\mathbf{E}) = \mathcal{K}(\mathbf{E},\mathbf{E})$ and (resp . $\mathcal{F}(\mathbf{E}) = \mathcal{F}(\mathbf{E},\mathbf{E})$) for short.

Remark 2.10. *:*

1. $\mathcal{K}(\mathbf{E},\mathbf{F})$ *is a closed subspace of* $\mathcal{B}(\mathbf{E},\mathbf{F})$.
2. $\mathcal{F}(\mathbf{E},\mathbf{F})$ *is a subspace of* $\mathcal{B}(\mathbf{E},\mathbf{F})$ *and* $\overline{\mathcal{F}(\mathbf{E},\mathbf{F})} \subset \mathcal{K}(\mathbf{E},\mathbf{F})$.

Theorem 2.9. *: Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces. Then, if* $\mathbf{T} \in \mathcal{K}(\mathbf{E},\mathbf{F})$, *then* $\mathbf{T}$ *is of finite rank if and only if* $\mathcal{R}(\mathbf{T})$ *is closed .*

▸**Proof:**

Clearly, each finite-rank operator has closed range.

For the converse, let $\mathbf{T} : \mathbf{E} \to \mathbf{F}$ be compact and $\mathcal{R}(\mathbf{T})$ closed. By the open mapping theorem, there is a positive constant k with $\mathbf{T}B_{\mathbf{E}} \supset k.B_{\mathcal{R}(\mathbf{T})}$. Since $\mathbf{T}$ is compact, we conclude that $k.B_{\mathcal{R}(\mathbf{T})}$ is compact. Hence dim $\mathcal{R}(\mathbf{T}) < \infty$.

■

Let $\mathbf{E}$ and $\mathbf{F}$ be two Banach spaces and $\mathbf{E} \otimes \mathbf{F}$ be the algebraic completion of the tensor product of $\mathbf{E}$ and $\mathbf{F}$. The tensor product of $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ and $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ on $\mathbf{E} \otimes \mathbf{F}$ is the operator defined as

$$(\mathbf{T} \otimes \mathbf{S})(\sum_i x_i \otimes y_i) = \sum_i \mathbf{T}x_i \otimes \mathbf{S}y_i,$$

for each $\sum_i x_i \otimes y_i \in \mathbf{E} \otimes \mathbf{F}$.

Now let $u \in \mathbf{F}$ and $f \in \mathbf{E}'$, we define

$$(u \otimes f)(x) = f(x)u, \ \ for\ all\ x \in \mathbf{E},$$

we can observe that $u \otimes f$ is bounded linear manifold, with $\|(u \otimes f)(x)\| = \|f\|\ \|u\|\|x\|$.

The operators $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ of finite-rank , with $\mathbf{rank}(\mathbf{T}) = 1$ writes from the form $\mathbf{T} = u \otimes f$.

Example 2.19. *: Let* $\mathbf{E}$, $\mathbf{F}$ *be Banach spaces,* $x' \in \mathbf{E}'$ *and* $y \in \mathbf{F}$. *Denote by* $y \otimes x' : \mathbf{E} \to \mathbf{E}$ *the operator defined by*

$$(y \otimes x')x = \langle x, x' \rangle y \ \ (x \in \mathbf{E}).$$

Obviously, $\|y \otimes x'\| = \|y\|.\|x'\|$ *and* $\textbf{dim}\mathcal{R}(y \otimes x') = 1$.

Finite-rank operators are precisely finite linear combinations of operators of this form.

Operators that can be expressed as $\sum_{i=1}^{\infty} y_i \otimes x_i'$ *for some* $y_i \in \mathbf{F}$ *and* $x_i' \in \mathbf{E}'$ *with* $\sum_i \|y_i\|\|x_i'\| < \infty$ *are called nuclear. It is easy to see that nuclear operators are norm-limits of finite-rank operators and therefore they are compact. Nuclear operators acting on* $\mathbf{E}$ *form a non-closed two-sided ideal.*

The next theorem shows that the compactness of dual map.

Proposition 2.5. ***(Schauder theorem)****: If* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$,$\lambda \in \mathbb{K}$, *then:*

$\mathbf{T}$ *is compact operator if and only if* $\mathbf{T}'$ *is also compact operator.*

▸**Proof:**

Suppose that $\mathbf{T}$ is compact and let $\varepsilon > 0$. We must show that there exists a finite subset $\{y_1', \ldots, y_p'\} \subset B_{\mathbf{F}'}$ such that for every $y' \in B_{\mathbf{F}'}$ there exists r, $1 \leqslant r \leqslant p$ with $\|\mathbf{T}'y' - \mathbf{T}'y_r'\| \leqslant \varepsilon$.

Since $\mathbf{T}$ is compact, there exists a finite subset $\{x_1, \ldots, x_n\} \subset B_{\mathbf{E}}$ such that $\min\{\|\mathbf{T}x - \mathbf{T}x_j\| : 1 \leqslant j \leqslant n\} \leqslant \frac{\varepsilon}{3}$ for every $x \in B_{\mathbf{E}}$.

The set $\{(\langle \mathbf{T}x_1, y' \rangle, \ldots, \langle \mathbf{T}x_n, y' \rangle) : y' \in B_{\mathbf{F}'}\}$ is a bounded subset of $\mathbb{C}^n$, therefore there exists a finite subset $\{y_1', \ldots, y_p'\}$ of $B_{\mathbf{F}'}$ such that for each $y' \in B_{\mathbf{F}'}$ there exists $r \in \{1, \ldots, p\}$ with the property

$$|\langle \mathbf{T}x_j \ , \ \ y' - y_r' \rangle| \leqslant \frac{\varepsilon}{3} \qquad (1 \leqslant j \leqslant n). \tag{2.3}$$

We show that $\{y_1', \ldots, y_p'\}$ is the required subset of $B_{\mathbf{F}'}$

Let $y' \in B_{\mathbf{F}'}$. Find $r \in \{1, \ldots, p\}$ with (2.3). Let $x \in B_{\mathbf{E}}$. Then there is a $j \in \{1, \ldots, n\}$ such that $\|\mathbf{T}x - \mathbf{T}x_j\| \leqslant \frac{\varepsilon}{3}$ and we have

$$\begin{aligned} |\langle x, \mathbf{T}'y' \rangle - \langle x, \mathbf{T}'y_r' \rangle| &\leqslant |\langle x - x_j, \mathbf{T}'y' | + |\langle xj, \mathbf{T}'(y' - y_r') \rangle| + |\langle x_j - x, \mathbf{T}'y_r' \rangle| \\ &= |\langle \mathbf{T}x - \mathbf{T}x_j, y' \rangle| + |\langle \mathbf{T}x_j, y' - y_r' \rangle| + |\langle \mathbf{T}x_j - \mathbf{T}x, y_r' \rangle| \leqslant \varepsilon. \end{aligned}$$

Thus

$$\|\mathbf{T}'y' - \mathbf{T}'y_r'\| = \sup |\langle x, \mathbf{T}'y' - \mathbf{T}'y_r' \rangle| : x \in\in B_{\mathbf{E}} \leqslant \varepsilon$$

and $\mathbf{T}'$ is compact.

Suppose now that $\mathbf{T}'$ is compact. Then $\mathbf{T}''$ is compact and so is $\mathbf{T} = \mathbf{T}''_{\backslash \mathbf{E}}$, since $\overline{\mathbf{T}B_{\mathbf{E}}} = \overline{\mathbf{T}''(B_{\mathbf{E}''} \cap \mathbf{E})} \subset \mathbf{T}''B_{\mathbf{E}''}$, which is compact.

■

Example 2.20. : *By* ***Example*** *2.3 and* ***Example*** *2.16, then*

$$(\mathbf{T}'x))(s) = \int_a^b \overline{k(t,s)}x(t)\,dt \;\; x \in C[a,b],$$

is also compact operator.

In the sequel we will need the following important lemma.

Lemma 2.2. *([25],* ***Lemma 10.2)(Riesz lemma):*** *Let $\mathcal{M}$ be a proper closed subspace of a normed space $\mathbf{E}$. Then for every $0 < \delta < 1$ there exists a vector $x_\delta \in \mathbf{E}$ such that $\|x_\delta\| = 1$ and*

$$\|y - x_\delta\| \geqslant \delta \text{ for all } y \in \mathcal{M}.$$

▸**Proof:**

Let $y \in \mathbf{E}$ such that $y \notin \mathcal{M}$. Set $\rho = \inf_{x \in \mathcal{M}} \|x - y\|$ and let (x_n) be a sequence such that $\|x_n - y\| \longrightarrow \rho$ as $n \longrightarrow \infty$. Since $\mathcal{M}$ is closed we have $\rho > 0$. Now, if $0 < \delta < 1$ then $\rho/\delta > \rho$, hence there exists $z \in \mathcal{M}$ such that $0 < \|z - y\| \leqslant \rho/\delta$. Setting $\gamma = \frac{1}{\|z-y\|}$ and $x_\delta = \gamma(y - z)$ we then obtain $\|x_\delta\| = 1$. Since $(\frac{1}{\gamma})x + z \in \mathcal{M}$ and $\gamma \geqslant \delta/\rho$ it then follows that

$$\|x - x_\delta\| = \gamma \|(\frac{1}{\gamma}x + z) - y\| \geqslant \frac{\delta}{\rho}.\rho = \delta,$$

as desired.

■

Riesz[4] **Lemma** has many important consequences. One of the most important is that the Bolzano[5]Weiestrass[6] theorem.

Theorem 2.10. ***(Bolzano-Weiestrass theorem)*** : *Suppose that $\mathbf{E}$ is a normed vector space. Then every bounded sequence contains a convergent subsequence precisely when the space $\mathbf{E}$ is finite-dimension.*

▸**Proof:**

See [25]. Theorem 10.1 . p 58.

[4]**Frigyes Riesz** was Hungarian mathematician . 22 January 1880- 28 February 1956.
[5]**Bernard Bolzano** was a Bohemain mathematician. 5 October 178166 18 December 1848.
[6]**Karl Theodor Wilhelm Weiestrass** was a German mathematician . 31 October 1815- 19 February 1897.

■

Theorem 2.11. *: If $\mathbf{T} \in \mathcal{K}(\mathbf{E})$ and $\lambda \in \mathbb{C}\backslash\{0\}$, then $\mathcal{R}(\lambda\mathbf{I}-\mathbf{T})$ is closed.*

▸**Proof:**

We can suppose $\lambda = 1$ since $\lambda\mathbf{I}-\mathbf{T} = \lambda(I - \frac{1}{\lambda}\mathbf{T})$ and $\frac{1}{\lambda}\mathbf{T}$ is compact. To show that $\mathcal{R}(\mathbf{I}-\mathbf{T})$ is closed, set $\mathbf{S} = \mathbf{I}-\mathbf{T}$. We show that $y_n = \mathbf{S}x_n \to y$ implies $y \in \mathcal{R}(\mathbf{S})$. Let

$$\lambda_n = \inf_{u \in Ker\mathbf{S}} \|x_n - u\|.$$

Then for every n there exists $u_n \in Ker\mathbf{S}$ such that $\|x_n - u_n\| \leqslant 2\lambda_n$ and if we set $v_n = x_n - u_n$ then $y_n = \mathbf{S}v_n$ and $\|v_n\| \leqslant 2\lambda_n$. We claim that the sequence (v_n) is bounded. Suppose that (v_n) is unbounded. Then it contains a subsequence, which will be denoted again by (v_n), such that $\|v_n\| \to \infty$. If we set $w_n = \frac{v_n}{\|v_n\|}$, it easily follows that $\mathbf{S}w_n \to 0$. Since $\|w_n\| = 1$, the compactness of $\mathbf{T}$ implies the existence of a convergent subsequence of $(\mathbf{T}w_n)$. Let $(\mathbf{T}w_{nj})$ be such a sequence and say that $\mathbf{T}w_{nj} \to z$. Clearly,

$$w_{n_j} = (\mathbf{I}-\mathbf{T})w_{nj} + \mathbf{T}w_nj = \mathbf{S}w_{nj} + \mathbf{T}w_{nj} \to z.$$

Consequently, $\mathbf{S}z = lim\ \ \mathbf{S}w_{nj} = 0$, thus $z \in Ker\mathbf{S}$. An easy estimate yields

$$\|w_n - z\| = \|\frac{x_n - u_n - z}{\|v_n\|}\| = \frac{1}{\|v_n\|}\|x_n - (u_n + \|v_n\|z)\| \geqslant \frac{\lambda_n}{\|v_n\|},$$

and this is impossible, since $w_{nj} \to z$. Thus, (v_n) is bounded and since $\mathbf{T}$ is compact then $(\mathbf{T}v_n)$ contains a convergent subsequence $(\mathbf{T}v_{nj})$. From $v_{nj} = \mathbf{S}v_{nj} + \mathbf{T}v_{nj} = y_{nj} + \mathbf{T}v_{nj}$ we see that (v_{nj}) converges to some $v \in \mathbf{E}$, so that

$$y = lim\ \ y_n = lim\ \ y_{nj} = lim\ \ v_{nj} = Sv \in \mathcal{R}(\mathbf{S}),$$

thus $\mathcal{R}(\mathbf{S}) = \mathcal{R}(\lambda\mathbf{I}-\mathbf{T})$ is closed.

■

Proposition 2.6. *: Suppose that $\mathbf{T}$ and $\mathbf{S}$ are commuting bounded linear operators on the Banach space $\mathbf{E}$. If $\mathbf{T}-\mathbf{S}$ is compact and $\mathbf{T}$ is onto, then $\mathbf{S}$ has finite descent.*

▸**Proof:**

For each nonnegative integer k, the range, $\mathcal{R}(\mathbf{S}^k)$, has finite codimension and the map induced by $\mathbf{T}$ on $\mathbf{E}/\mathcal{R}(\mathbf{S}^k)$ is onto. Therefore this induced map is one-to-one, so that the kernel $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{S}^k)$. Since $\mathbf{T}$ is onto, there is a positive number δ for which $\|\mathbf{T}x\| > \delta.\mathbf{dist}(x, Ker\mathbf{T})$ for all x in $\mathbf{E}$. Suppose that x belongs to $\mathbf{E}$ and z belongs to $\mathcal{R}(\mathbf{S}^k)$; then $\mathbf{T}(\mathcal{R}(\mathbf{S}^k)) = \mathcal{R}(\mathbf{S}^k\mathbf{T}) = \mathcal{R}(\mathbf{S}^k)$ so there is a y in $\mathcal{R}(\mathbf{S}^k)$ with $\mathbf{T}y = z$. Thus we have

$\|\mathbf{T}x - z\| = \|\mathbf{T}(x-y)\| > \delta.\mathbf{dist}(x-y, Ker\mathbf{T}) > \delta.\mathbf{dist}(x, \mathcal{R}(\mathbf{S}^k))$, since $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{S}^k)$. Since this holds for all z in $\mathcal{R}(\mathbf{S}^k)$, we obtain

$$\mathbf{dist}(\mathbf{T}x, \mathcal{R}(\mathbf{S}^k)) > \delta.\mathbf{dist}(x, \mathcal{R}(\mathbf{S}^k)),$$

Suppose $\mathbf{S}$ had infinite descent. Then there would be a bounded sequence $\{x_n\} with x_n \in \mathcal{R}(\mathbf{S})$ and $\mathbf{dist}(x_n, \mathcal{R}(\mathbf{S}^{k+1})) \geqslant 1$. Let $\mathbf{K} = \mathbf{T} - \mathbf{S}$ and suppose $m > n$. Then $\mathbf{K}x_m - \mathbf{K}x_n = (\mathbf{K} + (\mathbf{T} - \mathbf{K})x_n) - \mathbf{T}x_n$. So that

$$\|\mathbf{K}x_m - \mathbf{K}x_n\| > \mathbf{dist}(\mathbf{T}x_n, \mathcal{R}(\mathbf{S}^{n+1})) > \delta.\mathbf{dist}(x_n, \mathcal{R}(\mathbf{S}^{n+1})) > \delta.$$

But this contradicts the compactness of $\mathbf{K}$, so $\mathbf{S}$ must have finite descent.

■

Proposition 2.7. *: Suppose that* $\mathbf{T}$ *and* $\mathbf{S}$ *are commuting bounded linear operators on the Banach space* $\mathbf{E}$*. If* $\mathbf{T} - \mathbf{S}$ *is compact and* $\mathbf{T}$ *is bounded below, then* $\mathbf{S}$ *has finite ascent.*

▸**Proof:**

See [8] .Lemma 5.1 . p 332 .

■

The next results show us the **Fredholm alternative** properties.

Lemma 2.3. *: If* $\mathbf{T} \in \mathcal{K}(\mathbf{E})$, $\mathbf{E}$ *Banach space, then* $asc(\lambda\mathbf{I} - \mathbf{T}) < \infty$ *for all* $\lambda \neq 0$.

▸**Proof:**

Fix $\lambda \in \mathbb{C}\{\}$ By contradiction, let us suppose that $Ker(\lambda\mathbf{I} - \mathbf{T}^{n-1})$ is a proper subspace of $Ker(\lambda\mathbf{I} - \mathbf{T}^n)$ for every $n \in \mathbb{N}$. (where $(\lambda\mathbf{I} - \mathbf{T}^0) = \mathbf{I}$) Applying **Riesz's lemma**, we can infer that for every $n \in \mathbb{N}$ there exists $x_n \in Ker(\lambda\mathbf{I} - \mathbf{T}^n)$ such that $\|x_n\| = 1$ and $dist(x_n, Ker(\lambda\mathbf{I} - \mathbf{T}^{n-1})$. Note that:

$$\begin{aligned} |\lambda|^{-1} \| \mathbf{T}x_n - \mathbf{T}x_m \| &= |\lambda|^{-1} \| \lambda x_n - (\lambda\mathbf{I} - \mathbf{T})x_n + (\lambda\mathbf{I} - \mathbf{T}))x_m - \lambda x_m \| \\ &= \| x_n - (\lambda^{-1}(\lambda\mathbf{I} - \mathbf{T})x_n - \lambda^{-1}(\lambda\mathbf{I} - \mathbf{T})x_m + x_m) \| \geqslant \tfrac{1}{2} . \end{aligned}$$

For every $n, m \in \mathbb{N}$, with n¿m, since $\lambda^{-1}(\lambda\mathbf{I} - \mathbf{T})x_n - \lambda^{-1}(\lambda\mathbf{I} - \mathbf{T})x_m + x_m \in Ker(\lambda\mathbf{I} - \mathbf{T}^{n-1})$ This implies that no convergent subsequence of $(\mathbf{T})x_n)_{n\in\mathbb{N}}$ exists, which is a contradiction, since $\mathbf{T} \in \mathcal{K}(\mathbf{E})$ and $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence. We thus conclude that $asc(\lambda\mathbf{I} - \mathbf{T}) < \infty$.

■

Lemma 2.4. *: If* $\mathbf{T} \in \mathcal{K}(\mathbf{E})$, $\mathbf{E}$ *Banach space, then* $dsc(\lambda\mathbf{I} - \mathbf{T}) < \infty$ *for all* $\lambda \neq 0$.

▸**Proof:**

In a completely similar way of **Lemma**2.3 proof, we can show that $dsc(\lambda\mathbf{I}-\mathbf{T})<\infty$.

■

Corollary 2.6. *: If* $\mathbf{T}\in\mathcal{K}(\mathbf{E})$, $\mathbf{E}$ *Banach space, then* $asc(\lambda\mathbf{I}-\mathbf{T})=dsc(\lambda\mathbf{I}-\mathbf{T})<\infty$ *for all* $\lambda\neq 0$.

▸**Proof:**

The result follows immediately by **Theorem** 1.5.

■

Lemma 2.5. *: If* $\mathbf{T}\in\mathcal{K}(\mathbf{E})$ *and* $\lambda\in\mathbb{C}\backslash\{0\}$, *then* $\alpha(\lambda\mathbf{I}-\mathbf{T})<\infty$.

