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MASTER DISSERTATION

Left-invertible semigroups and exact observability on Hilbert spaces.

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Dedication

I dedicate this work :

To my dear mother Farida ALIAT and my dear father Ali who have never ceased to support me and encourage me during my years of study.

May they find here the testimony of my deep gratitude and the knowledge.

To my brother Oussama, my sisters Hanane and Chaima, my grandmothers and my family for giving me love and care.

To all those who helped me - directly or indirectly - and those who shared with me the emotional moments during the realization of this work and who warmly supported and encouraged me throughout my journey.

To all my friends who have always encouraged me, and to whom I wish more success.

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♡

Abstract

For strongly continuous semigroups on a Hilbert space, we present a short proof of the fact that the left inverse of a left-invertible semigroup can be chosen to be a C_0-semigroup as well. Furthermore, we show that this semigroup need not to be unique. Moreover, we concentrate to show the relation between left invertibility of C_0-semigroups and exact observability, and also we discuss the characteristic property of the left invertible semigroups on general Banach spaces and admissibility of the observation operators for such semigroups.

Keywords : Strongly continuous semigroup, Left inverse, Exact observability, Admissibile observation operator.

Résumé

Pour les semigroupes fortement continus sur un espace de Hilbert, nous présentons une courte preuve du fait que l'inverse gauche d'un semigroupe inversible à gauche peut également être choisi pour être un C_0-semigroupe. De plus, nous montrons que ce semigroupe n'est pas unique. Ensuite, nous donnons la relation entre l'inversibilité à gauche des C_0-semigroupes et l'observabilité exacte, et aussi nous discutons de la propriété caractéristique des semigroupes inversibles à gauche sur les espaces de Banach généraux et de l'admissibilité des opérateurs d'observation pour de tels semigroupes.

Mots clé : Semigroupe fortement continu, Inverse à gauche, Observabilité exacte, Opérateur d'observation admissible.

Contents

List of abbreviations

$\mathbb{R}$	:	The set of real numbers .
$\mathbb{C}$	:	The set of complex numbers .
$\mathbb{N}$	:	The set of natural numbers .
$H^1[a,b]$	:	The Sobolev space of $L^2[a,b]$.
$L^2[a,b]$	:	The space of square integrable functions on $[a,b]$.
X, Z, Y	:	Complex Banach or Hilbert spaces of infinite-dimensional.
$\mathcal{B}(Z,Y)$	:	The space of bounded linear operators from Z to Y.
$\mathcal{B}(Z)$	:	The space of bounded linear operators from Z to Z.
$\mathcal{D}(A)$	:	Domaine of the operator A.
X^*	:	The dual of the space X.
X^{**}	:	The bidual of the space X.
A^*	:	Adjoint of the operator A on the Banach or Hilbert space.
A^{-1}	:	The inverse of operator A.
$Im\ A$	:	The image of the operator A.
$ker\ A$	:	The kernel of the operator A.
$Gr(A)$	:	The graph of the operator A.
$\sigma(A)$	:	The spectrum of A.
$\sigma_p(A)$	:	The point spectrum of A.
$\sigma_c(A)$	:	The continuous spectrum of A.
$\sigma_r(A)$	:	The residual spectrum of A.
$\rho(A)$	:	The resolvent of A.
$\langle .,. \rangle$	:	The inner product in Z.
$\|A\|$	:	The norm of A.

Introduction

The purpose of this study is to develop a research paper of HANS ZWART entitled ”Left-invertible semigroup on Hilbert spaces” published in the Journal of Evolution Equation, 13(2013) 335-342.
The aim of this work is to study left invertible semigroups and its relation with exact observability. In [6], Louis and Wexler showed that if a strongly continuous semigroup on a Hilbert space is left invertible for one or equivalently all positive time instants, then there exists a left inverse which is also a strongly continuous semigroup. Their proof uses optimal control and Riccati equations. The present work uses Lyapunov equations. Furthermore, using this Lyapunov equation, the author showed that any left-invertible semigroup is a bounded perturbation of an isometric semigroup, the results obtained in this work complement those found in [4, 11] and [12], who mainly concentrate on the relation between left invertibility and exact observability.
The thesis is organized in three chapters.

Chapter 1 is a reminder on essential notions, first we present some basic and important properties of linear bounded and unbounded operators, second we introduce basic concepts and properties of semigroup. Finally we recall some notions from control theory.

In Chapter 2, We present some characteristic properties of the left invertible semigroups on general Banach spaces and define the concept of admissibility of observation operators for such semigroups.

In Chapter 3, We first establish necessary and sufficient conditions for a C_0-semigroup on Hilbert spaces to be left invertible. Then we present a proof based on Lyapunov equation of the fact that the left inverse of a left-invertible semigroup can be chosen to be a C_0-semigroup as well. Finally, we show that any left-invertible semigroup is a bounded perturbation of an isometric semigroup.