▸**Proof:**

We can suppose $\lambda=1$ since $\lambda\mathbf{I}-\mathbf{T}=\lambda(I-\frac{1}{\lambda}\mathbf{T})$ and $\frac{1}{\lambda}\mathbf{T}$ is compact. If (x_n) is a bounded sequence in $Ker(\lambda\mathbf{I}-\mathbf{T})$ we have $\mathbf{T}x_n=x_n$. Since $\mathbf{T}$ is compact then there exists a convergent subsequence of $(\mathbf{T}x_n)=(x_n)$, so from **Bolzano-Weiestrass theorem** we deduce that $Ker(\mathbf{I}-\mathbf{T})$ is finite-dimensional.

■

Corollary 2.7. *: If* $\mathbf{T}\in\mathcal{K}(\mathbf{E})$ *and* $\lambda\in\mathbb{C}\backslash\{0\}$, *then* $\beta(\lambda\mathbf{I}-\mathbf{T})<\infty$, *and* $ind(\lambda\mathbf{I}-\mathbf{T})=0$.

▸**Proof:**

The result follows immediately by **Theorem** 1.8 .

■

Theorem 2.12. *: If* $\mathbf{T}\in\mathcal{K}(\mathbf{E})$ *and* $\lambda\in\mathbb{C}\backslash\{0\}$, *then*

$$\begin{aligned}\alpha(\lambda\mathbf{I}-\mathbf{T})&=\alpha(\lambda\mathbf{I}-\mathbf{T}')\\&=\beta(\lambda\mathbf{I}-\mathbf{T})=\beta(\lambda\mathbf{I}-\mathbf{T}').\end{aligned}$$

▸**Proof:**

By **Proposition** 2.1 and **Proposition** 2.11, then

$$\alpha(\lambda\mathbf{I}-\mathbf{T})=\beta(\lambda\mathbf{I}-\mathbf{T}')\quad and\quad\beta(\lambda\mathbf{I}-\mathbf{T})=\alpha(\lambda\mathbf{I}-\mathbf{T}').$$

By the preceding theorem, we have $\alpha(\lambda\mathbf{I}-\mathbf{T})=\beta(\lambda\mathbf{I}-\mathbf{T})$,Hence

$$\alpha(\lambda\mathbf{I}-\mathbf{T})=\alpha(\lambda\mathbf{I}-\mathbf{T}')=\beta(\lambda\mathbf{I}-\mathbf{T})=\beta(\lambda\mathbf{I}-\mathbf{T}').$$

■

Remark 2.11. *: If* $\mathbf{T} \in \mathcal{K}(\mathbf{E})$, *then* $ind(\lambda \mathbf{I} - \mathbf{T}) = ind(\lambda \mathbf{I} - \mathbf{T}') = 0.$

Corollary 2.8. *: If* $\mathbf{T} \in \mathcal{K}(\mathbf{E})$ *and* $\lambda \in \mathbb{C} \backslash \{0\}$, *then*

$$\alpha(\lambda \mathbf{I} - \mathbf{T}^n) = \alpha(\lambda \mathbf{I} - \mathbf{T}'^n)$$

$$= \beta(\lambda \mathbf{I} - \mathbf{T}^n) = \beta(\lambda \mathbf{I} - \mathbf{T}'^n).$$

Remark 2.12. *: If* $\mathbf{T} \in \mathcal{K}(\mathbf{E})$, $\lambda \in \mathbb{C} \backslash \{0\}$ *and* $Ker(\lambda \mathbf{I} - \mathbf{T}) = \{0\}$ *then* $\mathcal{R}(\lambda \mathbf{I} - \mathbf{T}) = \mathbf{E}$.

2.4 The Kato decomposition property

In this section, we will study two important invariant subspace . As in the previous section, let $\mathbf{E}, \mathbf{F}$ be tow Banach spaces, and $\mathbf{T} \in \mathcal{B}(\mathbf{E})$.

2.4.1 About analytic core and quasi-nilpotent part of an operator.

The following subspace is in a certain sense, the analytic counterpart of the algebraic core $\mathrm{C}(\mathbf{T})$.

Definition 2.13. *: Let* $\mathbf{E}$ *be a Banach space and* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$. *The analytical core of* $\mathbf{T}$ *is the set* $\mathrm{K}(\mathbf{T})$ *of all* $x \in \mathbf{E}$ *such that there exists a sequence* $(u_n) \subset \mathbf{E}$ *and a constant* $\delta > 0$ *such that*

1. $x = u_0$, $\mathbf{T}u_{n+1} = u_n$ *for every* $n \in \mathbb{Z}^+$;
2. $\|u_n\| \leqslant \delta^n \|x\|$ *for every* $n \in \mathbb{Z}^+$.

We now introduce another important invariant subspace.

Definition 2.14. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space. The quasi-nilpotent part of* $\mathbf{T}$ *is defined to be the set*

$$\mathrm{H}_0(\mathbf{T}) := \{x \in \mathbf{E} \ : \ \lim_{n \to \infty} \|\mathbf{T}^n x\|^{1/n}\} = 0.$$

Remark 2.13. *:*

Clearly $\mathrm{K}(\mathbf{T})$ *and* $\mathrm{H}_0(\mathbf{T})$ *are linear subspace of* $\mathbf{E}$, *generally not closed.*

In the following theorems we collect some elementary properties of $\mathrm{K}(\mathbf{T})$ and $\mathrm{H}_0(\mathbf{T})$.

Theorem 2.13. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space. Then:*

1. $\mathrm{K}(\mathbf{T})$ *is a linear subspace of* $\mathbf{E}$;
2. $\mathbf{T}(\mathrm{K}(\mathbf{T})) = \mathrm{K}(\mathbf{T})$;
3. $\mathrm{K}(\mathbf{T}) \subseteq \mathrm{C}(\mathbf{T})$.

▸**Proof:**

(1) It is evident that if $x \in \mathrm{K}(\mathbf{T})$ then $\lambda x \in \mathrm{K}(\mathbf{T})$ for every $\lambda \in \mathbb{C}$. We show that if $x, y \in \mathrm{K}(\mathbf{T})$ then $x + y \in \mathrm{K}(\mathbf{T})$. If $x \in \mathrm{K}(\mathbf{T})$ there exists $\delta_1 > 0$ and a sequence $(u_n) \subset \mathbf{E}$ satisfying the condition (1) and which is such that $\|u_n\| \leqslant \delta_1^n \|x\|$ for all $n \in \mathbb{Z}^+$. Analogously, since $y \in \mathrm{K}(\mathbf{T})$ there exists $\delta_2 > 0$ and a sequence $(v_n) \subset \mathbf{E}$ satisfying the condition (1) of the definition of $\mathrm{K}(\mathbf{T})$ and such that $\|v_n\| \leqslant \delta_2^n \|y\|$ for every $n \in \mathbb{N}$.

Let $\delta = \max\{\delta_1, \delta_2\}$. We have

$$\|u_n + v_n\| \leqslant \|u_n\| + \|v_n\| \leqslant \delta_1^n \|x\| + \delta_2^n \|y\| \leqslant \delta^n (\|x\| + \|y\|).$$

Trivially, if $x + y = 0$ there is nothing to prove since $0 \in \mathrm{K}(\mathbf{T})$. Suppose then $x + y \neq 0$ and set

$$\mu = \frac{\|x\| + \|y\|}{\|x + y\|}$$

Clearly $\mu \geqslant 1$, so $\mu \geqslant \mu^n$ and therefore

$$\|u_n + v_n\| \leqslant (\delta)^n \mu \|x + y\| \leqslant (\delta\mu)^n \|x + y\| \text{ for all } n \in \mathbb{Z}^+,$$

which shows that also the property (2) of the definition of $\mathrm{K}(\mathbf{T})$ is verified for every sum $x + y$, with $x, y \in \mathrm{K}(\mathbf{T})$. Hence $x + y \in \mathrm{K}(\mathbf{T})$, and consequently $\mathrm{K}(\mathbf{T})$ is a linear subspace of $\mathbf{E}$.

The proof **(2)** of is analogous to that of **Theorem**1.4, whilst **(3)** is a trivial consequence of **(2)** and the definition of $\mathrm{C}(\mathbf{T})$.

■

Proposition 2.8. *: For every* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space, we have:*

1. $Ker(\mathbf{T}^m) \subseteq \mathcal{N}^\infty(\mathbf{T}) \subseteq \mathrm{H}_0(\mathbf{T})$ *for every* $m \in \mathbb{N}$;
2. $x \in \mathrm{H}_0(\mathbf{T}) \Leftrightarrow Tx \in \mathrm{H}_0(\mathbf{T})$;
3. $Ker(\lambda\mathbf{I} - \mathbf{T}) \cap \mathrm{H}_0(\mathbf{T}) = 0$ *for every* $\lambda \neq 0$.

▸**Proof:**

(1). If $\mathbf{T}^m x = 0$ then $\mathbf{T}^n x = 0$ for every $n \geqslant m$.

(2). If $x_0 \in \mathrm{H}_0(\mathbf{T})$ from the inequality $\|\mathbf{T}^n \mathbf{T}x\| \leqslant \|\mathbf{T}\|\|\mathbf{T}^n x\|$ it easily follows that $\mathbf{T}x \in \mathrm{H}_0(\mathbf{T})$. Conversely, if $\mathbf{T}x \in \mathrm{H}_0(\mathbf{T})$ from

$$\|\mathbf{T}^{n-1}\mathbf{T}x\|^{1/n-1} = (\|\mathbf{T}^n x\|^{1/n})^{n/n-1},$$

we conclude that $x \in \mathrm{H}_0(\mathbf{T})$.

(3). If $x \neq 0$ is an element of $Ker(\lambda\mathbf{I} - \mathbf{T})$ then $\mathbf{T}^n x = \lambda^n x$, so

$$\lim_{n\to\infty} \|\mathbf{T}^n x\|^{1/n} = \lim_{n\to\infty} \mid \lambda \mid \|x\|^{1/n} = \mid \lambda \mid$$

and therefore $x \notin \mathrm{H}_0(\mathbf{T})$.

■

Theorem 2.14. *: Suppose that* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*. Then we have*

1. *If* $\mathcal{M}$ *is a closed subspace of* $\mathbf{E}$ *such that* $\mathbf{T}(\mathcal{M}) = \mathcal{M}$ *then* $\mathcal{M} \subseteq \mathrm{K}(\mathbf{T})$*;*
2. *If* $\mathrm{C}(\mathbf{T})$ *is closed then* $\mathrm{C}(\mathbf{T}) = \mathrm{K}(\mathbf{T})$*.*

▸**Proof:**

(1) . Let $\mathbf{T}_0 : \mathcal{M} \longrightarrow \mathcal{M}$ denote the restriction of $\mathbf{T}$ on $\mathcal{M}$. By assumption $\mathcal{M}$ is a Banach space and $\mathbf{T}(\mathcal{M}) = \mathcal{M}$, so, by the open mapping theorem, $\mathbf{T}_0$ is open. This means that there exists a constant $\delta > 0$ with the property that for every $x \in \mathcal{M}$ there is $u \in \mathcal{M}$ such that $\mathbf{T}u = x$ and $\|u\| \leqslant \delta\|x\|$.

Now, if $x \in \mathcal{M}$, define $u_0 = x$ and consider an element $u_1 \in \mathcal{M}$ such that

$$\mathbf{T}u_1 = u_0 \text{ and } \|u_1\| \leqslant \delta\|u_0\|.$$

By repeating this procedure, for every $n \in \mathbb{N}$ we find an element $u_n \in \mathcal{M}$ such that

$$\mathbf{T}u_n = u_{n-1} \text{ and } \|u_n\| \leqslant \delta\|u_{n-1}\|.$$

From the last inequality we obtain the estimate

$$\|u_n\| \leqslant \delta^n\|u_0\| = \delta^n\|x\| \text{ for every } n \in \mathbb{N},$$

so $x \in \mathrm{K}(\mathbf{T})$. Hence $\mathcal{M} \subseteq \mathrm{K}(\mathbf{T})$.

(2) . Suppose that $\mathrm{C}(\mathbf{T})$ is closed. Since $\mathrm{C}(\mathbf{T}) = \mathbf{T}(\mathrm{C}(\mathbf{T}))$ the first part of the theorem shows that $\mathrm{C}(\mathbf{T}) \subseteq \mathrm{K}(\mathbf{T})$, and hence, since the reverse inclusion is always true, $\mathrm{C}(\mathbf{T}) = \mathrm{K}(\mathbf{T})$.

■

Theorem 2.15. *([4], **Theorem 1.24**): Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space, be Kato. Then* $\mathrm{C}(\mathbf{T})$ *is closed and*

$$\mathrm{C}(\mathbf{T}) = \mathrm{K}(\mathbf{T}) = \mathcal{R}^{\infty}(\mathbf{T}).$$

▸**Proof:**

The semi-regularity[7] of $\mathbf{T}$ gives, by definition, $Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^n)$ for all $n \in \mathbb{N}$. Hence by **Proposition**1.5 we have $\mathcal{R}^{\infty}(\mathbf{T}) = \mathrm{C}(\mathbf{T})$. By **Corollary**2.4 $\mathbf{T}^n$ is Kato for all $n \in \mathbb{N}$, so $\mathcal{R}(\mathbf{T}^n)$ is closed for all $n \in \mathbb{N}$ and hence $\mathcal{R}^{\infty}(\mathbf{T}) = \bigcap_{n=1}^{\infty} \mathcal{R}(\mathbf{T}^n)$ is closed. By part (2) of **Theorem** 2.14 then we conclude that $\mathrm{K}(\mathbf{T})$ coincides with $\mathrm{C}(\mathbf{T})$.

■

Theorem 2.16. *: For every bounded operator* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space, we have:*

$$\mathrm{H}_0(\mathbf{T}) \subseteq^{\perp} \mathrm{K}(\mathbf{T}') \ \ and \ \ \mathrm{K}(\mathbf{T}) \subseteq^{\perp} \mathrm{H}_0(\mathbf{T}').$$

▸**Proof:**

Consider an element $u \in \mathrm{H}_0(\mathbf{T})$ and $f \in \mathrm{K}(\mathbf{T}')$. From the definition of $\mathrm{K}(\mathbf{T}')$ we know that there exists $\delta > 0$ and a sequence (g_n), $n \in \mathbb{Z}^+$ of $\mathbf{E}'$ such that

$$g_0 = f, \ \mathbf{T}' g_{n+1} = g_n \text{ and } \|g_n\| \leqslant \delta^n \|f\|,$$

for every $n \in \mathbb{Z}^+$. These equalities entail that $f = (\mathbf{T}')^n g_n$ for every $n \in \mathbb{Z}^+$, so that

$$f(u) = (\mathbf{T}')^n g_n(u) = g_n(\mathbf{T}n_u) \ \ for \ every \ \ n \in \mathbb{Z}^+.$$

From that it follows that $| f(u) | \leqslant \|\mathbf{T}^n u\| \ \|g_n\|$ for every $n \in \mathbb{Z}^+$ and therefore

$$| f(u) | \leqslant \delta^n \|f\| \ \|\mathbf{T}^n u\| \ \ for \ every \ \ n \in \mathbb{Z}^+. \tag{2.4}$$

From $u \in \mathrm{H}_0(\mathbf{T})$ we now obtain that $\lim_{n \to n} \|\mathbf{T}^n u\|^{1/n} = 0$ and hence by taking the n-th root in (2.4) we conclude that $f(u) = 0$. Therefore $\mathrm{H}_0(\mathbf{T}) \subseteq^{\perp} \mathrm{K}(\mathbf{T}')$.

The inclusion $\mathrm{K}(\mathbf{T}) \subseteq^{\perp} \mathrm{H}_0(\mathbf{T}')$. is proved in a similar way.

■

2.4.2 The generalized Kato decomposition

We introduce an important property of decomposition for bounded operators which involves the concept of Kato operators.

[7] The semi-regular is an identical term for kato operator

Recall that spectral radius of $\mathbf{T}$ denoted $r(\mathbf{T})$, is given by

$$r(\mathbf{T}) := \inf_{n\in\mathbb{N}} \|\mathbf{T}^n\|^{1/n} = \lim_{n\to\infty} \|\mathbf{T}^n\|^{1/n}.$$

Definition 2.15. $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is said to be quasi-nilpotent if its spectral radius*

$$r(\mathbf{T}) := \inf_{n\in\mathbb{N}} \|\mathbf{T}^n\|^{1/n} = \lim_{n\to\infty} \|\mathbf{T}^n\|^{1/n} = 0.$$

Example 2.21. *: An example of quasi-nilpotent element is* $\mathbf{T} : \ell^2 \to \ell^2$ *given by*

$$\mathbf{T}(x_1, x_2, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{4}, \ldots, \frac{x_n}{2^n}, \ldots).$$

With $\|\mathbf{T}^n\|^{1/n} \leqslant \dfrac{1}{2^{\frac{(n+1)}{2}}}$. *Thus , since* $\dfrac{1}{2^{\frac{(n+1)}{2}}}$ *goes to zero as n increases. Therefore,* $r(\mathbf{T}) := \inf_{n\in\mathbb{N}} \|\mathbf{T}^n\|^{1/n} = 0.$

Definition 2.16. *: An operator* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space, is said to admit a generalized Kato decomposition, abbreviated as **GKD**, if there exists a pair of* $\mathbf{T}$*-invariant closed subspaces* (M, N) *such that* $\mathbf{E} = M \oplus N$*, the restriction* $\mathbf{T}_{|M}$ *is Kato and* $\mathbf{T}_{|N}$ *is quasi-nilpotent.*

Example 2.22. *: Clearly, every Kato operator has a **GKD*** $M = \mathbf{E}$ *and* $N = \{0\}$*. and a quasi-nilpotent operator has a **GKD*** $M = \{0\}$ *and* $N = \mathbf{E}$*, Kato type operators. In addition to essentially Kato operators with N is finit-dimensional and* $\mathbf{T}_{/N}$ *is .*

Example 2.23. *:Riesz operator. If* $\mathbf{T}$ *is a Riesz operator then* $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$*, with* $\mathbf{T}_1$ *is compact and* $\mathbf{T}_2$ *is quasi-nilpotent operator.*

Quasi-polar and polar operator. If $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is said to be quasi-polar (resp: polar) operator if there is a projection P commuting with* $\mathbf{T}$ *for which* $\mathbf{T}$ *has a matrix representation :*

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix} : \quad \mathcal{R}(\mathbf{T}) \oplus Ker\mathbf{T} \longrightarrow \mathcal{R}(\mathbf{T}) \oplus Ker\mathbf{T}.$$

Notes 2.1. *: A relevant case is obtained if we assume in the definition above that* $\mathbf{T}_{|N}$ *is nilpotent, there exists* $d \in N$ *for which* $(\mathbf{T}_{|N})\ d = 0$ *. In this case* $\mathbf{T}$ *is said to be of Kato type of operator of order* d*.*

An operator is said to be essentially Kato if it admits a ***GKD*** (M,N) *such that* N *is finite-dimensional. Note that if* $\mathbf{T}$ *is essentially Kato then* $\mathbf{T}_{|N}$ *is nilpotent, since every quasi-nilpotent operator on a finite dimensional space is nilpotent.*

Hence we have the following implications:

$$\mathbf{T}\ \textit{Kato} \Rightarrow \mathbf{T}\ \textit{essentially Kato} \Rightarrow \mathbf{T}\ \textit{of Kato type} \Rightarrow \mathbf{T}\ \textit{admits a}\ \boldsymbol{GKD}.$$

Remark 2.14. *: If (M,N) is a* ***GKD*** *for* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*. Then we have:*

1. $\mathrm{K}(\mathbf{T}) = \mathrm{K}(\mathbf{T}_{/M})$ *and* $\mathrm{K}(\mathbf{T})$ *is closed;*
2. $Ker\mathbf{T}_{/M} = Ker\mathbf{T} \cap M = \mathrm{K}(\mathbf{T}) \cap Ker\mathbf{T}$.

Theorem 2.17. *([4],* ***Theorem 1.43*** *): Assume that* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*,* $\mathbf{E}$ *a Banach space, admits a* ***GKD*** (M,N)*. Then* $(N^{\perp}, M^{\perp})$ *is a* ***GKD*** *for* $\mathbf{T}'$*. Furthermore, if* $\mathbf{T}$ *is of Kato type then* $\mathbf{T}'$ *is of Kato type*

▸**Proof:**

Suppose that $\mathbf{T}$ has a **GKD** (M,N). Clearly both subspaces $N^{\perp}$ and $M^{\perp}$ are invariant under $\mathbf{T}'$. Let P_M denote the projection[8] of $\mathbf{E}$ onto M along N. Trivially, P'_M is idempotent and from the equalities $M = \mathcal{R}(P_M)$, $N = KerP_M$ we obtain that

$$\mathcal{R}(P'_M) = (KerP_M)^{\perp} = N^{\perp} \quad \textit{and} \quad KerP'_M = [\mathcal{R}(P_M)]^{\perp} = M^{\perp}.$$

Hence

$$\mathbf{E}' = \mathcal{R}(P'_M) \oplus KerP'_M = N' \oplus M'.$$

Now, if $P_N := \mathbf{I} - P_M$ then $\mathbf{T}P_N = P_N\mathbf{T}$ is quasi-nilpotent and therefore also $\mathbf{T}'P'_N = P'_N\mathbf{T}'$ is quasi-nilpotent, from which we conclude that the restriction $\mathbf{T}'_{|M^{\perp}}$ is quasi-nilpotent.

To end the proof of the first assertion we need only to show that $\mathbf{T}'_{|N^{\perp}}$ is Kato, that is $\mathbf{T}'(N^{\perp})$ is closed and $Ker(\mathbf{T}'_{|N^{\perp}})^n \subseteq \mathbf{T}'(N^{\perp})$ for all positive integer $n \in \mathbb{N}$.

From assumption $\mathbf{T}(M) = \mathcal{R}(P_M)$ is closed, and therefore, by **Corollary ??**, $\mathcal{R}(P'_M)$ is closed. From the equality $\mathbf{T}'P'_M = P'_M\mathbf{T}'$ it then follows that

$$\mathcal{R}((\mathbf{T}P_M)') = \mathcal{R}(\mathbf{T}'P'_M) = \mathbf{T}'(N^{\perp}).$$

is closed. Furthermore, for all $n \in \mathbb{N}$ we have

[8] We said that P_M is projection or idempotent, if $P_M^2 = P_M$.

$$Ker(\mathbf{T}'_{|N^{\perp}})^n = Ker(\mathbf{T}')^n \cap N^{\perp} = \mathcal{R}(\mathbf{T}^n)^{\perp} \cap N^{\perp} = [\mathcal{R}(\mathbf{T}^n) + N]^{\perp}.$$

From the equalities

$$Ker(\mathbf{T}P_M) = Ker\mathbf{T}_{|M} + N \subseteq \mathcal{R}(\mathbf{T}^n) + N \subseteq \mathcal{R}(\mathbf{T}^n) + N,$$

we then conclude that

$$Ker(\mathbf{T}'_{|N^{\perp}})^n = [\mathcal{R}(\mathbf{T}^n) + N]^{\perp} \subseteq [Ker(\mathbf{T}P_M)]^{\perp} == \mathcal{R}(\mathbf{T}'P'_M) = \mathbf{T}'(N^{\perp}),$$

for all $n \in \mathbb{N}$, thus $\mathbf{T}'_{|N^{\perp}}$ is Kato. This shows that if $\mathbf{T}$ has a **GKD** (M, N) then $\mathbf{T}'$ has the **GKD** $(N^{\perp}, M^{\perp})$. Evidently, if additionally $\mathbf{T}_{|N}$ is nilpotent then $\mathbf{T}_{|M^{\perp}}$ is nilpotent, so that $\mathbf{T}'$ is of Kato type.

■

Remark 2.15. *: If $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ is quasi-nilpotent if and only if $H_0(\mathbf{T}) = \mathbf{E}$.*

The next result describes the quasi-nilpotent part of an operator T which admits a **GKD**.

Corollary 2.9. *([4], **Corollary 1.69**): Assume that $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ a Banach space, admits a **GKD** (M, N). Then*

$$H_0(\mathbf{T}) = H_0(\mathbf{T}_{/M}) \oplus H_0(\mathbf{T}_{/N}) = H_0(\mathbf{T}_{/M}) \oplus N.$$

▸**Proof:**

We know that $N = H_0(\mathbf{T}_{/N})$. The inclusion $H_0(\mathbf{T}) \supseteq H_0(\mathbf{T}_{/M}) + H_0(\mathbf{T}_{/N})$ is clear. In order to show the opposite inclusion, consider an arbitrary element $x \in H_0(\mathbf{T})$ and let $x = u + v$, with $u \in M$ and $v \in N$. Evidently $N = H_0(\mathbf{T}_{/N}) \subseteq H_0(\mathbf{T})$. Consequently $u = x - v \in H_0(\mathbf{T}) \cap M = H_0(\mathbf{T}_{/M})$ and hence $H_0(\mathbf{T}) \subseteq H_0(\mathbf{T}_{/M}) + H_0(\mathbf{T}_{/N})$. Clearly the sum $H_0(\mathbf{T}_{/M}) + N$ is direct since $M \cap N = \{0\}$.

■

2.5 Basics of closed operators on Banach spaces

Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces (over the complex numbers). By a subspace of $\mathbf{E}$ or $\mathbf{F}$ we shall always mean a linear subspace, not necessarily closed. Let $\mathbf{T}$ be a linear operator (either bounded or not) with domain $\mathcal{D}(\mathbf{T}) \subset \mathbf{E}$ and range $\mathcal{R}(\mathbf{T}) \subset \mathbf{F}$. This implies that $\mathcal{D}(\mathbf{T})$ and $\mathcal{R}(\mathbf{T})$ are linear subspaces, but we do neither assume that $\mathcal{D}(\mathbf{T})$ is a closed subspace of $\mathbf{E}$ nor that $\mathcal{D}(\mathbf{T})$ is dense in $\mathbf{E}$. The subspace $\mathcal{G}(\mathbf{T}) = \{(x, \mathbf{T}x) : x \in \mathcal{D}(\mathbf{T})\}$ of the product space $\mathbf{E} \times \mathbf{F}$ is called the graph of $\mathbf{T}$.

Given the positive constants α and β, the product space $\mathbf{E} \times \mathbf{F}$ is a Banach space with respect to the norm $\|(x,y)\| = \alpha\|x\| + \beta\|y\|$. Evidently, the norm topology in $\mathbf{E} \times \mathbf{F}$ induced by $\|(x,y)\|$ is identical to the product topology in $\mathbf{E} \times \mathbf{F}$. The following definition is well-known.

Definition 2.17. *: The linear operator $\mathbf{T}$ with domain $\mathcal{D}(\mathbf{T}) \subset \mathbf{E}$ and range $\mathcal{R}(\mathbf{T}) \subset \mathbf{F}$ is said to be closed whenever $\mathcal{G}(\mathbf{T})$ is a closed subspace of $\mathbf{E} \times \mathbf{F}$. Equivalently, $\mathbf{T}$ it closed whenever it follows from $x_n \in \mathcal{D}(\mathbf{T})$ for $n = 1, 2, \dots, x_n \to x$ and $\mathbf{T}x_n \to y$ that $x \in \mathcal{D}(\mathbf{T})$ and $\mathbf{T}x = y$.*

The set of all closed operators from $\mathbf{E}$ to $\mathbf{F}$ will be denoted by $\mathcal{C}(\mathbf{E}, \mathbf{F})$. Also we write $\mathcal{C}(\mathbf{E}, \mathbf{E}) = \mathcal{C}(\mathbf{E})$.

In particular, every $\mathbf{T} \in \mathcal{B}(\mathbf{E}, \mathbf{F})$ is closed : $\mathcal{B}(\mathbf{E}, \mathbf{F}) \subset \mathcal{C}(\mathbf{E}, \mathbf{F})$.

Example 2.24. *: **(Differential Operator)** Let $\mathbf{E} = \mathbf{F} = \mathbf{C}[0,1]$ and let $\mathbf{C}^1[0,1]$ be the subspace of $\mathbf{E}$ consisting of the functions with continuous first derivatives. Define the linear differential operator $\mathbf{T}$ mapping $\mathbf{C}^1[0,1]$ into $\mathbf{F}$ by $(\mathbf{T}x)(t) = x'(t)$, $t \in [0,1]$. $\mathbf{T}$ is closed; for if $x_n \to x$ and $\mathbf{T}x_n \to y$, then $\{x_n\}$ converges uniformly to x and $\{x'_n\}$ converges uniformly to y on $[0,1]$. It follows from taking antiderivatives of x'_n and y that x is in $\mathbf{C}^1[0,1]$ and that $\mathbf{T}x = x' = y$ on $[0,1]$. Thus $\mathbf{T}$ is closed. However, $\mathbf{T}$ is unbounded, since the sequence $\{x_n(t)\} = \{t_n\}$ has the properties $\|\mathbf{T}x_n\| = n$ and $\|x_n\| = 1$.*

Remark 2.16. *:*

1. *If $\mathbf{T}$ is injective and closed, then $\mathbf{T}^{-1}$ is closed.*
2. *The null manifold of a closed operator is closed.*
3. *If $\mathcal{D}(\mathbf{T})$ is closed and T is continuous, then $\mathbf{T}$ is closed.*
4. *The continuity of $\mathbf{T}$ does not necessarily imply that $\mathbf{T}$ is closed. Conversely, $\mathbf{T}$ closed does not necessarily imply that $\mathbf{T}$ is continuous. This statement can be verified by the following example.*

Example 2.25. *: Let $\mathcal{D}(\mathbf{T})$ be any proper dense subspace of $\mathbf{E} = \mathbf{F}$ and let $\mathbf{T}$ be the identity map. $\mathbf{T}$ is obviously continuous but not closed.*

Theorem 2.18. *: A closed operator* $\mathbf{T}$ *from* $\mathbf{E}$ *to* $\mathbf{F}$ *with domain* $\mathbf{E}$ *is bounded. In other words,* $T \in \mathcal{C}(\mathbf{E},\mathbf{F})$ *and* $\mathcal{D}(\mathbf{T}) = \mathbf{E}$ *imply* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$.

▸**Proof:**
See [9]. Theorem 5.20. p 166.

■

The linear operator $\mathbf{T_1}$ with domain $\mathcal{D}(\mathbf{T_1}) \subset \mathbf{E}$ and range $\mathcal{R}(\mathbf{T_1}) \subset \mathbf{F}$ is called an extension of T if $\mathcal{D}(\mathbf{T}) \subset \mathcal{D}(\mathbf{T_1})$ and $\mathbf{T_1}x = \mathbf{T}x$ for all $x \in \mathcal{D}(\mathbf{T_1})$. If, in addition, $\mathbf{T_1}$ is a closed linear operator, then $\mathbf{T_1}$ is called a closed linear extension of $\mathbf{T}$.

Definition 2.18. *: An operator* $\mathbf{T}$ *is called **closable** if it has a closed extensions . the smallest closed extensions of* $\mathbf{T}$ *whose graph equals* $\overline{\mathcal{G}(\mathbf{T})}$ *is denoted by* $\overline{\mathbf{T}}$ *and called the **closure** of* $\mathbf{T}$ *. Every closable has a closure.*

Remark 2.17. *: The product of closed (resp: closable) operator* $\mathbf{T}$*, with bounded operator* $\mathbf{S}$ *give a closed (resp : closable) operator* $\mathbf{TS}$*, and* $\mathcal{D}(\mathbf{TS}) = \left\{ x \in \mathcal{D}(\mathbf{S}) \ : \mathbf{T}x \in \mathcal{D}(\mathbf{T}) \right\}$*. But the product of two closed operator need not be closed operator .*

Example 2.26. *: Let* $\mathbf{E} = \mathbf{C}[0,1]$*,* $\mathbf{T} = f'$ *with* $\mathcal{D}(\mathbf{T}) = \mathbf{C}^1[0,1]$*, and* $\varphi \in \mathbf{C}[0,1]$ *such that* $\varphi = 0$ *on* $[0,1/2]$*. Define* $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ *by* $\mathbf{S}f = \varphi f$ *for all* $f \in \mathbf{E}$*. Then the operator* $\mathbf{ST}$ *with* $\mathcal{D}(\mathbf{ST}) = \mathcal{D}(\mathbf{T})$ *is not closed .To see this take functions* $f_n \in \mathcal{D}(\mathbf{T})$ *such that* $f_n = 1$ *on* $[1/2,1]$ *and* $f_n \to f \in \mathbf{E}$ *with* $f \notin \mathbf{C}^1[0,1]$*. Then* $\mathbf{ST}f_n = \varphi f_n' = 0$ *converges to* 0*, but* $f \notin \mathcal{D}(\mathbf{T})$.

Definition 2.19. *:(**The adjoint operator**): Let the domain of* $\mathbf{T}$ *be dense in* $\mathbf{E}$*. The Adjoint* $\mathbf{T}'$ *of* $\mathbf{T}$ *is defined as follows.*
$\mathcal{D}(\mathbf{T}') = \left\{ y' \in \mathbf{F}', y' \ \mathbf{T} \text{ continuous on } \mathcal{D}(\mathbf{T}) \right\}$*. For* $y' \in \mathcal{D}(\mathbf{T}')$*, let* $\mathbf{T}'$ *be the operator which takes* $y' \in \mathcal{D}(\mathbf{T}')$ *to* $\overline{y'\mathbf{T}}$ *, where* $\overline{y'\mathbf{T}}$ *is the unique continuous linear extension of* $y'\mathbf{T}$ *to all of* $\mathbf{E}$*.*
$\mathcal{D}(\mathbf{T}')$ *is a subspace of* $\mathbf{F}'$*, and* $\mathbf{T}'$ *is linear.*
$\mathbf{T}'y'$ *is taken to be* $\overline{y'\mathbf{T}}$ *rather than* $y'\mathbf{T}$ *in order that* $\mathcal{R}(\mathbf{T}')$ *be contained in* $\mathbf{E}'$.

Example 2.27. *: Let* $\mathbf{E} = \mathbf{F} = \ell^p$, $1 \leqslant p < \infty$*, and let*

$$u_1 = (1,0,0,...), \quad u_2 = (0,1,0,...), ...etc.$$

be the unit vectors in ℓ^p*. Define* $\mathbf{T}$ *by*

$$\mathcal{D}(\mathbf{T}) = span\{u_k\}.$$

$$\mathbf{T}(x_1, x_2, ..., x_n, 0, 0, ...) = \Big(\sum_j x_j, x_2, x_3, ..., x_n, 0, 0, ...\Big).$$

Suppose $y\prime = (a_1, a_2, ...) \in \mathcal{D}(\mathbf{T}')$. *Then for* $k \geqslant 1$,

$$| y'\mathbf{T}u_k | = | a_1 k - a_k | \geqslant | a_1 | k + | a_k | \geqslant | a_1 | k - \|y'\|.$$

Since $\|u_k\| = 1 and y'\mathbf{T}$ *is bounded on* $\mathcal{D}(\mathbf{T}), a_1 = 0$. *Also, any element* $(0, b_1, b_2, ...) \in \ell^{p'} = \ell'^p$ *is in* $\mathcal{D}(\mathbf{T}')$. *Hence the domain of* $\mathbf{T}'$ *consists of all the elements in* $\ell^{p'}$ *which have zero as their first term. Suppose* $\mathbf{T}'y' = (c_1, c_2, ...)$, *where* $y' = (0, a_2, a_3, ...) \in \mathcal{D}(\mathbf{T}')$. *Then*

$$c_k = \mathbf{T}'y'u_k = y'\mathbf{T}u_k = a_k \quad k \geqslant 2.$$

and $c_1 = 0$. *Thus* $\mathbf{T}'y' = y'$.

Example 2.28. *: Let* $\mathbf{T} : \mathcal{D}(\mathbf{T}) \subseteq L^2[a,b] \to L^2[a,b]$ *defined by*

$$\mathbf{T}x = x', \;\; x \in \mathcal{D}(\mathbf{T}),$$

where

$$\mathcal{D}(\mathbf{T}) := \Big\{x \in L^2[a,b] \;:\; x \text{ absolutely continuous}, \;\; x(a) = 0 \text{ and } x' \in L^2[a,b]\Big\}.$$

It can shown that $\mathcal{D}(\mathbf{T})$ *is dense in* $L^2[a,b]$. *Taking*

$$Y_0 := \Big\{y \in L^2[a,b] \;\; : y \;\; \text{absolutely continuous}, \;\; y(b) = 0 \;\; \text{and} \;\; y' \in L^2[a,b]\Big\}.$$

we see that for $x \in \mathcal{D}(\mathbf{T})$ *and* $y \in Y_0$,

$$\langle \mathbf{T}x, y\rangle = \int_a^b x'(t)\overline{y(t)}dt = [\overline{y(t)}x(t)]_a^b - \int_a^b \overline{y'(t)}x(t)dt = \langle x, z\rangle,$$

where $z = -y'$. *Thus,* $\mathbf{T}'y = -y'$ *with* $\mathcal{D}(\mathbf{T}') := Y_0$.

Theorem 2.19. *([28], **II. 2.6 Theorem.**):* $\mathbf{T}'$ *is a closed linear operator in* $\mathbf{F}'$.

▸**Proof:**

Suppose $y'_n \to y'$ and $\mathbf{T}'y'_n \to x'$. Then for each $x \in \mathcal{D}(\mathbf{T})$, $y'_n\mathbf{T}x \to y'\mathbf{T}x$ and $y'_n\mathbf{T}x = \mathbf{T}'y'_n x \to x'x$. Thus $y'\mathbf{T} = x'$ on $\mathcal{D}(\mathbf{T})$. Hence, by the definition of $\mathbf{T}'$, $y' \in \mathcal{D}(\mathbf{T}')$ and $\mathbf{T}'y' = x'$. Therefore $\mathbf{T}'$ is closed.

■

Remark 2.18. : *$\mathcal{D}(\mathbf{T}') = \mathbf{F}'$ if and only if $\mathbf{T}$ is continuous. If that is the case, then $\mathbf{T}'$ is also continuous and $\|\mathbf{T}'\| = \|\mathbf{T}\|$.*

We can see that, the notion of a closed linear operator is an extension of the notion of a bounded linear operator. Therefore, most of concepts previously mentioned in **section1.2** apply with closed linear operators. For further clarifications see [27] and [28].

Proposition 2.9. : *Suppose that $\mathbf{T}_1$ is a linear extension of $\mathbf{T}$ such that $\infty > n = \boldsymbol{dim}\mathcal{D}(\mathbf{T}_1)/\mathcal{D}(\mathbf{T})$.*

1. *If $\mathbf{T}$ is closed, then $\mathbf{T}_1$ is closed;*
2. *If $\mathbf{T}$ has a closed range, then $\mathbf{T}_1$ has a closed range;*
3. *If $\mathbf{T}$ has an index, then $ind(\mathbf{T}_1) = ind(\mathbf{T}) + n$.*

▸**Proof:**

(1) .By hypothesis, $\mathcal{D}(\mathbf{T}_1) = \mathcal{D}(\mathbf{T})\oplus\mathbf{N}$, where $\mathbf{N}$ is a finite-dimensional subspace. Hence, $\mathcal{G}(\mathbf{T}_1) = \mathcal{G}(\mathbf{T}) + Z$, where $\mathcal{G}(\mathbf{T})$ and $\mathcal{G}(\mathbf{T}_1)$ are the graphs of $\mathbf{T}$ and $\mathbf{T}_1$, respectively, and $Z = \{(n, \mathbf{T}_1 n) : n \in \mathbf{N}\}$. Thus, if $\mathcal{G}(\mathbf{T})$ is closed, then $\mathcal{G}(\mathbf{T}_1)$ is closed, since Z is finite-dimensional.

(2). If $\mathcal{R}(\mathbf{T})$ is closed, then $\mathcal{R}(\mathbf{T}_1)$ is closed, since

$$\mathcal{R}(\mathbf{T}_1) = \mathcal{R}(\mathbf{T}) + \mathbf{T}_1(\mathbf{N})$$

and $\mathbf{T}_1(\mathbf{N})$ is finite-dimensional.