Chapter 1

Preliminaries

In this introductory chapter, we will introduce some basic concepts and well-known results that facilitate the understanding of this work, in particular we recall

1. Bounded linear operators on Banach and Hilbert spaces.

2. Unbounded linear operators on Hilbert spaces.

3. Semigroups of linear operators.

4. Notions from control theory.

1.1 Bounded linear operators on a Banach and Hilbert spaces

Let Z and Y be two complex Banach spaces, and $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1.1 (Linear Operators)
We call linear operator any application $A : Z \longrightarrow Y$ which satisfies

1. $\forall x, y \in Z,\ A(x+y) = Ax + Ay.$

2. $\forall \lambda \in \mathbb{K},\ \forall x \in Z,\ A(\lambda x) = \lambda A(x).$

Definition 1.1.2 (Bounded operators)
We say that a linear operator $A : Z \longrightarrow Y$ is bounded if there exists a constant $M \in \mathbb{R}_+^*$ such that

$$\|Ax\| \leq M \|x\|,\ \forall x \in Z.$$

if A is a bounded linear operator, then its norm is defined by

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The set of bounded linear operators defined from Z to Y is a Banach space denoted by $\mathcal{B}(Z,Y)$, if $Z=Y$ we denote $\mathcal{B}(Z)$ instead of $\mathcal{B}(Z,Z)$.

Proposition 1.1.1
Let $A,\ B \in \mathcal{B}(Z)$ and $\lambda \in \mathbb{K}$, then $A+B$, λA and AB are also bounded operators. Moreover

$$\|AB\| \leq \|A\|\ \|B\|.$$

Definition 1.1.3 (Inverse of an operator)
Let $A \in \mathcal{B}(Z)$, we say that A is invertible if there exists an operator $B \in \mathcal{B}(Z)$, which satisfies

$$AB = BA = I.$$

B is said to be the inverse of A and we denote it by $B = A^{-1}$.

Definition 1.1.4 (Left and Right inverse of an operator)
Let $A \in \mathcal{B}(Z)$.

1. *We say that A is left invertible if there exists an operator $B \in \mathcal{B}(Z)$ such that*

$$BA = I.$$

2. *We say that A is right invertible if there exists an operator $B \in \mathcal{B}(Z)$ such that*

$$AB = I.$$

Theorem 1.1.1 (Banach isomorphism)
Let $T : Z \longrightarrow Y$ be a linear bounded and bejective operator, then $T^{-1} \in \mathcal{B}(Y,Z)$.

Definition 1.1.5 (The resolvent set of an operator)
Let $A \in \mathcal{B}(Z)$, we say that $\lambda \in \mathbb{C}$ belongs to the resolvent set of A if $A-\lambda I$ is a bijection from Z into Z and $(A-\lambda I)^{-1} \in \mathcal{B}(Z)$. The resolvent set of A is denoted by $\rho(A)$.

$$\rho(A) = \{\lambda \in \mathbb{C} : A-\lambda I \ \ \textit{invertible}\ \}.$$

Definition 1.1.6 (The spectrum of an operator)
Let $A \in \mathcal{B}(Z)$. The spectrum of A denoted by $\sigma(A)$ is the complement in $\mathbb{C}$ of $\rho(A)$.

$$\sigma(A) = \{\lambda \in \mathbb{C} : A-\lambda I \ \ \textit{non invertible}\ \}.$$

Definition 1.1.7
The point spectrum of A is the set of eigenvalues of A, denoted by $\sigma_p(A)$, then

$$\sigma_p(A) = \{\lambda \in \sigma(A) : A - \lambda I \ \ \textit{non injective}\ \}.$$

The continuous spectrum of A denoted by $\sigma_c(A)$ is the set

$$\sigma_c(A) = \Big\{\lambda \in \sigma(A) : A - \lambda I \ \ \textit{injective} \ \ \textit{and} \ \ Im(A - \lambda I) \neq \overline{Im(A - \lambda I)} = Z\Big\}.$$

The residual spectrum of A denoted by $\sigma_r(A)$ is the set

$$\sigma_r(A) = \Big\{\lambda \in \sigma(A) : A - \lambda I \ \ \textit{injective} \ \ \textit{and} \ \ \overline{Im(A - \lambda I)} \neq Z\Big\}.$$

Remark 1.1.1

1. *The spectrum $\sigma(A)$ is the disjoint union of three sets*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

2. *If the dimension of the space Z is finite, then $\sigma_c(A) = \sigma_r(A) = \emptyset$. Hence $\sigma(A) = \sigma_p(A)$.*

3. *If the dimension of the space Z is infinite, then $\sigma_p(A)$ can be empty.*

Proposition 1.1.2
Let $A \in \mathcal{B}(Z)$, then $\sigma(A)$ is a closed and bounded nonempty subset of $\mathbb{C}$. Moreover,

$$\sigma(A) \subset B(0, \|A\|).$$

Where $B(0, \|A\|)$ is the closed disc of $\mathbb{C}$ with center zero and radius $\|A\|$.