(3). It is easy to see that it suffices to prove (3) for the case when $n = 1$. Suppose that. $\mathcal{D}(\mathbf{T}_1) = \mathcal{D}(\mathbf{T}) \oplus span\{x\}$, for some $x \in \mathcal{D}(\mathbf{T}_1)$ Then $\mathbf{T}_1\mathbf{X} = \mathbf{TX} \oplus \mathbf{V}$, where $\mathbf{V} = span\{\mathbf{T}_1 x_1\}$ when $\mathbf{T}_1 x \notin \mathcal{R}(\mathbf{T})$ or $\mathbf{V} = \{0\}$ when $\mathbf{T}_1 x \in \mathcal{R}(\mathbf{T})$.

If $\mathbf{T}_1 x \notin \mathcal{R}(\mathbf{T})$, then it is easy to verify that $\beta(\mathbf{T}) = \beta(\mathbf{T}_1) + 1$ and that $\mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{T}_1)$. Therefore, $ind(\mathbf{T}_1) = ind(\mathbf{T}) + 1$.

If $\mathbf{T}_1 x \in \mathcal{R}(\mathbf{T})$, then $\mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{T}_1)$ and there exists a $z \in \mathcal{D}(\mathbf{T})$ such that $\mathbf{T}z = \mathbf{T}_1 x$. Hence, $\mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{T}_1) \oplus span\{x - z\}$. Thus $\alpha(\mathbf{T}) = \alpha(\mathbf{T}_1) + 1$ and $ind(\mathbf{T}_1) = ind(\mathbf{T}) + 1$.

■

Definition 2.20. : *The linear operator $\mathbf{S}$ is said to be bounded with respect to $\mathbf{T}$ If*

1. *$\mathcal{D}(\mathbf{T}) \subset \mathcal{D}(\mathbf{S}) \subset \mathbf{E}$ and $\mathcal{R}(\mathbf{S}) \subset \mathbf{F}$;*
2. *there exist positive constant α and β, such that $\|\mathbf{S}x\| \leqslant \alpha\|x\| + \beta\|\mathbf{T}x\|$ for all $x \in \mathcal{D}(\mathbf{T})$.*

Definition 2.21. *: The linear operator* $\mathbf{S}$ *is said to be a finite perturbation of* $\mathbf{T}$*, if* $\mathbf{S}$ *is bounded with respect to* $\mathbf{S}$ *and if the range* $\mathcal{R}(\mathbf{S})$ *is finite dimensional.*

Theorem 2.20. *: Let* $\mathbf{T}$ *be a closed linear operator with domain* $\mathcal{D}(\mathbf{T})$ *in* $\mathbf{E}$ *and range* $\mathcal{R}(\mathbf{T})$ *in* $\mathbf{F}$*, and let* $\mathbf{S}$ *be a finite perturbation of* $\mathbf{T}$*. Then* $\mathbf{T} + \lambda\mathbf{S}$ *is a closed linear operator for all* λ *such that*

1. *for each* λ *the range* $\mathcal{R}(\mathbf{T} + \lambda\mathbf{S})$ *is closed if and only If the range* $\mathcal{R}(\mathbf{T})$ *is closed,*
2. $\alpha(\mathbf{T} + \lambda\mathbf{S})$ *and* $\beta(\mathbf{T} + \lambda\mathbf{S})$ *are constant except for a finite number of value of* λ*.*

▸**Proof:**
See[27] Theorem 19.6. p 67.

■

Definition 2.22. *: A closed linear operator with closed range is called also* ***normally solvable****.*

Remark 2.19. *: Let* $\mathcal{D}(\mathbf{T}) \subset \mathcal{D}(\mathbf{S})$*, and* $\mathbf{T}$ *is normally solvable and has an index. with* $\|\mathbf{S}\| \leqslant \gamma(\mathbf{T})$ *then*

- $\mathbf{T} + \mathbf{S}$ *is normally solvable;*
- $\alpha(\mathbf{T} + \mathbf{S}) \leqslant \alpha(\mathbf{T})$*;* $\beta(\mathbf{T} + \mathbf{S}) \leqslant \beta(\mathbf{T})$*.*
- $ind(\mathbf{T} + \mathbf{S}) = ind(\mathbf{T})$*.*

CHAPTER 3

FREDHOLM THEORY

We now introduce some important classes of operators in Fredholm theory. Let $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces. In the sequel, for every bounded operator $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, we shall denote by $\alpha(\mathbf{T})$ the nullity of $\mathbf{T}$, defined as $\alpha(\mathbf{T}) := \mathbf{dim} Ker\mathbf{T}$ whilst the deficiency $\beta(\mathbf{T})$ of $\mathbf{T}$ is defined $\beta(\mathbf{T}) := \mathbf{codim}\mathcal{R}(\mathbf{T})$, with same definition in **chapter 1**.

3.1 Fredholm and semi-Fredholm operators and Perturbations

Consider the Calkin[1] algebra $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ the quotient algebra of $\mathcal{B}(\mathbf{E})$ modulo the ideal $\mathcal{K}(\mathbf{E})$ of all compact operators. If dim $\mathbf{E} < \infty$, then all operators are compact, and so $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ is trivially null. Thus if the Calkin algebra is brought into play, then the space $\mathbf{E}$ is assumed infinite-dimensional (i.e., $\mathbf{dimE} = \infty$). Since $\mathcal{K}(\mathbf{E})$ is a subspace of $\mathcal{B}(\mathbf{E})$, then $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ is a unital Banach algebra whenever $\mathbf{E}$ is infinite-dimensional. Moreover, consider the natural map (or the natural quotient map) $\pi : \mathcal{B}(\mathbf{E}) \to \mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ which is defined by

$$\pi(\mathbf{T}) = [\mathbf{T}] = \Big\{\mathbf{S} \in \mathcal{B}(\mathbf{E}) : \mathbf{S} = \mathbf{T} + \mathbf{K} \;\; for\ some\ \mathbf{K} \in \mathcal{K}(\mathbf{E})\Big\} = \mathbf{T} + \mathcal{K}(\mathbf{E}).$$

for every $\mathbf{T} \in \mathcal{B}(\mathbf{E})$. The origin of the linear space $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ is

$$[0] = \mathcal{K}(\mathbf{E}) \in \mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E}),$$

the kernel of the natural map π is

$$Ker(\pi) = \Big\{\mathbf{T} \in \mathcal{B}(\mathbf{E}) : \pi(\mathbf{T}) = [0]\Big\} = \mathcal{K}(\mathbf{E}) \subseteq \mathcal{B}(\mathbf{E}),$$

[1] John Williams Calkin, American mathematician. 11 October 1909- 5 August 1964.

and π is a unital homomorphism. Indeed, since $\mathcal{K}(\mathbf{E})$ is an ideal of $\mathcal{B}(\mathbf{E})$,

$$\pi(\mathbf{T}+\mathbf{T}') = (\mathbf{T}+\mathbf{T}') + \mathcal{K}(\mathbf{E}) = (\mathbf{T}+\mathcal{K}(\mathbf{E})) + (\mathbf{T}'+\mathcal{K}(\mathbf{E})) = \pi(\mathbf{T}+\pi(\mathbf{T}'),$$

$$\pi(\mathbf{T}\mathbf{T}') = (\mathbf{T}\mathbf{T}') + \mathcal{K}(\mathbf{E}) = (\mathbf{T}+\mathcal{K}(\mathbf{E})(\mathbf{T}'+\mathcal{K}(\mathbf{E})) = \pi(\mathbf{T})\pi(\mathbf{T}'),$$

for every $\mathbf{T},\mathbf{T}' \in \mathcal{B}(\mathbf{E})$, and $\pi(\mathbf{I}) = [\mathbf{I}]$ is the identity element of the algebra $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$. Furthermore, the norm on $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ is given by $\| [T] \| = \underset{K\in\mathcal{K}(\mathbf{E})}{inf}\ \mathbf{K} \in \mathcal{K}(\mathbf{E}) \subseteq \|\mathbf{T}+\mathbf{K}\| \leqslant \|\mathbf{T}\|$, so that π is a contraction.

Now we defined the following sets.

Definition 3.1. *: Given two Banach spaces $\mathbf{E}$ and $\mathbf{F}$, the set of all upper semi-Fredholm operators is defined by*

$$\Phi_+(\mathbf{E},\mathbf{F}) := \Big\{\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F}) : \alpha(\mathbf{T}) < \infty \ \ and \ \ \mathcal{R}(\mathbf{T}) \ \ closed\ \Big\},$$

The set of all lower semi-Fredholm operators is defined by

$$\Phi_-(\mathbf{E},\mathbf{F}) := \Big\{\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F}) : \beta(\mathbf{T}) < \infty \ \ and \ \ \mathcal{R}(\mathbf{T}) \ \ closed\ \Big\},$$

The set of all semi-Fredholm operators is defined by

$$\Phi_\pm(\mathbf{E},\mathbf{F}) := \Phi_+(\mathbf{E},\mathbf{F}) \cup \Phi_-(\mathbf{E},\mathbf{F}),$$

The class $\Phi(\mathbf{E},\mathbf{F})$ of all Fredholm operators from $\mathbf{E}$ into $\mathbf{F}$ is defined by

$$\Phi(\mathbf{E},\mathbf{F}) := \Phi_+(\mathbf{E},\mathbf{F}) \cap \Phi_-(\mathbf{E},\mathbf{F}).$$

If $\mathbf{E}=\mathbf{F}$ then $\Phi_+(\mathbf{E},\mathbf{F})$, $\Phi_-(\mathbf{E},\mathbf{F})$, $\Phi_\pm(\mathbf{E},\mathbf{F})$ and $\Phi(\mathbf{E},\mathbf{F})$ are replaced, respectively by $\Phi_+(\mathbf{E})$, $\Phi_-(\mathbf{E})$, $\Phi_\pm(\mathbf{E})$ and $\Phi(\mathbf{E})$.

Remark 3.1. *:*

- *If $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ and $\mathcal{R}(\mathbf{T})$ is closed, we say that $\mathbf{T}$ is a semi-Fredholm operator if either $\alpha(\mathbf{T}) < \infty$ or $\beta(\mathbf{T}) < \infty$.*
- *If $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ and $\mathcal{R}(\mathbf{T})$ is closed, we say that $\mathbf{T}$ is a Fredholm operator if either $\alpha(\mathbf{T}) < \infty$ and $\beta(\mathbf{T}) < \infty$.*

We again define the same previous concept, of the index that was studied in the previous chapter.

Definition 3.2. *: The index of a semi-Fredholm operator* $\mathbf{T} \in \Phi_{\pm}(\mathbf{E}, \mathbf{F})$ *is defined by*

$$ind(\mathbf{T}) := \alpha(\mathbf{T}) - \beta(\mathbf{T}).$$

Clearly, $ind(\mathbf{T})$ *is an integer or* $\pm\infty$ *(i.e:* $ind(\mathbf{T}) \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ *) .*

We fellow that $\Phi_{+}(\mathbf{E}, \mathbf{F})$ and $\Phi_{-}(\mathbf{E}, \mathbf{F})$ are open subsets of $\mathcal{B}(\mathbf{E}, \mathbf{F})$ and the index function

$$ind : \mathbf{T} \in \Phi_{\pm}(\mathbf{E}) \longrightarrow ind(\mathbf{T}) \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\},$$

is continuous and therefore constant on the connected components of the open set $\Phi_{\pm}(\mathbf{E}, \mathbf{F})$.

Consequently , we have the function

$$\alpha(.) : \mathbf{T} \in \Phi_{+}(\mathbf{E}) \longrightarrow \alpha(\mathbf{T}) \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\},$$

$$\beta(.) : \mathbf{T} \in \Phi_{-}(\mathbf{E}) \longrightarrow \beta(\mathbf{T}) \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}.$$

Remark 3.2. *: If* $\mathbf{T} \in \mathcal{B}(\mathbf{E}, \mathbf{F})$ *with closed rang, then we have*

$$\mathbf{T} \in \Phi_{+}(\mathbf{E}, \mathbf{F}) - \Phi_{-}(\mathbf{E}, \mathbf{F}) \Leftrightarrow ind(\mathbf{T}) = +\infty,$$

$$\mathbf{T} \in \Phi_{-}(\mathbf{E}, \mathbf{F}) - \Phi_{+}(\mathbf{E}, \mathbf{F}) \Leftrightarrow ind(\mathbf{T}) = -\infty,$$

$$\mathbf{T} \in \Phi(\mathbf{E}, \mathbf{F}) \Leftrightarrow ind(\mathbf{T}) \in \mathbb{Z}.$$

Example 3.1. *: The operator defined in* ***Example*** *1.7 is a Fredholm, with zero index .*

Example 3.2. *:From the previous studies, we can provide these theoretical examples :*

1. *All operators which is invertible are Fredholm with zero index.*
2. *By* ***section 2.1****, if* $\mathbf{T} \in \mathcal{K}(E)$ *with* $\lambda \neq 0$*, then we have* $\mathcal{R}(\lambda\mathbf{I} - \mathbf{T})$ *is closed and* $\alpha(\lambda\mathbf{I} - \mathbf{T}) = \beta(\lambda\mathbf{I} - \mathbf{T}) < \infty$ *. Then* $\lambda\mathbf{I} - \mathbf{T}$ *is Fredholm operator, with* $ind(\lambda\mathbf{I} - \mathbf{T}) = 0$.

*In particularly. by **Example** 2.16 The classical Fredholm integral equation,*

$$\lambda f(s) - \int_a^b k(s,t)f(t)\,dt = g(s) \qquad (a \leqslant s \leqslant b),$$

which we can write as $(\lambda \mathbf{I} - \mathbf{T})f = g$, *gives a Fredholm operator of index 0 is a consequence of the compactness of*

$$(\mathbf{T}f)(s) = \int_a^b k(s,t)f(t)\,dt \qquad (a \leqslant s \leqslant b).$$

Theorem 3.1. : *Upper and lower semi-Fredholm operators are dual each other, and we have*

$$\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F}) \Leftrightarrow \mathbf{T}' \in \Phi_-(\mathbf{E},\mathbf{F}),$$

$$\mathbf{T} \in \Phi_-(\mathbf{E},\mathbf{F}) \Leftrightarrow \mathbf{T}' \in \Phi_+(\mathbf{E},\mathbf{F}),$$

$$\mathbf{T} \in \Phi(\mathbf{E},\mathbf{F}) \Leftrightarrow \mathbf{T}' \in \Phi(\mathbf{E},\mathbf{F}).$$

If $\mathbf{T} \in \Phi_\pm(\mathbf{E})$, *then* $ind(\mathbf{T}') = -ind(\mathbf{T})$.

▸**Proof:**
We have by **Proposition**2.1 that:

$$\alpha(\mathbf{T}) = \beta(\mathbf{T}') \quad and \quad \beta(\mathbf{T}) = \alpha(\mathbf{T}').$$

So, the previous equivalences are verified, and $ind(\mathbf{T}') = -ind(\mathbf{T})$.

■

Example 3.3. : *Going back to **Example** 2.5 and **Example** 2.10, we can see that the **right shift** and **left shift** operators are adjoint with each other, and Fredholm fulfills **Theorem**3.1 , then*

$$ind(\mathbf{T}_l) = 1 \qquad and \qquad ind(\mathbf{T}_r) = -1.$$

With $\mathbf{T}_r$ *and* $\mathbf{T}_l$, *we can build a Fredholm operator whose index is equal to an arbitrary prescribed integer. Indeed if*

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_r^p & 0 \\ 0 & \mathbf{T}_l^q \end{pmatrix} \quad : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2.$$

then $\mathbf{T}$ *is Fredholm,* $\alpha(\mathbf{T}) = q$, $\beta(\mathbf{T}) = p$, *and hence* $ind(\mathbf{T}) = q - p$.

Lemma 3.1. *: Suppose that* $\mathbf{E}$, $\mathbf{F}$ *and* $\mathbf{G}$ *are Banach spaces , let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$. *Then we have*

1. *If* $\mathbf{T} \in \Phi_-(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \Phi_-(\mathbf{F},\mathbf{G})$, *then* $\mathbf{ST} \in \Phi_-(\mathbf{E},\mathbf{G})$ *and* $ind(\mathbf{ST}) = ind(\mathbf{T}) + ind(\mathbf{S})$;
2. *If* $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \Phi_+(\mathbf{F},\mathbf{G})$, *then* $\mathbf{ST} \in \Phi_+(\mathbf{E},\mathbf{G})$ *and* $ind(\mathbf{ST}) = ind(\mathbf{T}) + ind(\mathbf{S})$.

▸**Proof:**

$(\mathbf{1})$. Let $\mathbf{F}_0 \subset \mathbf{F}$ and $\mathbf{G}_0 \subset \mathbf{G}$ be finite-dimensional subspaces such that $\mathcal{R}(\mathbf{T}) + \mathbf{F}_0 = \mathbf{F}$ and $\mathcal{R}(\mathbf{S}) + \mathbf{G}_0 = \mathbf{G}$. Then $\mathbf{G} = \mathbf{G}_0 + \mathbf{S}(\mathbf{F}) = \mathbf{G}_0 + \mathbf{S}(\mathcal{R}(\mathbf{T})) + S(\mathbf{F}_0) = \mathcal{R}(\mathbf{ST}) + (\mathbf{G}_0 + \mathbf{S}(\mathbf{F}_0))$, where $\mathbf{dim}(\mathbf{G}_0 + \mathbf{SF}_0) < \infty$. Thus $\mathbf{ST} \in \Phi_-(\mathbf{E},\mathbf{G})$.

$(\mathbf{2})$. If $\mathbf{S},\mathbf{T}$ are upper semi-Fredholm, then $\mathbf{T}',\mathbf{S}'$ are lower semi-Fredholm and, by (1), $\mathbf{T}'\mathbf{S}'$ is lower semi-Fredholm. Thus $\mathbf{S},\mathbf{T}$ is upper semi-Fredholm.

By **Theorem** 1.7 we have $ind(\mathbf{ST}) = ind(\mathbf{T}) + ind(\mathbf{S})$.

■

Corollary 3.1. *: Suppose that* $\mathbf{E}$, $\mathbf{F}$ *and* $\mathbf{G}$ *are Banach spaces , let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$, *we have. If* $\mathbf{T} \in \Phi(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \Phi(\mathbf{F},\mathbf{G})$, *then* $\mathbf{ST} \in \Phi(\mathbf{E},\mathbf{G})$ *and* $ind(\mathbf{ST}) = ind(\mathbf{T}) + ind(\mathbf{S})$.

▸**Proof:**

The result follows from part (1) and (2) of **Lemma** 3.1.

■

Lemma 3.2. *: Suppose that* $\mathbf{E}$, $\mathbf{F}$ *and* $\mathbf{G}$ *are Banach spaces, let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$. *Then we have*

1. *If* $\mathbf{ST} \in \Phi_-(\mathbf{E},\mathbf{G})$ *then* $\mathbf{T} \in \Phi_-(\mathbf{F},\mathbf{G})$;
2. *If* $\mathbf{ST} \in \Phi_+(\mathbf{E},\mathbf{G})$ *then* $\mathbf{S} \in \Phi_+(\mathbf{F},\mathbf{G})$.

▸**Proof:**

$(\mathbf{1})$. We have $\mathcal{R}(\mathbf{S}) \supset \mathcal{R}(\mathbf{ST})$, so $\beta(\mathbf{S}) \leqslant \beta(\mathbf{ST}) < \infty$.

$(\mathbf{2})$. If $\mathbf{ST}$ is upper semi-Fredholm, then its adjoint $(\mathbf{ST})' = \mathbf{T}'\mathbf{S}'$ is lower semi-Fredholm, so $\mathbf{T}'$ is lower semi-Fredholm and $\mathbf{T}$ is upper semi-Fredholm.

■

Corollary 3.2. *: Suppose that* $\mathbf{E}$, $\mathbf{F}$ *and* $\mathbf{G}$ *are Banach spaces , let* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *and* $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$, *we have. If* $\mathbf{ST} \in \Phi(\mathbf{E},\mathbf{G})$ *then* $\mathbf{S} \in \Phi_-(\mathbf{F},\mathbf{G})$ *and* $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$.

▸**Proof:**

The result follows from part (1) and (2) of **Lemma** 3.2.

■

Remark 3.3. : *If* $\mathbf{T} \in \Phi_+(\mathbf{E})$ *(respectively,* $\mathbf{T} \in \Phi_-(\mathbf{E})$*) then* $\mathbf{T}^n \in \Phi_+(\mathbf{E})$ *for every* $n \in \mathbb{N}$ *(respectively,* $\mathbf{T}^n \in \Phi_-(\mathbf{E})$*). Moreover* $ind(\mathbf{T}^n) = n\, ind(\mathbf{T})$.

Proposition 3.1. *If* $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$ *then* $asc(\mathbf{T}) = dsc(\mathbf{T}')$ *and* $dsc(\mathbf{T}) = asc(\mathbf{T}')$.

▸**Proof:**

If $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$ then $\mathbf{T}^n \in \Phi_{\pm}(\mathbf{E})$, and hence the range of $\mathbf{T}^n$ is closed for all n. Analogously, also $\mathbf{T}'^n$ has closed range, and therefore for every $n \in \mathbb{N}$,

$$Ker\mathbf{T}^{n\prime} = \mathcal{R}(\mathbf{T}^n)^{\perp}, Ker\mathbf{T}^n =^{\perp} \mathcal{R}(\mathbf{T}^{n\prime}) =^{\perp} \mathcal{R}(\mathbf{T}'^n).$$

Obviously these equalities imply that $asc(\mathbf{T}) = dsc(\mathbf{T}')$ and $dsc(\mathbf{T}) = asc(\mathbf{T}')$.

■

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed linear subspaces of a Banach space $\mathbf{E}$ and define, if $M \neq \{0\}$,

$$\delta(\mathcal{M},\mathcal{N}) := sup\{dist(x,\mathcal{N}) :, x \in \mathcal{M}, \|x\| = 1\},$$

while $\delta(\mathcal{M},\mathcal{N}) = 0$ if $\delta(\mathcal{M},\mathcal{N}) = \{0\}$. The gap between $\mathcal{M}$ and $\mathcal{N}$ is defined as

$$\Theta(\mathcal{M},\mathcal{N}) := max\{\delta(\mathcal{M},\mathcal{N}), \delta(\mathcal{N},\mathcal{M})\}.$$

It is clear that $0 \leqslant \Theta(\mathcal{M},\mathcal{N}) \leqslant 1$, while $\Theta(\mathcal{M},\mathcal{N}) = 0$ precisely when $\mathcal{M} = \mathcal{N}$ and $\Theta(\mathcal{M},\mathcal{N}) = \Theta(\mathcal{N},\mathcal{M})$. Moreover, $\Theta(\mathcal{M},\mathcal{N}) = \Theta(\mathcal{M}^{\perp},\mathcal{N}^{\perp})$. And if $\Theta(\mathcal{M},\mathcal{N}) < 1$, then either $\mathcal{M}$ and $\mathcal{N}$ are both infinite-dimensional or $\mathbf{dim}\mathcal{M} = \mathbf{dim}\ \mathcal{N} < \infty$.

Theorem 3.2. *([16],* ***(4.2.1) Theorem****): Let* $\mathbf{T} \in \Phi_+(\mathbf{E})$. *Then there exists a number* $\varepsilon > 0$ *such that* $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ *and* $\|\mathbf{S}\| < \varepsilon$ *implies* $\mathbf{T} + \mathbf{S} \in \Phi_+(\mathbf{E})$. *Moreover,*

$$\alpha(\mathbf{T}+\mathbf{S}) \leqslant \alpha(\mathbf{T}), \text{ and if } \beta(\mathbf{T}) = \infty \text{ then } \beta(\mathbf{T}+\mathbf{S}) = \infty.$$

▸**Proof:**

$Ker(\mathbf{T})$ is finite dimensional, so there exists a closed subspace Q such that

$$\mathbf{E} = Ker(\mathbf{T}) \oplus Q.$$

Then $\mathbf{T}_{/Q} = \mathbf{T}_Q$ (recall $\mathbf{T}_Q$ denotes the restriction of $\mathbf{T}$ to Q) is one to one and has closed range $\mathcal{R}(\mathbf{T})$, so $\mathbf{T}_Q^{-1}$ is continuous and therefore bounded. Hence there exists $\delta > 0$ such that $\|\mathbf{T}x\| \leqslant \eta \, \|x\|$ for all $x \in Q$.

Let $\varepsilon = \frac{\eta}{3}$ If $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ with $\|\mathbf{S}x\| \leqslant \varepsilon$, we have

$$0 \leqslant \|\mathbf{S}x\| < \varepsilon \, \|x\| = \tfrac{\eta}{3}\|x\| \text{ for all } x \in Q$$

From this we get

$$\|(\mathbf{T}+\mathbf{S})x\| \geqslant \|\mathbf{T}x\| - \|\mathbf{S}x\| \geqslant \eta\|x\| - \frac{\eta}{3}\|x\| = \frac{2\eta}{3}\|x\|, \quad for \ all \ x \in Q. \tag{3.1}$$

So $(\mathbf{T}+\mathbf{S})_Q^{-1}$ exists and is continuous. Now $\mathbf{T}+\mathbf{S}$ is one to one on Q and $\mathcal{R}[(\mathbf{T}+\mathbf{S})_Q]$ is closed. We must show that $\alpha(\mathbf{T}+\mathbf{S}) < \infty$ and $\mathcal{R}(\mathbf{T}+\mathbf{S})$ is closed. Let $\alpha(\mathbf{T}) = p$. Suppose $Ker(\mathbf{T}+\mathbf{S})$ had $p+1$ linearly independent elements, $x_1, x_2, x_3, ..., x_{p+1}$, Then since $Ker(\mathbf{T}+\mathbf{S}) \cap Q = \{0\}$ we would have that $x_1, x_2, x_3, ..., x_{p+1}$ are linearly independent modulo Q. Since $\mathbf{E} = Ker\mathbf{T} \oplus Q$, it follows that $\mathbf{E}/Q$ is p dimensional, so there cannot exist $p+1$ elements of $\mathbf{E}$ which are linearly independent modulo Q. This contradiction implies that $\alpha(\mathbf{T}+\mathbf{S}) \leqslant \alpha(\mathbf{T})$.

Now to show that $\mathcal{R}(\mathbf{T}+\mathbf{S})$ is closed. Since $\mathbf{E} = [Q \oplus Ker(\mathbf{T}+\mathbf{S})] + Ker(\mathbf{T})$, there exists a finite dimensional subspace K such that

$$\mathbf{E} = [Q \oplus Ker(\mathbf{T}+\mathbf{S})] \oplus Ker(\mathbf{T}).$$

Therefore

$$\mathcal{R}(\mathbf{T}+\mathbf{S}) = (\mathbf{T}+\mathbf{S})K + (\mathbf{T}+\mathbf{S})(Q \oplus Ker(\mathbf{T}+\mathbf{S})).$$

Since $(\mathbf{T}+\mathbf{S})K$ is finite dimensional, it remains to show that $(\mathbf{T}+\mathbf{S})(Q \oplus Ker(\mathbf{T}+\mathbf{S}))$ is closed. But this reduces to showing that $(\mathbf{T}+\mathbf{S})_Q$ is closed and we noted this earlier.

Now to get that $\beta(\mathbf{T}) = \beta(\mathbf{T}+\mathbf{S})$ if $\beta(\mathbf{T}) = \infty$.

$$\|(\mathbf{T}+\mathbf{S})x\| = \|\mathbf{S}x\| \leqslant \varepsilon\|x\| = \frac{\eta}{3}\|x\| \leqslant \frac{1}{3}\|\mathbf{T}x\| \quad for \ all \ x \in \mathbf{E},$$

and using (3.1) we get

$$\|\mathbf{T}x - (\mathbf{T}+\mathbf{S})x\| = \|\mathbf{S}x\| \leqslant \|\mathbf{S}\| \ \|x\| \leqslant \frac{3\|\mathbf{S}\|}{2\eta}\|(\mathbf{T}+\mathbf{S})x\| \quad for \ all \ x \in \mathbf{E},$$

Note that $\frac{3\|\mathbf{S}\|}{2\eta} < \frac{3\varepsilon}{2\eta} = \frac{1}{2}$ for all, $x \in Q$. So the above two inequalities give us an estimate of the gap between $R_1 = \mathcal{R}[(\mathbf{T}+\mathbf{S})_Q]$ and $R_2 = \mathcal{R}(\mathbf{T}_Q)$ of $\Theta(R_1, R_2) < \frac{1}{2}$.

We get that

$$\mathbf{dim} R_1^{\perp} = \mathbf{dim} R_2^{\perp} = \beta(\mathbf{T}), \quad since \quad R_2 = \mathcal{R}(\mathbf{T}).$$

Inasmuch as $\mathcal{R}(\mathbf{T}+\mathbf{S}) = R_1 \oplus W$, where W is a suitable finite dimensional subspace, we see that $\beta(\mathbf{T}+\mathbf{S}) = \infty$ if $\beta(\mathbf{T}) = \infty$.

■

Theorem 3.3. *([16], **(4.2.2) Theorem**): Analogously, if $\mathbf{T} \in \Phi_-(\mathbf{E})$ then there exists a number $\varepsilon > 0$ such that for every $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ with $\|\mathbf{S}\| < \varepsilon$ we have $\mathbf{T}+\mathbf{S} \in \Phi_-(\mathbf{E},\mathbf{F})$ and*

$$\beta(\mathbf{T}+\mathbf{S}) \leqslant \beta(\mathbf{T}) \text{ and if } \alpha(\mathbf{T}) = \infty \text{ then } \alpha(\mathbf{T}+\mathbf{S}) = \infty.$$

▸**Proof:**

Use the relationships between $\mathbf{T}$ and $\mathbf{T}'$, and in the same way as the previous proof, we find that the results are valid.

■

Remark 3.4. *: The relation between minimum Modulus and upper, (resp :lower) semi-Fredholm operators , as follows:*

- *If $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$, then*

$$\gamma(\mathbf{T}) = sup\{s > 0 \ : \ \alpha(\mathbf{T}+\mathbf{S}) \leqslant \alpha(\mathbf{T}) \ for \ every \ \mathbf{S} \ with \ \ \|\mathbf{S}\| < s\}.$$

- *If $\mathbf{T} \in \Phi_-(\mathbf{E},\mathbf{F})$, then*

$$\gamma(\mathbf{T}) = sup\{s > 0 \ : \ \beta(\mathbf{T}+\mathbf{S}) \leqslant \beta(\mathbf{T}) \ for \ every \ \mathbf{S} \ with \ \ \|\mathbf{S}\| < s\}.$$

We also have the following results products of operators.

Theorem 3.4. *([35], **Theorem 5.31**): If $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$ and $\mathbf{ST} \in \Phi_-(\mathbf{E},\mathbf{G})$, then $\mathbf{S} \in \Phi_-(\mathbf{F},\mathbf{G})$.*

▸**Proof:**

Since $\mathbf{dim}\mathcal{R}(\mathbf{ST})^{\perp} < \infty$, there is a subspace $\mathbf{G}_0$ such that $\mathbf{dim}\mathbf{G}_0 < \infty$ and $\mathbf{G} = \mathcal{R}(\mathbf{ST}) \oplus \mathbf{G}_0$. Since $\mathcal{R}(\mathbf{S}) \supset \mathcal{R}(\mathbf{ST})$, we know that $\mathcal{R}(\mathbf{S})$ is closed and $\mathcal{R}(\mathbf{S})^{\perp} \subset \mathcal{R}(\mathbf{ST})^{\perp}$. Thus, $\mathbf{dim}\mathcal{R}(\mathbf{S})^{\perp} < \infty$. Consequently, $\mathbf{S} \in \Phi_-(\mathbf{F},\mathbf{G})$.

■

Theorem 3.5. *([35], **Theorem** 5.32): If* $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, $\mathbf{S} \in \mathcal{B}(\mathbf{F},\mathbf{G})$ *and* $\mathbf{ST} \in \Phi_+(\mathbf{E},\mathbf{G})$, *then* $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$.

▸**Proof:**

We merely note that $\mathbf{T}'\mathbf{S}' \in \Phi_-(\mathbf{E}',\mathbf{G}')$. **Theorem** 3.4 now implies that $\mathbf{T}' \in \Phi_-(\mathbf{F}',\mathbf{E}')$, which means that $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$.

■

We remind again that, two closed subspaces $\mathbf{E}_1$, $\mathbf{E}_2$ of a Banach space $\mathbf{E}$ are called complementary when $\mathbf{E}_1 \cap \mathbf{E}_2 = \{0\}$ and $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2$. Either subspace is called a complement of the other. We say that a subspace $\mathbf{E}_1 \subset \mathbf{E}$ is complemented if it has a complement. Some Banach spaces contain subspaces which are not complemented. We note, if $\mathbf{E}_1$ is a closed complemented subspace of a Banach space $\mathbf{E}$, then there is a bounded projection P on $\mathbf{E}$ with $\mathcal{R}(P) = \mathbf{E}_1$.

Theorem 3.6. *([35], **Theorem** 5.34): If* $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$ *and* $\mathcal{R}(\mathbf{T})$ *is complemented in* $\mathbf{F}$, *then there is an* $\mathbf{T}_0 \in \mathcal{B}(\mathbf{E},\mathbf{F})$ *such that* $\mathbf{T}_0\mathbf{T} \in \Phi(\mathbf{E})$.

▸**Proof:**

Let P be a bounded projection from $\mathbf{F}$ to $\mathcal{R}(\mathbf{T})$. There is a closed subspace $\mathbf{E}_0 \in \mathbf{E}$ such that

$$\mathbf{E} = \mathbf{E}_0 \oplus Ker\mathbf{T}.$$

Then $\mathbf{T}$ has a bounded inverse $\hat{\mathbf{T}}$ from $\mathcal{R}(\mathbf{T})$ to $\mathbf{E}_0$. Let $\mathbf{T}_0 = \hat{\mathbf{T}}P$. Then $\mathbf{T}_0 \in \mathcal{B}(\mathbf{E},\mathbf{F})$, and

$$\mathbf{T}_0\mathbf{T} = \begin{cases} \mathbf{I} & on \quad \mathbf{E}_0, \\ 0 & on \quad Ker\mathbf{T}. \end{cases}$$

Thus, $\mathbf{T}_0\mathbf{T} \in \Phi(\mathbf{E})$.

■

Theorem 3.7. *([35], **Theorem 5.35**): If* $\mathbf{T} \in \Phi_-(\mathbf{E},\mathbf{F})$ *and* $Ker\mathbf{T}$ *is complemented, then there is an* $\mathbf{T}_0 \in \mathcal{B}(\mathbf{F},\mathbf{E})$ *such that* $\mathbf{T}\mathbf{T}_0 \in \Phi(\mathbf{F})$.

▸**Proof:**

There is a finite dimensional subspace $\mathbf{F}_0 \in \mathbf{F}$ such that

$$\mathbf{F} = \mathbf{F}_0 \oplus \mathcal{R}(\mathbf{T}).$$

Let P be the bounded projection onto $\mathcal{R}(\mathbf{T})$ which vanishes on $\mathbf{F}_0$, and define $\mathbf{T}_0$ as above. Then

$$\mathbf{T}\mathbf{T}_0 = \begin{cases} \mathbf{I} & on \quad \mathcal{R}(\mathbf{T}), \\ 0 & on \quad \mathbf{F}_0. \end{cases}$$

Thus, $\mathbf{T}\mathbf{T}_0 \in \Phi(\mathbf{E})$.

■

Theorem 3.8. *([35], **Theorem 5.36**) : If* $\mathbf{T}$ *is in* $\mathcal{B}(\mathbf{E},\mathbf{F})$, $\mathbf{S}$ *is in* $\mathcal{B}(\mathbf{F},\mathbf{G})$ *and* $\mathbf{ST} \in \Phi(\mathbf{E},\mathbf{G})$, *then* $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$, *and* $\mathbf{S} \in \Phi_-(\mathbf{F},\mathbf{G})$. *Moreover,* $\mathcal{R}(\mathbf{T})$ *and* $Ker\mathbf{S}$ *are complemented.*

▸**Proof:**

The first statement follows from **Theorems** 3.6 and **Theorems** 3.7. Consequently, there is a closed subspace $\mathbf{E}_0 \subset \mathbf{E}$ such that

$$\mathbf{E} = \mathbf{E}_0 \oplus Ker\mathbf{T}.$$

Let

$$\mathbf{F}_1 = \mathcal{R}(\mathbf{T}) \cap Ker\mathbf{S}, \qquad \mathbf{E}_1 = \mathbf{T}^{-1}(\mathbf{F}_1) \cap \mathbf{E}_0.$$

Since $\mathbf{E}_1 \subset Ker(\mathbf{ST})$, $\mathbf{dim}\mathbf{E}_1 \leqslant \alpha(\mathbf{ST}) < \infty$. Since $\mathbf{T}$ is one-to-one from $\mathbf{E}_1$ onto $\mathbf{F}_1$, we see that $\mathbf{dim}\mathbf{F}_1 = \mathbf{dim}\mathbf{E}_1 < \infty$. Hence, there are subs paces $\mathbf{F}_2 \subset \mathcal{R}(\mathbf{T}), \mathbf{F}_3 \subset Ker\mathbf{S}$ such that

$$\mathcal{R}(\mathbf{T}) = \mathbf{F}_1 \oplus \mathbf{F}_2, \qquad Ker\mathbf{S} = \mathbf{F}_1 \oplus \mathbf{F}_3.$$

We also know that

$$\mathbf{G} = \mathcal{R}(\mathbf{ST}) \oplus \mathbf{G}_0,$$

where $\mathbf{dim}\mathbf{G}_0 < \infty$. Let $\mathbf{G}_4 = \mathcal{R}(\mathbf{S}) \cap \mathbf{G}_0$. Then

$$\mathcal{R}(\mathbf{S}) = \mathcal{R}(\mathbf{ST}) \oplus \mathbf{G}_4,$$

and there is a subspace $\mathbf{G}_5 \subset \mathbf{G}_0$ such that $\mathbf{G}_0 = \mathbf{G}_4 \oplus \mathbf{G}_5$. Consequently

$$\mathbf{G} = \mathcal{R}(\mathbf{ST}) \oplus \mathbf{G}_4 \oplus \mathbf{G}_5, \qquad = \mathcal{R}(\mathbf{S}) = \mathcal{R}(\mathbf{ST}) \oplus \mathbf{G}_4.$$

Let $g_1, \ldots, g_2$ be a basis for $\mathbf{G}_4$. Then there are $y_1, \ldots, y_n \in \mathbf{F}$ such that

$$\mathbf{S}y_j = z_j, 1 \leqslant j \leqslant n.$$

Let $\mathbf{F}_4$ be the subspace of $\mathbf{F}$ spanned by $y_1, \ldots, y_n$. Then $\mathbf{dim}\mathbf{F}_4 < \mathbf{dim}\mathbf{G}_4 < \infty$. We note that

$$\mathbf{F}_2 \cap Ker\mathbf{S} = \{0\}, \quad \mathbf{F}_4 \cap Ker\mathbf{S} = \{0\}, \quad \mathbf{F}_2 \cap \mathbf{F}_4 = \{0\}.$$

Thus,

$$Ker\mathbf{S} \cap [\mathbf{F}_2 \oplus \mathbf{F}_4] = \{0\}.$$

Since $\mathbf{dim}\, \mathbf{F}_4 < \infty$, the subspace $\mathbf{F}_2 \oplus \mathbf{F}_4$ is closed. Let y be any element of $\mathbf{F}$. Then $\mathbf{S}y \in \mathcal{R}(\mathbf{S}) = \mathcal{R}(\mathbf{ST}) \oplus \mathbf{G}_4$. Hence, there are $z_2 \in \mathcal{R}(\mathbf{ST})$, $z_4 \in \mathbf{G}_4$ such that $\mathbf{S}y = z_2 + z_4$. There are $y_2 \in \mathbf{F}_2, y_4 \in \mathbf{F}_4$ such that $\mathbf{S}y_2 = z_2$, $\mathbf{S}y_4 = z_4$. Then

$$\mathbf{S}(y - y_2 - y_4) = \mathbf{S}y - z_2 - z_4 = 0.$$

Thus, $(y - y_2 - y_4 \in Ker\mathbf{S}$, and consequently

$$\mathbf{F} = \mathbf{F}_2 \oplus \mathbf{F}_4 \oplus Ker\mathbf{S}.$$

This shows that $Ker\mathbf{S}$ is complemented. Moreover,

$$\mathbf{F} = \mathbf{F}_2 \oplus \mathbf{F}_4 \oplus \mathbf{F}_1 \oplus \mathbf{F}_3 = \mathbf{F}_3 \oplus \mathbf{F}_4 \oplus \mathcal{R}(\mathbf{T}),$$

showing that $\mathcal{R}(\mathbf{T})$ is also complemented.