1.1.1 Dual of a normed spaces

Definition 1.1.8
Let Z be a normed space over the field $\mathbb{K}$, The space $\mathcal{B}(Z, \mathbb{K})$ of continous linear functions of Z in the field $\mathbb{K}$ is called the topological dual of Z, denoted by Z^*. Similarly, we denote by Z^{**} the dual of Z^*, which we call bidual of Z. We define a duality between Z and Z^* by $\langle x, f\rangle = f(x) \in \mathbb{C}$, and we have

$$\|f\|_{Z^*} = \sup_{\|x\| \leq 1} |\langle x, f\rangle| = \|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

Theorem 1.1.2
Let Z and Y be two Banach spaces, for all $A \in \mathcal{B}(Z, Y)$, there exists a unique $A^* \in \mathcal{B}(Y^*, Z^*)$, such that

$$(A^* f)(x) = f(Ax), \ \forall x \in Z, \ \forall f \in Y^*.$$

Moreover, $\|A\|_{\mathcal{B}(Z,Y)} = \|A^*\|_{\mathcal{B}(Y^*,Z^*)}$.

Proposition 1.1.3
Let Z be a Banach space, for all $A,B \in \mathcal{B}(Z)$, and for all $\alpha \in \mathbb{K}$, we have

1. $(A+B)^* = A^* + B^*$, *and* $(\alpha A)^* = \alpha A^*$.
2. $(AB)^* = B^* A^*$.
3. *If* A^{-1} *exists and* $A^{-1} \in \mathcal{B}(Z)$, *then* $(A^*)^{-1}$ *exists and* $(A^*)^{-1} \in \mathcal{B}(Z^*)$, *and* $(A^*)^{-1} = (A^{-1})^*$.

Theorem 1.1.3

1. *Let Z be a normed space and Y a Banach space, then $(\mathcal{B}(Z,Y), \|.\|)$ is a Banach space.*
2. *The topolgical dual of any normed space Z is a Banach space.*

1.1.2 Adjoint of a bounded opeartors on a Hilbert spaces

In this part we suppose that Z and Y are Hilbert spaces on $\mathbb{K}$.

Theorem 1.1.4 (Riesz frechet theorem)
Let $f \in Z^*$ be a linear form, then there exists a unique vector y such that

$$f(x) = \langle x, y \rangle \text{ , for all } x \in Z \text{ and } \| f \|_{Z^*} = \| y \|_Z .$$

Definition 1.1.9
Let $A \in \mathcal{B}(Z,Y)$, then there exists a unique operator $A^* \in \mathcal{B}(Y,Z)$ such that

$$\forall x \in Z, \forall y \in Y : \langle Ax, y \rangle = \langle x, A^* y \rangle.$$

The operator A^* is called the adjoint of A.

Example 1.1.1
Let $Z = L^2([a,b])$ and let $A : Z \to Z$ defined by

$$(Af)(s) = \int_a^b k(s,t) f(t) dt, \text{ with } k \in L^2([a,b] \times [a,b]),$$

we shall compute the adjoint of A. $\forall f, g \in Z$

$$\begin{aligned}
\langle Af, g \rangle_Z &= \int_a^b Af(x) \cdot \overline{g(x)} dx = \int_a^b \int_a^b k(x,t) f(t) dt \cdot \overline{g(x)} dx \\
&= \int_a^b f(t) \int_a^b k(x,t) \cdot \overline{g(x)} dx dt \\
&= \int_a^b f(t) \int_a^b \overline{\overline{k(x,t)} g(x)} dx dt \\
&= \langle f, A^*(g) \rangle .
\end{aligned}$$

Thus, $(Af)(s) = \int_a^b \overline{k(s,t)} g(x) dt$

Proposition 1.1.4
Let $A, B \in \mathcal{B}(Z)$, then we have

1. $(A+B)^* = A^* + B^*$, *and* $(\alpha A)^* = \overline{\alpha} A^*$.

2. $(AB)^* = B^* A^*$.

3. *If* A^{-1} *exists and* $A^{-1} \in \mathcal{B}(Z)$, *then* $(A^*)^{-1}$ *exists and* $(A^*)^{-1} \in \mathcal{B}(Z)$, *and* $(A^*)^{-1} = (A^{-1})^*$.