■

Now we will define other Fredholm sets.

Definition 3.3. : *The set of left invertible and right invertible operators are denoted by* $\mathfrak{G}_l(\mathbf{E})$ *and* $\mathfrak{G}_r(\mathbf{E})$, *respectively. Note that* $\mathbf{T}$ *is invertible, if* $\mathbf{T}$ *is left and right invertible. The set of left Fredholm operators is defined by*

$$\Phi_l(\mathbf{E},\mathbf{F}) = \Big\{\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F}) \text{ such that } \mathcal{R}(\mathbf{T}) \text{ is closed, complemented subspaces and } \alpha(\mathbf{T}) < \infty\Big\};$$

and the set of right Fredholm operators is defined by

$$\Phi_r(\mathbf{E},\mathbf{F}) = \Big\{\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F}) \text{ such that } Ker\mathbf{T} \text{ is complemented subspaces and } \beta(\mathbf{T}) < \infty\Big\}.$$

Thus, we have the following inclusions

$$\Phi(\mathbf{E},\mathbf{F}) \subseteq \Phi_l(\mathbf{E},\mathbf{F}) \subseteq \Phi_+(\mathbf{E},\mathbf{F});$$

and

$$\Phi(\mathbf{E},\mathbf{F}) \subseteq \Phi_r(\mathbf{E},\mathbf{F}) \subseteq \Phi_-(\mathbf{E},\mathbf{F}).$$

Clearly, if $\mathbf{T} \in \Phi(\mathbf{E},\mathbf{F})$, then

$$ind(\mathbf{T}) < \infty.$$

If $\mathbf{T} \in \Phi_l(\mathbf{E},\mathbf{F}) \backslash \Phi(\mathbf{E},\mathbf{F})$, then

$$ind(\mathbf{T}) = -\infty,$$

and, if $\mathbf{T} \in \Phi_r(\mathbf{E},\mathbf{F}) \backslash \Phi(\mathbf{E},\mathbf{F})$, then

$$ind(\mathbf{T}) = +\infty.$$

In next theorem, It should be noted that in the characterization below the ideal $\mathcal{F}(\mathbf{E})$ may be replaced by the ideal $\mathcal{K}(\mathbf{E})$ of all compact operators.

Theorem 3.9. *: ([4], **Theorem 1.53 . p 33**)(Atkinson characterization of Fredholm operators): If $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$, then $\mathbf{T} \in \Phi(\mathbf{E},\mathbf{F})$ if and only there exist $\mathbf{U}_1$, $\mathbf{U}_2 \in \mathcal{B}(\mathbf{E},\mathbf{F})$ and finite-dimensional operators $\mathbf{K}_1 \in \mathcal{F}(\mathbf{E})$, $\mathbf{K}_2 \in \mathcal{F}(\mathbf{F})$ such that*

$$\mathbf{U}_1\mathbf{T} = \mathbf{I}_\mathbf{E} - \mathbf{K}_1 \text{ and } \mathbf{T}\mathbf{U}_2 = \mathbf{I}_\mathbf{F} - \mathbf{K}_2 .$$

In particular, $\mathbf{T} \in \Phi(\mathbf{E})$ if and only if $\mathbf{T}$ is invertible in $\mathcal{B}(\mathbf{E})$ modulo the ideal of finite-dimensional operators $\mathcal{F}(\mathbf{E})$.

In the following theories we will study the compact perturbations .

Lemma 3.3. *: Let $\mathbf{T} \in \mathcal{B}(\mathbf{E},\mathbf{F})$ and $\mathbf{K} \in \mathcal{K}(\mathbf{E},\mathbf{F})$. Then the following statements hold*

1. *If $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$ then $\mathbf{T}+\mathbf{K} \in \Phi_+(\mathbf{E},\mathbf{F})$;*
2. *If $\mathbf{T} \in \Phi_-(\mathbf{E},\mathbf{F})$ then $\mathbf{T}+\mathbf{K} \in \Phi_-(\mathbf{E},\mathbf{F})$.*

▸**Proof:**

(1). Let $\mathbf{T} \in \Phi_+(\mathbf{E},\mathbf{F})$ and let M_1 be a closed subspace of $\mathbf{E}$ such that $\mathbf{codim} M_1 < \infty$ and $inf\{ \|\mathbf{T}x\| : x \in M_1, \|x\| = 1 \} = c > 0$. Since $\mathbf{K}$ is compact, there exists a closed subspace $M_2 \subset \mathbf{E}$ with $\mathbf{codim} M_2 < \infty$ and $sup\{\|\mathbf{T}x\| : x \in M_2, \|x\| = 1\} < \frac{c}{2}$. Set $M = M_1 \cap M_2$. Then $\mathbf{codim} M < \infty$ and $inf\{\|(\mathbf{T}+\mathbf{K})x\| : x \in M, \|x\| = 1\} \geqslant inf\{\|\mathbf{T}x\| - \|\mathbf{K}x\| : x \in M, \|x\| = 1\} \geqslant \frac{c}{2}$. Hence $\mathbf{T}+\mathbf{K} \in \Phi_+(\mathbf{E},\mathbf{F})$.

(2). If $\mathbf{T} \in \Phi_-(\mathbf{E},\mathbf{F})$ and $\mathbf{K} \in \mathcal{K}(\mathbf{E},\mathbf{F})$, then $\mathbf{T}'$ is upper semi-Fredholm and $\mathbf{K}'$ is compact. By **(1)**, $\mathbf{T}'+\mathbf{K}'$ is upper semi-Fredholm, and so $\mathbf{T}+\mathbf{K}$ is lower. semi-Fredholm.

■

Theorem 3.10. *: If $\mathbf{T} \in \Phi(\mathbf{E},\mathbf{F})$ and $\mathbf{K} \in \mathcal{K}(\mathbf{E},\mathbf{F})$, then $\mathbf{T}+\mathbf{K} \in \Phi(\mathbf{E},\mathbf{F})$. Moreover*

$$ind(\mathbf{T}+\mathbf{K}) = ind(\mathbf{T}).$$

▸**Proof:**

By **Lemma 3.3**, $\Phi(\mathbf{E},\mathbf{F})$, $\Phi_+(\mathbf{E},\mathbf{F})$ and $\Phi_-(\mathbf{E},\mathbf{F})$ are invariant under compact perturbations. Let $\mathbf{T}\in\Phi(\mathbf{E},\mathbf{F})$. By **Theorem 3.9**, there exist $\mathbf{U}\in\mathcal{B}(\mathbf{E},\mathbf{F})$ and $\mathbf{K}_1\in\mathcal{K}(\mathbf{E})$ such that $\mathbf{UT}=\mathbf{I_E}+\mathbf{K}_1$. By **Lemma 3.1**, $ind(\mathbf{T})+ind(\mathbf{U})=ind(\mathbf{I_E}+\mathbf{K}_1)=0$, so $ind(\mathbf{T})=-ind(\mathbf{U})$. Further, $\mathbf{U}(\mathbf{T}+\mathbf{K})=\mathbf{I_E}+(\mathbf{K}_1+\mathbf{UK})$, where $\mathbf{K}_1+\mathbf{UK}\in\mathcal{K}(\mathbf{E})$, and so $ind(\mathbf{U})+ind(\mathbf{T}+\mathbf{K})=0$. Hence $ind(\mathbf{T}+K)=-ind(\mathbf{U})=ind(\mathbf{T})$. If $\mathbf{T}\in\Phi_+(\mathbf{E},\mathbf{F})\backslash\Phi(\mathbf{E},\mathbf{F})$, then $\mathbf{T}+\mathbf{K}\in\Phi_+(\mathbf{E},\mathbf{F})\backslash\Phi(\mathbf{E},\mathbf{F})$, and so $ind(\mathbf{T}+\mathbf{K})=ind\mathbf{T}=-\infty$.

■

Example 3.4. *:Let* $\mathbf{E}=\ell^p$, $1\leqslant p\leqslant\infty$, *the space of all sequences* $x=(x_1,x_2,x_3,\ldots)$ *with finite norm*

$$\|x\|_p=\begin{cases}(\sum_{n=1}^{\infty}\mid x_n\mid^p)^{\frac{1}{p}} & if \quad 1\leqslant p\leqslant\infty\\ sup_{n\geqslant 1}\mid x_n\mid & if \quad p=\infty\end{cases}$$

we define the following operators on ℓ^p *by*

$$\mathbf{T}_0x=(0,x_1,x_2,x_3,\ldots),$$

$$\mathbf{T}_1x=(x_2,x_3,x_4,\ldots),$$

$$\mathbf{T}_2x=(x_1,\frac{1}{2}x_2,\frac{1}{3}x_3,\ldots),$$

$$\mathbf{T}_3x=(x_2,\frac{1}{2}x_3,\frac{1}{3}x_4,\ldots),$$

$$\mathbf{T}_4x=(0,x_1,\frac{1}{2}x_2,\frac{1}{3}x_3,\ldots).$$

We can see that $Ker\mathbf{T}_1$ *consists of those elements of the form*

$$(x_1,0,\ldots).$$

so that $ind(\mathbf{T}_1)=1$.

The operator $\mathbf{T}_0$ *is a Fredholm operator with* $ind(\mathbf{T}_0)=-1$ *because* $\mathbf{T}_0$ *is injective and* $\mathcal{R}(\mathbf{T}_0)=\mathbf{E}_0$. *(*$\mathbf{T}_0$ *and* $\mathbf{T}_1$ *are left and right shift operators.)*

Since the operators $\mathbf{T}_2,\mathbf{T}_3$ *and* $\mathbf{T}_4$ *are compact, it follows therefore, that* $\mathbf{T}_0+\mathbf{T}_i$ *and* $\mathbf{T}_1+\mathbf{T}_i$ *are Fredholm operators with* $ind(\mathbf{T}_0+\mathbf{T}_i)=ind(\mathbf{T}_0)=-1$ *and* $ind(\mathbf{T}_1+\mathbf{T}_i)=ind(\mathbf{T}_1)=1$, *with* $i=2,3,4$.

Proposition 3.2. *: Let* $\mathbf{T}\in\mathcal{B}(\mathbf{E})$, *then the following assertions are equivalent:*

1. $\mathbf{T}\in\Phi(\mathbf{E})$ *;*
2. $[\mathbf{T}]$ *is invertible element of Calkin algebra* $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$ *.*

▸**Proof:**
(1) $\Rightarrow$ **(2)** . Suppose that (1) is true, then there exist $\mathbf{S} \in \mathcal{B}(\mathbf{E}), \mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}(\mathbf{E})$ such that

$$\mathbf{ST} = \mathbf{I} + \mathbf{K}_1, \quad \textit{and} \quad \mathbf{TS} = \mathbf{I} + \mathbf{K}_2.$$

Then we have

$$\begin{aligned}[\mathbf{S}].[\mathbf{T}] = [\mathbf{ST}] &= [\mathbf{I} + \mathbf{K}_1] \\ &= [\mathbf{I}][\mathbf{K}_1] \\ &= \mathbf{I} + 0 \\ &= \mathbf{I}.\end{aligned}$$

In the same way

$$\begin{aligned}[\mathbf{T}].[\mathbf{S}] = [\mathbf{TS}] &= [\mathbf{I} + \mathbf{K}_2] \\ &= [\mathbf{I}][\mathbf{K}_2] \\ &= \mathbf{I} + 0 \\ &= \mathbf{I}.\end{aligned}$$

Hence $[\mathbf{T}]$ is invertible in $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$, with $[\mathbf{T}]^{-1} = [\mathbf{S}]$
(2) $\Rightarrow$ **(1)** . Suppose that (1) is true, then there exist $\mathbf{S} \in \mathcal{B}(\mathbf{E})$ such that

$$[\mathbf{T}].[\mathbf{S}] = [\mathbf{S}].[\mathbf{T}] = \mathbf{I}.$$

Then we have

$$[\mathbf{ST} - \mathbf{I}] = [\mathbf{TS} - \mathbf{I}] = 0,$$

in another way we say that $\mathbf{ST} - \mathbf{I}$ and $\mathbf{TS} - \mathbf{I}$ are compact, then we have $\mathbf{ST} - \mathbf{I} = \mathbf{K}_1$ and $\mathbf{TS} - \mathbf{I} = \mathbf{K}_2$. Therefore

$$\mathbf{ST} = \mathbf{I} + \mathbf{K}_1, \quad \textit{and} \quad \mathbf{TS} = \mathbf{I} + \mathbf{K}_2,$$

with $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}(\mathbf{E})$, then by **Theorem** 3.9 we conclude that $\mathbf{T} \in \Phi(\mathbf{E})$.

■

Proposition 3.3. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, *then we have . If* $\mathbf{T} \in \Phi_-(\mathbf{E})$ *then there exists* $\varepsilon > 0$ *such that* $\lambda\mathbf{I} + \mathbf{T} \in \Phi_+(\mathbf{E})$ *and* $\alpha(\lambda\mathbf{I} - \mathbf{T})$ *is constant on the punctured neighbourhood* $0 <| \lambda |< \varepsilon$. *Moreover*

$$\alpha(\lambda\mathbf{I} - \mathbf{T}) \leqslant \alpha(\mathbf{T}) \quad for \ \ all \quad | \lambda |< \varepsilon, \tag{3.2}$$

and

$$ind(\lambda\mathbf{I} - \mathbf{T}) = ind(\mathbf{T}) \quad for \ \ all \quad | \lambda |< \varepsilon.$$

▸**Proof:**
See [6], Theorem 1.64 .p 40.

■

Analogously;

Proposition 3.4. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, *then we have . If* $\mathbf{T} \in \Phi_{-}(\mathbf{E})$ *then there exists* $\varepsilon > 0$ *such that* $\lambda\mathbf{I} - \mathbf{T} \in \Phi_{-}(\mathbf{E})$ *and* $\beta(\lambda\mathbf{I} + \mathbf{T})$ *is constant on the punctured neighbourhood* $0 <| \lambda |< \varepsilon$. *Moreover*

$$\beta(\lambda\mathbf{I} - \mathbf{T}) \leqslant \beta(\mathbf{T}) \quad for \ \ all \quad | \lambda |< \varepsilon, \tag{3.3}$$

and

$$ind(\lambda\mathbf{I} - \mathbf{T}) = ind(\mathbf{T}) \quad for \ \ all \quad | \lambda |< \varepsilon$$

.

▸**Proof:**
See [6] Theorem 1.64 .p 40.

■

Definition 3.4. *: Let* $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$, $\mathbf{E}$ *a Banach space. Let* $\varepsilon > 0$ *as in (3.2) or (3.3). If* $\mathbf{T} \in \Phi_{+}(\mathbf{E})$, ***the jump*** $j(\mathbf{T})$ *is defined by*

$$j(\mathbf{T}) = \alpha(\mathbf{T}) - \alpha(\lambda\mathbf{I} - \mathbf{T}), \qquad 0 <| \lambda |< .$$

while, if $\mathbf{T} \in \Phi_{-}(\mathbf{E})$, *the jump* $j(\mathbf{T})$ *is defined by*

$$j(\mathbf{T}) = \beta(\mathbf{T}) - \beta(\lambda\mathbf{I} - \mathbf{T}), \qquad 0 <| \lambda |< .$$

Remark 3.5. *: Clearly* $j(\mathbf{T}) \geqslant 0$ *and the continuity of the index ensures that both definitions of* $j(\mathbf{T})$ *coincide whenever* $\mathbf{T} \in \Phi(\mathbf{E})$, *so* $j(\mathbf{T})$ *is unambiguously defined. An immediate consequence by **Proposition**3.3 , **Proposition**3.4 and **Theorem**3.1 is that if* $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$ *then* $j(\mathbf{T}) = j(\mathbf{T}')$.

In the sequel we shall denote by $\mathbf{T}_\infty$ the restriction $\mathbf{T}_{\backslash \mathcal{R}^\infty(\mathbf{T})}$ of $\mathbf{T}$ to the invariant subspace $\mathcal{R}^\infty(\mathbf{T})$ of a linear space $\mathcal{X}$. Let $\hat{x} = x + \mathcal{R}^\infty(\mathbf{T})$ be the coset corresponding to x in the quotient space $\hat{\mathcal{X}} = \mathcal{X}/\mathcal{R}^\infty(\mathbf{T})$. If $\mathcal{Y}$ is a subset of $\mathcal{X}$, we set $\hat{\mathcal{Y}} := \{\hat{y} : y \in \mathcal{Y}\}$. Obviously $\hat{\mathcal{Y}}$ coincides with the quotient $[\hat{\mathcal{Y}} + \mathcal{R}^\infty(\mathbf{T})] / \mathcal{R}^\infty(\mathbf{T})$.

Let $\widehat{\mathbf{T}_\infty} : \hat{\mathcal{X}} \longrightarrow \hat{\mathcal{X}}$ denote the quotient operator defined by

$$\widehat{\mathbf{T}_\infty}\hat{x} = \hat{\mathbf{T}}x, \quad x \in \mathcal{X}.$$

It is easily seen that $\widehat{\mathbf{T}_\infty}$ is well defined.

In the next lemma we collect some elementary properties of $\widehat{\mathbf{T}_\infty}$.

Lemma 3.4. *([4], **Lemma 1.56**): Let* $\mathbf{T}$ *be a linear operator on a vector space* $\mathcal{X}$*, and assume that* $\alpha(\mathbf{T}) < \infty$ *or* $\beta(\mathbf{T}) < \infty$*. Then:*

1. $\mathcal{N}^\infty(\widehat{\mathbf{T}_\infty}) = \widehat{\mathcal{N}^\infty(\mathbf{T})}$*;*
2. $\mathcal{R}^\infty(\widehat{\mathbf{T}_\infty}) = \{0\}$.

▸**Proof:**

(1). We know that $\mathbf{T}(\mathcal{R}^\infty(\mathbf{T})) = \mathcal{R}^\infty(\mathbf{T})$. Let $\hat{x} \in Ker\widehat{\mathbf{T}_\infty}$. Then $\mathbf{T}x \in \mathcal{R}^\infty(\mathbf{T}) = \mathbf{T}(\mathcal{R}^\infty(\mathbf{T}))$. Consider an element $u \in \mathcal{R}^\infty(\mathbf{T})$ such that $\mathbf{T}x = \mathbf{T}u$. Clearly $x - u \in Ker\mathbf{T}$, so $x = u + v$ for some $v \in Ker\mathbf{T}$, $x \in Ker\mathbf{T} + \mathcal{R}^\infty(\mathbf{T})$ and hence $\hat{x} \in \widehat{Ker\mathbf{T}} + \widehat{\mathcal{R}^\infty(\mathbf{T})} = \widehat{Ker\mathbf{T}}$. This shows the inclusion $Ker\widehat{\mathbf{T}_\infty} \subseteq \widehat{Ker\mathbf{T}}$. The opposite inclusion is obvious, so $Ker\widehat{\mathbf{T}_\infty} = \widehat{Ker\mathbf{T}}$. Similarly $Ker(\widehat{\mathbf{T}_\infty})^n = \widehat{Ker\mathbf{T}^n}$ for every $n \in \mathbb{N}$, and from this the equality (1) easily follows.

(2). It is easy to check that $\mathcal{R}(\hat{\mathbf{T}}^n_\infty) = \widehat{\mathcal{R}(\mathbf{T}^n)}$ for all $n \in \mathbb{N}$, and from we obtain that $\mathcal{R}^\infty(\hat{\mathbf{T}}^n_\infty) = \widehat{\mathcal{R}^\infty(\mathbf{T}^n)} = \hat{0}$.

■

Lemma 3.5. *([4], **lemma 1.57**): Let* $\mathbf{T} \in \Phi_+(\mathbf{E})$, $\mathbf{E}$ *a Banach space. Then:*

1. $\mathbf{T}_\infty$ *is a Fredholm operator;*
2. $\widehat{\mathbf{T}_\infty}$ *is an upper semi-Fredholm operator.*

▸**Proof:**

(1). Since $\alpha(\mathbf{T}) < \infty$, from **Proposition 1.5** and **Proposition 1.12** we have $\beta(\mathbf{T}_\infty) = 0$, and from the inclusion $Ker\mathbf{T}_\infty \subseteq Ker\mathbf{T}$ we conclude that $Ker\mathbf{T}$ is finite-dimensional, hence $\mathbf{T}_\infty$ is a Fredholm operator

(2). From **Lemma 3.4** we have $Ker\widehat{\mathbf{T}_\infty} = \widehat{Ker\mathbf{T}}$ and hence $\alpha(\widehat{\mathbf{T}_\infty}) < \infty$. Moreover, it is easy to see that $\mathcal{R}(\hat{\mathbf{T}}_\infty) = \widehat{\mathcal{R}(\mathbf{T})}$ is a closed subspace of $\hat{\mathbf{E}}$, thus $\hat{\mathbf{T}}_\infty \in \Phi_+(\hat{\mathbf{E}})$.

■

Theorem 3.11. *: Let* $\mathbf{T} \in \Phi_\pm(\mathbf{E})$, $\mathbf{E}$ *a Banach space. Then* $j(\mathbf{T}) = 0$ *if and only if is Kato.*

▸**Proof:**

Since $\mathcal{R}(\mathbf{T})$ is closed it suffices to show the equivalence

$$j(\mathbf{T}) = 0 \Leftrightarrow \mathcal{N}^{\infty}(\mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T}).$$

Assume first $\mathbf{T} \in \Phi_{+}(\mathbf{E})$ and $\mathcal{N}^{\infty}(\mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T})$. Observe first that

$$\alpha(\lambda\mathbf{I} + \mathbf{T}) = \alpha(\lambda\mathbf{I} + \mathbf{T}\infty) \quad for \;\; all \;\; \lambda \in \mathbb{C}.$$

For $\lambda = 0$ this is clear, since $Ker\mathbf{T} \subseteq \mathcal{N}^{\infty}(\mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T})$ implies that $Ker\mathbf{T} = Ker\mathbf{T}_{\infty}$. For $\lambda \neq 0$ we have, by part (2) of **Corollary** 1.7,

$$Ker\mathbf{T} \subseteq \mathcal{N}^{\infty}(\lambda\mathbf{I} + \mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T}),$$

so that $Ker(\lambda\mathbf{I} + \mathbf{T}) = Ker(\lambda\mathbf{I} + \mathbf{T}_{\infty})$.

Now, from**Proposition** 1.5 and **Proposition** 1.12 we know that $\beta(\mathbf{T}_{\infty}) = 0$ and hence there exists $\varepsilon > 0$ such that $\beta(\lambda\mathbf{I} + \mathbf{T}_{\infty}) = 0$ for all $|\lambda| < \varepsilon$, see **Lemma** 2.8. From **Lemma** 3.4 we know that $\mathbf{T}_{\infty}$ is Fredholm, so we can assume ε such that

$$ind(\lambda\mathbf{I} + \mathbf{T}_{\infty}) = ind(\mathbf{T}_{\infty}) \quad for \;\; all \quad |\lambda| < \varepsilon.$$

Therefore $\alpha(\lambda\mathbf{I} + \mathbf{T}_{\infty}) = \alpha(\mathbf{T}_{\infty})$ for all $|\lambda| < \varepsilon$ and hence $\alpha(\lambda\mathbf{I} + \mathbf{T}) = \alpha(\mathbf{T})$ for all $|\lambda| < \varepsilon$, so that $j(\mathbf{T}) = 0$.

Conversely, suppose that $\mathbf{T} \in \Phi_{+}(\mathbf{E})$ and $j(\mathbf{T}) = 0$, namely there exists $\varepsilon > 0$ such $\alpha(\lambda\mathbf{I} + \mathbf{T})$ is constant for $|\lambda| < \varepsilon$. Then

$$\alpha(\mathbf{T}_{\infty}) \leqslant \alpha(\mathbf{T}) = \alpha(\lambda\mathbf{I} + \mathbf{T}) = \alpha(\lambda\mathbf{I} + \mathbf{T}_{\infty}) \quad for \;\; all 0 < |\lambda| < \varepsilon.$$

But $\mathbf{T}_{\infty}$ is Fredholm by **Lemma** 3.4, and hence, see **Proposition** 3.3 and **Proposition** 3.4, we can choose $\varepsilon > 0$ such that $\alpha(\lambda\mathbf{I} + \mathbf{T}_{\infty}) \leqslant \alpha(\mathbf{T}_{\infty})$ for all $|\lambda| < \varepsilon$. This shows that $\alpha(\mathbf{T}_{\infty}) = \alpha(\mathbf{T})$ and consequently, $\mathcal{N}^{\infty}(\mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T})$. Consider now the case that $\mathbf{T} \in \Phi_{-}(\mathbf{E})$ and $j(\mathbf{T}) = 0$. Then $\mathbf{T}' \in \Phi_{+}(\mathbf{E}')$ and $j(\mathbf{T}) = j(\mathbf{T}') = 0$. From the first part of the proof we deduce that $\mathcal{N}^{\infty}(\mathbf{T}') \subseteq \mathcal{R}^{\infty}(\mathbf{T}')$. From **Corollary** 1.3 it follows that $Ker\mathbf{T}'^{n} \subseteq \mathcal{R}(\mathbf{T}'^{n})$ for all $n \in \mathbb{N}$, or equivalently $\mathcal{R}(\mathbf{T}^{n})^{\perp} \subseteq Ker\mathbf{T}$ for all $n \in \mathbb{N}$. Since all these subspaces are closed then $\mathcal{R}(\mathbf{T}^{n}) \supseteq Ker\mathbf{T}$ for all $n \in \mathbb{N}$, so by **Corollary** 1.3 we conclude that $\mathcal{N}^{\infty}(\mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T})$.

■

Theorem 3.12. *: If* $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$ *then* $\mathbf{T}$ *is essentially Kato .*

▸**Proof:**

Let $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$. If $\mathbf{T}$ is Kato then the pair (M, N), with $M = \mathbf{E}$ and $N = 0$, is a Kato decomposition which verifies the desired properties. If $\mathbf{T}$ is not Kato then $j(\mathbf{T}) > 0$, by

Theorem 3.11 and hence $\mathcal{N}^{\infty}(\mathbf{T}) \subseteq \mathcal{R}^{\infty}(\mathbf{T})$. Let $P = \sum_{j=0}^{n-1} \mathbf{T}'^{j} f \otimes \mathbf{T}^{n-j-1} y$ with $y \in Ker\mathbf{T}^{n}$ but $y \notin \mathcal{R}(\mathbf{T})$ and $f \in Ker\mathbf{T}'^{n}$. be the non-zero finite-rank projection . P commutes with $\mathbf{T}$. The restriction $\mathbf{T}_{|}Ker\mathbf{P}$ is semi-Fredholm and $j(\mathbf{T}_{|Ker\mathbf{P}}) = j(\mathbf{T}) - 1$. Continuing this process a finite number of times reduces the jump of the residual operator to zero.

■

Remark 3.6. *: We have already noted that if $\mathbf{T} \in \Phi_{\pm}(\mathbf{E})$ then $\lambda\mathbf{I} - \mathbf{T}$ is still semi-Fredholm near 0. By **Theorem 3.12** every semi-Fredholm operator is of Kato type and therefore, there exists a punctured open disc $\mathbb{D}_{\varepsilon}$ centered at 0 for which $\lambda\mathbf{I} - \mathbf{T}$ is semi-regular for all $\lambda \in \mathbb{D}_{\varepsilon}$. From **Theorem 3.11** we then conclude that if a semi-Fredholm operator has jump $j(\mathbf{T}) > 0$ then there is an open disc $\mathbb{D}_{\varepsilon}$ centered at 0 for which $j(\mathbf{T}) = 0$ for all $\lambda \in \mathbb{D}_{\varepsilon}\{0\}$.*

Let us begin by trying to enlarge the set $\Phi(\mathbf{E},\mathbf{F})$ of Fredholm operators to include unbounded ones, and going back to the concepts of the previous chapter (section 2.5) . We can attempt to define unbounded Fredholm operators. If you recall, in , we used the closed graph theorem (or its equivalent, the bounded inverse theorem) on a few occasions. Thus, it seems reasonable to define Fredholm operators in the following way: Let $\mathbf{E}$, $\mathbf{F}$ be Banach spaces. Then the set $\Phi(\mathbf{E},\mathbf{F})$ consists of linear operators from $\mathbf{E}$ to $\mathbf{F}$ such that

1. $\mathcal{D}(\mathbf{T})$ is dense in $\mathbf{E}$;
2. $\mathbf{T}$ is closed;
3. $\alpha(\mathbf{T}) < \infty$;
4. $\mathcal{R}(\mathbf{T})$ is closed in $\mathbf{F}$;
5. $\beta(\mathbf{T}) < \infty$.

And the most classes bounded Fredholm theorems are valid for this expanded set of unbounded Fredholm operators.

Example 3.5. *:we have that for function taking values in Banach space of dimension n, the derivative $(\mathbf{T}x)(t) = x'(t) : \mathbf{C}^{k}([0,1]) \longrightarrow \mathbf{C}^{k-1}([0,1])$ is a surjective Fredholm operator with index n, but imposing the boundary condition f(0)= f(1)=0 produces an injective Fredholm operator*

$$\left\{ f \in \mathbf{C}^{k}([0,1]) \mid f(0) = f(1) = 0 \right\} \xrightarrow{x'(t)} \mathbf{C}^{k-1}([0,1]),$$

with index $-n$.

Remark 3.7. *: shows that, in a Banach space, a Fredholm operator is **normally solvable**.*

3.2 Some operators related to Fredholm operators

In this section, we will define some of the operators that emerged from the emergence of Fredholm operators , or related to them.

Definition 3.5. *: A Weyl operator is a Fredholm operator with null index (equivalently, a semi-Fredholm operator with null index). Let*

$$\mathcal{W}(\mathbf{E}) = \left\{ \mathbf{T} \in \Phi(\mathbf{E}) \quad : ind(\mathbf{T}) = 0 \right\}.$$

Definition 3.6. *: A bounded operator* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is said to be upper semi-Weyl if* $\mathbf{T} \in \Phi_+(\mathbf{E})$ *and* $ind(\mathbf{T}) \leqslant 0$. $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is said to be lower semi-Weyl if* $\mathbf{T} \in \Phi_-(\mathbf{E})$ *and* $ind(\mathbf{T}) \geqslant 0$*. The set of all upper semi-Weyl operators will be denoted by* $\mathcal{W}_+(\mathbf{E})$*, while the set of all lower semi-Weyl operators will be denoted by* $\mathcal{W}_-(\mathbf{E})$*, and we write that*

$$\mathcal{W}_+(\mathbf{E}) = \left\{ \ \mathbf{T} \in \Phi_+(\mathbf{E}) \quad : ind(\mathbf{T}) \leqslant 0 \ \right\},$$

and

$$\mathcal{W}_-(\mathbf{E}) = \left\{ \ \mathbf{T} \in \Phi_-(\mathbf{E}) \quad : ind(\mathbf{T}) \geqslant 0 \ \right\}.$$

Hence

$$\mathcal{W}(\mathbf{E}) = \mathcal{W}_+(\mathbf{E}) \cap \mathcal{W}_-(\mathbf{E}).$$

Example 3.6. *: Let defined the following operator*

$$\mathbf{T} = \begin{pmatrix} U & 0 \\ 0 & U' \end{pmatrix} \quad : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2.$$

Where U *is the unilateral shift (* $U = \mathbf{T}_l$ *or* $U = \mathbf{T}_r$ *resp:* $U' = \mathbf{T}_r$ *or* $U' = \mathbf{T}_l$ *). Evidently* $\mathbf{T}$ *is Fredholm and* $ind(\mathbf{T}) = ind(U) + ind(U') = 0$*. Which says that* $\mathbf{T}$ *is weyl operator In the same time is upper semi-Weyl and lower semi-Weyl.*

Because weyl operators are also Fredholm operators then the most properties of section 3.1 are valid and we write this note

Notes 3.1. *:*

- ***a-*** $\mathbf{T}$ *is weyl operator if and only if* $\mathbf{T}'$ *is also weyl.*
- ***b-*** *If* $\boldsymbol{dim}\mathbf{E} < \infty$ *, then* $\mathcal{W}(\mathbf{E}) = \mathcal{B}(\mathbf{E})$*.*

c- *the Fredholm Alternative can be rephrased as:*

$$\mathbf{K} \in \mathcal{K}(\mathbf{E}) \;\; and \;\; \lambda \neq 0 \implies \lambda \mathbf{I} - \mathbf{K} \in \mathcal{W}(\mathbf{E}).$$

d- *every nonzero multiple of a Weyl operator is again a Weyl operator,*

$$\mathbf{T} \in \mathcal{W}(\mathbf{E}) \implies \lambda \mathbf{T} \in \mathcal{W}(\mathbf{E}) \;\; for \;\; every \;\; \lambda \neq 0.$$

e- *Every nonzero scalar operator is a Weyl operator. In fact, the product of two Weyl operators is again a Weyl operator,*

$$\mathbf{T}, \mathbf{S} \in \mathcal{W}(\mathbf{E}) \implies \mathbf{TS} \in \mathcal{W}(\mathbf{E}).$$

Thus integral powers of Weyl operators are Weyl operators

$$\mathbf{T} \in \mathcal{W}(\mathbf{E}) \implies \mathbf{T}^n \in \mathcal{W}(\mathbf{E}) \;\; for \;\; every \;\; n \in \mathbb{N}_0.$$

f- *Since* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *and for every compact* $\mathbf{K} \in \mathcal{K}(\mathbf{E})$,

$$\mathbf{T} \in \mathcal{W}(\mathbf{E}) \implies \mathbf{T} + \mathbf{K} \in \mathcal{W}(\mathbf{E}).$$

Example 3.7. *: Let* $\mathbf{E} = \ell^p$, $1 \leqslant p \leqslant \infty$, *and we consider the operator*

$$\mathbf{T}x = (0, x_3, x_2, x_5, x_4, x_7, x_6, ...).$$

We have $Ker\mathbf{T} = \{(x_1, 0, ...)\}$ *and* $\mathcal{R}(\mathbf{T}) = \{(0, x_1, x_2, x_3, x_4, ...)\}$ *and* $ind(\mathbf{T}) = 0$. *So* $\mathbf{T}$ *is Weyl operator. we know that the operator* $\mathbf{K}$ *given by*

$$\mathbf{K}x = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ...),$$

is compact, than the operator $\mathbf{T} + \mathbf{K}$ *is Weyl operator.*

Let $\Delta(\mathbf{T}) := \{n \in \mathbb{N} : m \geqslant n, m \in N \Rightarrow \mathcal{R}(\mathbf{T}^n) \cap Ker\mathbf{T} \subseteq \mathcal{R}(\mathbf{T}^m) \cap Ker\mathbf{T}\}$.

The degree of stable iteration is defined as $dis(\mathbf{T}) := inf \Delta(\mathbf{T})$ if $\Delta(\mathbf{T}) \neq \varnothing$, while $dis(\mathbf{T}) = \infty$ if $\Delta(\mathbf{T}) = \varnothing$.

Definition 3.7. *:* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is said to be* ***quasi-Fredholm*** *of degree* d *if there exists a* $d \in \mathbb{N}$ *such that:*

1. $dis(\mathbf{T}) = d$,
2. $\mathcal{R}(\mathbf{T}^n)$ *is a closed subspace of* $\mathbf{E}$ *for each* $n \geqslant d$,
3. $\mathcal{R}(\mathbf{T}) + Ker\mathbf{T}^d$ *is a closed subspace of* $\mathbf{E}$.

By $\mathcal{QF}(d)$ we denote the class of all quasi-Fredholm operators of degree d.

Theorem 3.13. *: If* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *then the following implication hold:*

$$\mathbf{T} \in \Phi_{\pm}(\mathbf{E}) \Longrightarrow \mathbf{T} \quad quasi - Fredholm.$$

▸**Proof:**
See [5] . Theorem 1.96. p 64.

■

Remark 3.8. *: If* $\mathbf{T} \in \mathcal{QF}(d)$ *if and only if* $\mathbf{T}' \in \mathcal{QF}(d)$.

Definition 3.8. *: An operator* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space, is said to be* ***B-Fredholm****, (respectively,* ***semi B-Fredholm****,* ***upper semi B-Fredholm****,* ***lower semiB-Fredholm****), if for some integer* $n \geqslant 0$ *the range* $\mathcal{R}(\mathbf{T}^n)$ *is closed and* $\mathbf{T}^n$ *is a* ***Fredholm*** *operator (respectively,* ***semi-Fredholm****,* ***upper semi-Fredholm*** *,* ***lower semi-Fredholm****).*

Example 3.8. *: It is easily seen that every nilpotent operator, as well as any idempotent bounded operator, is B-Fredholm. Therefore the class of B-Fredholm operators contains the class of Fredholm operators as a proper subclass.*

Definition 3.9. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *be semi B-Fredholm and let* $n \in \mathbb{N}$ *be such that* $\mathbf{T}^n$ *is a Fredholm operator. Then the index* $ind(\mathbf{T})$ *of* $\mathbf{T}$ *is defined as the index of* $\mathbf{T}^n$ *.*

The upper semi-Fredholm operators (respectively, the lower semi-Fredholm operators) are exactly the upper semi B-Fredholm operators (respectively, the lower semi B-Fredholm operators) for which we have $\alpha(\mathbf{T}) < \infty$ (respectively, $\beta(\mathbf{T}) < \infty$).

Theorem 3.14. *([5], **Theorem 1.114**): Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *. Then we have:*

1. $\mathbf{T}$ *is upper semi B-Fredholm and* $\alpha(\mathbf{T}) < \infty$ *if and only if* $\mathbf{T} \in \Phi_+(\mathbf{E})$*;*
2. $\mathbf{T}$ *is lower semi B-Fredholm and* $\beta(\mathbf{T}) < \infty$ *if and only if* $\mathbf{T} \in \Phi_-(\mathbf{E})$*.*

▸**Proof:**

(1). If $\mathbf{T}$ is upper semi B-Fredholm then there exists an $n \in \mathbb{N}$ such that $\mathcal{R}(\mathbf{T}^n)$ is closed and $\mathbf{T}^n$ is upper semi-Fredholm. Since $\alpha(\mathbf{T}) < \infty$ then $\alpha(\mathbf{T}^n) < \infty$ hence $\mathbf{T}^n$ is upper semi-Fredholm. From the classical Fredholm theory then $\mathbf{T}$ is also upper semi-Fredholm. The converse is obvious.

Part **(2)** may be proved in a similar way.

■

Corollary 3.3. *: If* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$ *is injective and upper semi B-Fredholm then* $\mathbf{T}$ *is bounded below.*

Remark 3.9. *: Every semi B-Fredholm operator is quasi- Fredholm.*

Definition 3.10. *: A **Browder** operator is a Fredholm operator with finite ascent and finite descent. Let* $\mathfrak{B}(\mathbf{E})$ *denote the class of all Browder operators from* $\mathcal{B}(\mathbf{E})$*:*

$$\mathfrak{B}(\mathbf{E}) = \left\{ \mathbf{T} \in \Phi(\mathbf{E}) : \ asc(\mathbf{T}) < \infty \ \ and \ \ dsc(\mathbf{T}) < \infty \right\};$$

*equivalently, according to **Theorem** 1.5:*

$$\mathfrak{B}(\mathbf{E}) = \left\{ \mathbf{T} \in \Phi(\mathbf{E}) : \ asc(\mathbf{T}) = dsc(\mathbf{T}) < \infty \right\};$$

We say that also an operator $\mathbf{T}$ is upper semi-Browder if it is upper semi-Fredholm and has finite ascent , denoted by

$$\mathfrak{B}_+(\mathbf{E}) = \left\{ \mathbf{T} \in \Phi_+(\mathbf{E}) : \ asc(\mathbf{T}) < \infty \right\}.$$

Similarly, $\mathbf{T}$ is lower semi-Browder if it is lower semi-Fredholm and has finite descent , denoted by

$$\mathfrak{B}_-(\mathbf{E}) = \left\{ \mathbf{T} \in \Phi_-(\mathbf{E}) : \ dsc(\mathbf{T}) < \infty \right\}.$$

Remark 3.10. *: by **Theorem** 3.1 and **Proposition** 3.1 , we can conclude that :*

$$\mathbf{T} \ \textit{is upper semi-Browder} \ \Leftrightarrow \ \mathbf{T}' \ \textit{is lower semi-Browder;}$$

$$\mathbf{T} \text{ is lower semi-Browder } \Leftrightarrow \mathbf{T}' \text{ is upper semi-Browder;}$$

$$\mathbf{T} \text{ is Browder } \Leftrightarrow \mathbf{T}' \text{ is Browder.}$$

Example 3.9. : *By* ***Example 1.3*** *and* ***Example 1.4*** *, the* ***right shift operator*** *is upper-semi-Browder and the* ***left shift operator*** *is lower-semi-Browder, but they are not Browder.*

Proposition 3.5. *([19],* ***Proposition 8****): Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*. Then:*

1. $\mathbf{T}$ *is upper semi-Browder if and only if* $\mathcal{R}(\mathbf{T})$ *is closed and* $\mathbf{dim}\mathcal{N}^{\infty}(\mathbf{T}) < \infty$*;*
2. $\mathbf{T}$ *is lower semi-Browder if and only if* $\mathbf{codim}\ \mathcal{R}^{\infty}(\mathbf{T}) < \infty$*;*
3. $\mathbf{T}$ *is Browderif and only if* $\mathbf{dim}\mathcal{N}^{\infty}(\mathbf{T}) < \infty$ *and* $\mathbf{codim}\ \mathcal{R}^{\infty}(\mathbf{T}) < \infty$*.*

▸**Proof:**

(1). If $\mathbf{T}$ is upper semi-Browder, then $\mathcal{R}(\mathbf{T})$ is closed and $k = \alpha(\mathbf{T}) < \infty$. Since $\mathbf{T}^k$ is upper semi-Fredholm, we have $\mathbf{dim}\mathcal{N}^{\infty}(\mathbf{T}) = \mathbf{dim}Ker\mathbf{T}^k < \infty$.

Conversely, if $\mathcal{R}(\mathbf{T})$ is closed and $\mathbf{dim}\mathcal{N}^{\infty}(\mathbf{T}) < \infty$, then $\mathbf{T}$ is upper semi-Fredholm. Further, $Ker\mathbf{T} \subset Ker\mathbf{T}^2 \subset ... \subset \mathcal{N}^{\infty}(\mathbf{T})$, and so there exists k with $Ker\mathbf{T}^{k+1} = Ker\mathbf{T}^k$. Hence $\alpha(\mathbf{T}) < \infty$.

The remaining statements can be proved similarly.

■

Lemma 3.6. *([19],* ***Lemma 9****): Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*. be upper semi-Browder and Kato. Then* $\mathbf{T}$ *is bounded below. If* $\mathbf{T}$ *is lower semi-Browder and Kato, then* $\mathbf{T}$ *is onto.*

▸**Proof:**

Suppose that there exists a non-zero vector $x_0 \in Ker\mathbf{T}$. Since $Ker\mathbf{T} \subset \mathcal{R}(\mathbf{T})$, there exists $x_1 \in \mathbf{E}$ such that $\mathbf{T}x_1 = x_0$. Further, $x_1 \in Ker\mathbf{T}^2 \subset \mathcal{R}(\mathbf{T})$ and we can construct inductively vectors $x_i \in \mathbf{E}$ satisfying $\mathbf{T}x_i = x_{i-1}(i \geqslant 1)$. It is easy to show that the vectors x_i are linearly independent and $x_i \in \mathcal{N}^{\infty}$, a contradiction with **Proposition**3.5.

The second statement can be proved by duality.

■

The following theorem shows that the relation between quasi-nilpotent part, analytic core , semi Fredholm and Browder operators, With λ_0 is isolated point of $\sigma(\mathbf{T})$.

Theorem 3.15. *: Let λ_0 be an isolated point of $\sigma(\mathbf{T})$. Then the following assertions are equivalent:*

1. $\lambda_0\mathbf{I}-\mathbf{T}\in\Phi_{\pm}(\mathbf{E})$;
2. $\lambda_0\mathbf{I}-\mathbf{T}\in\mathfrak{B}(\mathbf{E})$;
3. $\mathrm{H}_0(\lambda_0\mathbf{I}-\mathbf{T})$ *is finite-dimensional;*
4. $\mathrm{K}(\lambda_0\mathbf{I}-\mathbf{T})$ *is finite-codimensional.*

▸**Proof:**
See [6], Theorem 2.66 . p 86.

■

Proposition 3.6. *([34], **Proposition 3.7.1**): Let $\mathbf{T}\in\mathcal{B}(\mathbf{E})$ be an arbitrary operator on a Banach space $\mathbf{E}$, and let $\mathcal{M}\subseteq\mathbf{E}$ be a $\mathbf{T}$-invariant closed linear subspace of finite codimension in $\mathbf{E}$. Then $\mathbf{T}$ is a Fredholm operator on $\mathbf{E}$ if and only if $\mathbf{T}_{\setminus\mathcal{M}}$ is a Fredholm operator on $\mathcal{M}$. Moreover, in this case, $ind(\mathbf{T})=ind(\mathbf{T}_{\setminus\mathcal{M}})$.*

▸**Proof:**
Choose a projection $P\in\mathcal{B}(\mathbf{E})$ with range $\mathcal{M}$, and observe that $\mathbf{I}-P$ projects onto the finite-dimensional space $KerP$. Since $\mathbf{T}(\mathbf{I}-P)$ is a finite-rank operator, and $\mathbf{T}=\mathbf{T}P+\mathbf{T}(\mathbf{I}-P)$, it follows that $\mathbf{T}$ is a Fredholm operator precisely when $\mathbf{T}P$ is Fredholm, and that, in this case, $ind(\mathbf{T})=ind(\mathbf{T}P)$. Moreover, it is readily seen that

$$Ker(\mathbf{T}P)=Ker(\mathbf{T}_{\setminus\mathcal{M}})\oplus KerP \quad and \quad \mathcal{R}(\mathbf{T})=(\mathbf{T}_{\setminus\mathcal{M}})(\mathcal{M}).$$

Consequently, $\mathbf{T}P$ and $\mathbf{T}_{\setminus\mathcal{M}}$ are simultaneously Fredholm operators, and, when they are, they will have the same index.

■

We saw in the previous chapter that **Kato type** operators admit a generalised Kato decomposition and Kato proved that a bounded Fredholm operator is of Kato type, then semi-Fredholm operators also admit a generalised Kato decomposition. We saw also that Weyl operators are Fredholm with zero index then it have a generalised Kato decomposition .

Browder operator admit a generalised Kato decomposition, if $\mathbf{T}$ is Browder operator, then $\mathbf{T}=\mathbf{T}_1\oplus\mathbf{T}_2$ with $\mathbf{T}_1$ is invertible and $\mathbf{T}_2$ is nilpotent operator.

Quasi-Fredholm operator. In Hilbert spaces, this class coincide with Kato type operators, when there exist a pair of closed subspaces (M,N) of $\mathcal{H}$ such that $\mathcal{H} = M \oplus N$ and $\mathbf{T}(M) \subset M$ and $\mathbf{T}_{\backslash M}$ is Kato operator, $\mathbf{T}(N) \subset N$ and $\mathbf{T}_{\backslash N}$ is nilpotent operator. the pair (M,N) is said to be Kato decomposition of $\mathbf{T}$. the same decomposition exists also for quasi-Fredholm operators on Banach spaces under the additional assumption that the subspaces $\mathcal{R}(\mathbf{T}^d) \cap Ker\mathbf{T}$ and $\mathcal{R}(\mathbf{T}) + Ker\mathbf{T}^d$ are complemented .

3.3 Essential Fredholm , Browder and Weyl spectrum

We assume in this section that $\mathbf{E}$ is an infinite-dimensional Banach space (for finite-dimensional spaces all results would be trivial). In Section 3.1 we showed that the sets of all Fredholm, upper (lower) and left (right) semi-Fredholm operators in $\mathbf{E}$ form regularities.

An element a in a unital algebra $\mathcal{A}$ left invertible if there is an element a_ℓ in $\mathcal{A}$ (a left inverse of a) such that $a_\ell a = 1$ where 1 stands for the identity in $\mathcal{A}$ and it is right invertible if there is an element a_r in $\mathcal{A}$ (a right inverse of a) such that $a\, a_r = 1$. An element a in $\mathcal{A}$ is invertible if there is an element $a - 1$ in $\mathcal{A}$ (the inverse of a) such that $a^{-1}a = aa^{-1} = 1$. Thus a in $\mathcal{A}$ is invertible if and only if it has a left inverse a_ℓ in $\mathcal{A}$ and a right inverse a_r in $\mathcal{A}$, which coincide with its inverse a^{-1} in $\mathcal{A}$ (since $a_r = a_\ell a\, a_r = a_\ell$)
Recall that the corresponding spectra - the essential spectrum, essential approximate point spectrum, essential surjective spectrum and left (right) essential spectrum , – were defined by

$$\sigma_{ef}(\mathbf{T}) = \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \Phi(\mathbf{E})\Big\},$$

$$\sigma_{uf}(\mathbf{T}) = \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \Phi_+(\mathbf{E})\Big\},$$

$$\sigma_{lf}(\mathbf{T}) = \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \Phi_-(\mathbf{E})\Big\},$$

$$\sigma_{le}(\mathbf{T}) = \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \;\; is \;\; not \;\; left \;\; essentially \;\; invertible\Big\},$$

$$\sigma_{re}(\mathbf{T}) = \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \;\; is \;\; not \;\; right \;\; essentially \;\; invertible\Big\}.$$

the Weyl spectrum defined by:

$$\sigma_w(\mathbf{T}) := \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \mathcal{W}(\mathbf{E})\Big\},$$

the upper semi-Weyl spectrum defined by

$$\sigma_{uw}(\mathbf{T}) := \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \mathcal{W}_+(\mathbf{E})\Big\},$$

and the lower sem-Weyl spectrum defined by

$$\sigma_{lw}(\mathbf{T}) := \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \mathcal{W}_-(\mathbf{E})\Big\}.$$

Of course, also the classes of Browder operators generate spectra.
The Browder spectrum defined by

$$\sigma_b(\mathbf{T}) := \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \mathfrak{B}(\mathbf{E})\Big\},$$

the upper semi-Browder spectrum defined by

$$\sigma_{ub}(\mathbf{T}) := \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \notin \mathfrak{B}_+\mathbf{E})\Big\},$$

and the lower semi-Browder spectrum defined by

$$\sigma_{lb}(\mathbf{T}) := \Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \ mathbf T \notin \mathfrak{B}_-(\mathbf{E})\Big\}.$$

Proposition 3.7. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*. Then* $\sigma_{uf}(\mathbf{T}) \subset \sigma_{le}(\mathbf{T})$, $\sigma_{lf}(\mathbf{T}) \subset \sigma_{re}(\mathbf{T})$, $\sigma_{ef}(\mathbf{T}) = \sigma_{le}(\mathbf{T}) \cup \sigma_{re}(\mathbf{T}) = \sigma_{uf}(\mathbf{T}) \cup \sigma_{lf}(\mathbf{T})$.

▸**Proof:**
All statements with the exception of the last one are trivial.

■

It is easy to find an example of operator for which $\sigma_{uf}(\mathbf{T}) \neq \sigma_{lf}(\mathbf{T})$

Example 3.10. *: Let* $\mathbf{T}$ *be defined on* ℓ^2 *by*

$$\mathbf{T}x = (x_1, 0, x_2, 0, x_3, 0, \ldots),$$

obviously, $\mathbf{T}$ *is injective with closed range of infinite-codimension, so that* $0 \in \sigma_{lf}(\mathbf{T})$ *but* $0 \notin \sigma_{uf}(\mathbf{T})$.

Definition 3.11. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$*. The essential spectral radius of* $\mathbf{T}$ *is defined by* $r_e(\mathbf{T}) = max|\ \lambda\ | : \lambda \in \sigma_{ef}(\mathbf{T})$ *and the essential norm by* $\|\mathbf{T}\|_e = inf\|\mathbf{T} + \mathbf{T}\| : \mathbf{T} \in \mathcal{K}(\mathbf{E})$.

Clearly, $\|\mathbf{T}\|_e$ is the norm of the class $\mathbf{T}+\mathcal{K}(\mathbf{E})$ in the Calkin algebra $\mathcal{B}(\mathbf{E})/\mathcal{K}(\mathbf{E})$, and $\sigma_{ef}(\mathbf{T})$ is the spectrum of the class $\mathbf{T}+\mathcal{K}(\mathbf{E})$ in this algebra.

Proposition 3.8. *: Let* $\mathbf{T}\in\mathcal{B}(\mathbf{E})$*. Then we have that:*

$$\sigma_{ef}(\mathbf{T})\subset\sigma(\mathbf{T}).$$

▸**Proof:**

If $\mathbf{T}\in\mathcal{B}(\mathbf{E})$ and $\lambda\in\mathbb{K}$, suppose that $\lambda\notin\sigma(\mathbf{T})$, then there exist $\mathbf{S}\in\mathcal{B}(\mathbf{E})$ such that

$$\mathbf{S}(\lambda\mathbf{I}-\mathbf{T})=(\lambda\mathbf{I}-\mathbf{T})\mathbf{S}.$$

Hence

$$[\mathbf{S}].[\lambda\mathbf{I}-\mathbf{T}]=[\mathbf{S}(\lambda\mathbf{I}-\mathbf{T})]=[\mathbf{I}]=\mathbf{I}$$

and

$$[\lambda\mathbf{I}-\mathbf{T}].[\mathbf{S}]=[(\lambda\mathbf{I}-\mathbf{T})\mathbf{S}]=[\mathbf{I}]=\mathbf{I}.$$

Then we have

$$[\mathbf{S}].(\lambda\mathbf{I}-[\mathbf{T}])=(\lambda\mathbf{I}-[\mathbf{T}]).[\mathbf{S}]=\mathbf{I}.$$

Thus, we proved that $\lambda\notin\sigma_{ef}(\mathbf{T})$, so the proof is complete and $\sigma_{ef}(\mathbf{T})\subset\sigma(\mathbf{T})$.

■

Remark 3.11. *: Clearly all spectra* σ_{ef}*,* σ_{uf}*,* σ_{lf}*,* σ_{le} *and* σ_{re} *are invariant with respect to compact perturbations. Thus*

$$\sigma_{ef}(\mathbf{T})\subset\bigcap\Big\{\sigma(\mathbf{T}+\mathbf{K}):\mathbf{K}\in\mathcal{K}(\mathbf{E})\Big\};$$

$$\sigma_{uf}(\mathbf{T})\subset\bigcap\Big\{\sigma_{uf}(\mathbf{T}+\mathbf{K}):\mathbf{K}\in\mathcal{K}(\mathbf{E})\Big\};$$

$$\sigma_{lf}(\mathbf{T})\subset\bigcap\Big\{\sigma_{lf}(\mathbf{T}+\mathbf{K}):\mathbf{K}\in\mathcal{K}(\mathbf{E})\Big\};$$

$$\sigma_{le}(\mathbf{T})\subset\bigcap\Big\{\sigma_{le}(\mathbf{T}+\mathbf{K}):\mathbf{K}\in\mathcal{K}(\mathbf{E})\Big\};$$

$$\sigma_{re}(\mathbf{T})\subset\bigcap\Big\{\sigma_{re}(\mathbf{T}+\mathbf{K}):\mathbf{K}\in\mathcal{K}(\mathbf{E})\Big\}.$$

Example 3.11. *: Going back to* ***Example*****3.7***, we have* $\sigma_w(\mathbf{T}+\mathbf{K})=\sigma_w(\mathbf{T})$.

The results of the following theorem easily follows from duality.

Theorem 3.16. *([6], **Theorem 4.1**): Let $\mathbf{T} \in \mathcal{B}(\mathbf{E})$. Then we have:*

1. $\sigma_w(\mathbf{T}) = \sigma_w(\mathbf{T}')$,
2. $\sigma_{uw}(\mathbf{T}) = \sigma_{lw}(\mathbf{T}')$, *and* $\sigma_{lw}(\mathbf{T}) = \sigma_{uw}(\mathbf{T}')$.

Moreover,

$$\sigma_w(\mathbf{T}) = \sigma_{uw}(\mathbf{T}) \cup \sigma_{lw}(\mathbf{T}).$$

Theorem 3.17. *([6], **Theorem 4.3**): Let $\mathbf{T} \in \mathcal{B}(\mathbf{E})$. Then we have:*

1. $\sigma_b(\mathbf{T}) = \sigma_b(\mathbf{T}')$,
2. $\sigma_{ub}(\mathbf{T}) = \sigma_{lb}(\mathbf{T}')$, *and* $\sigma_{lb}(\mathbf{T}) = \sigma_{ub}(\mathbf{T}')$.

Moreover,

$$\sigma_b(\mathbf{T}) = \sigma_{ub}(\mathbf{T}) \cup \sigma_{lb}(\mathbf{T}).$$

Now let us defined some of the spectre classes, accompanying the various concepts studied above, through this note.

Notes 3.2. *: Let* $\mathbf{T} \in \mathcal{B}(\mathbf{E})$, $\mathbf{E}$ *a Banach space:*

a- *The Kato spectrum :* $\sigma_k(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ Kato\Big\}$,

b- *The essentially Kato spectrum :* $\sigma_{ek}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ essentially \ Kato\Big\}$,

c- *The Kato-type spectrum :* $\sigma_{kt}(\mathbf{T})\Big\{\lambda \in \mathbb{C} : \lambda\mathbf{I} - \mathbf{T} \ is \ not \ Kato \ type\Big\}$,

d- *The generalaised Kato spectrum :*
$\sigma_{gk}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ does \ not \ admit \ a \ generalized \ Kato \ decomposition\Big\}$,

e- *The Saphar spectrum :* $\sigma_{sa}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ Saphar\Big\}$,

f- *The essentially Saphar spectrum :*$\sigma_{esa}(\mathbf{T})\Big\{\lambda \in \mathbb{C} : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ essentially \ Saphar\Big\}$,

g- *The descent spectrum :* $\sigma_d(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ dsc(\lambda\mathbf{I} - \mathbf{T}) = \infty\Big\}$,

h- *The ascent spectrum :* $\sigma_a(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ asc(\lambda\mathbf{I} - \mathbf{T}) = \infty\Big\}$,

i- *The approximate point spectrum :* $\sigma_{ap}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ bounded \ below\Big\}$, *j-* *The surjectivity spectrum :* $\sigma_{srj}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ surjective,\Big\}.$

k- *The upper B-Fredholm spectrum :* $\sigma_{srj}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ upper \ B-Fredholm,\Big\}.$

l- *The lower B-Fredholm spectrum :* $\sigma_{srj}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ lower \ B-Fredholm,\Big\}.$

m- *The B-Fredholm spectrum :* $\sigma_{srj}(\mathbf{T})\Big\{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{I} - \mathbf{T} \ is \ not \ B-Fredholm,\Big\}.$

:

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CONCLUSION

The main thrust of this thesis is in the spirit of the Fredholm theory and operator theory; its aime to give a survey of various characteristic perturbation properties of different notions of Fredholm and semi Fredholm operators , we also provided a detailed study of the Kernel , the range , the nullity , the deficiency , ascent and descent of an operator in order to build a coherent and integrated work. And we have seen that we have to study the theory of operators with closed range , and the most classes of operators that enter into the same context. We give also a survey of various characteristic of different notions of essential spectrum of different class operator (Fredholm, semi-Fredholm, weyl, Browder and quasi-Fredholm ...ect).