4. $\| A \| = \| A^* \|$

Definition 1.1.10
Let $A \in \mathcal{B}(Z)$, then

1. *A is said to be self adjoint if $A = A^*$.*

2. *A is said to be positive if for all $x \in Z$, $\langle Ax, x \rangle \geq 0$ and we denote the postive operator by $A \geq 0$. We write $A \geq B$ if $A - B$ is positive.*

3. *A is said to be isometric if $\|Ax\| = \|x\|$, for all $x \in Z$.*

4. *A is said to be unitary if $A^* A = A A^* = I$.*

Remark 1.1.2
The condition $\langle Ax, x \rangle \geq 0$ automatically implies that A is self adjoint, since in this case A is self adjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$.

1.2 Unbounded linear operators on Hilbert spaces

We suppose in this section that Z and Y are Hilbert spaces

Definition 1.2.1 (The Domain of an operator)
A linear operator A from Z to Y is a linear map defined on a vector subspace $\mathcal{D}(A)$ of Z called domain of A such that

$$\mathcal{D}(A) = \{x \in Z, \quad Ax \in Y\}.$$

Definition 1.2.2
- *The graph of A is the vector subspace of $Z \times Y$ denoted by $Gr(A)$ and defined by*

$$Gr(A) = \{(u, Au);\ u \in \mathcal{D}(A)\}.$$

- *The kernel of A is the subspace of Z denoted $ker(A)$ and defined by*

$$ker(A) = \{u \in \mathcal{D}(A);\ Au = 0\},$$

and the image of A is the subspace of Y denoted $Im(A)$ and defined by

$$Im(A) = \{Ax;\ x \in \mathcal{D}(A)\},$$

- *We say that A is injective if $ker(A) = \{0\}$ and that A is surjective if $Im(A) = Y$.*
- *The operator A is bijective if it is both injective and surjective.*

Definition 1.2.3 (Unbounded operators)
An unbounded linear operator from Z to Y is the pair $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a subspace vector of Z and A is a linear map from $\mathcal{D}(A) \subset Z$ to Y.

Proposition 1.2.1 *[9]*
Let A and B be two unbounded linear operators and $\alpha \in \mathbb{K}$. Then we have the following properties

1. $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$;
2. $\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$;
3. - *If $\alpha = 0$, then $\mathcal{D}(\alpha A) = Z$ and $\alpha A = 0$;*
   - *If $\alpha \neq 0$, then $\mathcal{D}(\alpha A) = \mathcal{D}(A)$ and $(\alpha A)x = \alpha(Ax), \forall x \in \mathcal{D}(A)$.*

Definition 1.2.4 (Closed operators)
An unbounded linear operator $A : \mathcal{D}(A) \subset Z \to Y$ is closed if it's graph

$$Gr(A) = \{(x, Ax);\ x \in \mathcal{D}(A)\},$$

is closed in $Z \times Y$.

Remark 1.2.1
An operator A is closed if and only if for any sequence $(x_n)_n$ in $\mathcal{D}(A)$ such that $\lim_{n \to +\infty} x_n = x$ and $\lim_{n \to +\infty} Ax_n = y$, then $x \in \mathcal{D}(A)$ and $y = Ax$.

Definition 1.2.5 (Extension of an operator)
We say that $(A, \mathcal{D}(A))$ is an extension of $(B, \mathcal{D}(B))$ if $\mathcal{D}(B) \subset \mathcal{D}(A)$ and $Bx = Ax$,$\forall x \in \mathcal{D}(B)$ we denote $B \subset A$. Moreover $B \subset A$ if and only if $Gr(B) \subset Gr(A)$.

Definition 1.2.6 (Adjoint of an unbounded operator)
Let $A : Z \to Y$ be an unbounded operator of dense domain $\mathcal{D}(A)$ and let the domain

$$\mathcal{D}(A^*) = \{y \in Y \text{ such that } f : x \to \langle Ax, y\rangle \text{ is continuous on } \mathcal{D}(A)\}$$

If $y \in \mathcal{D}(A^*)$, then there exists a unique vector $z \in Z$, such that

$$\langle Ax, y\rangle = \langle x, z\rangle, \text{ for all } x \in \mathcal{D}(A).$$

We denote the unique vector $z \in Z$ by $z = A^*y$, the linear operator A^* is called the adjoint of A.

Remark 1.2.2
From the defenition of the adjoint we deduce that if A is the extension of B, then B^* is the extension of A^* and $\mathcal{D}(A^*) \subset \mathcal{D}(B^*)$.

Theorem 1.2.1 *[9]*
Let A, B and AB be densely defined operators on the Hilbert space Z. Then

(a) $B^*A^* \subset (AB)^*$.

(b) If $\mathcal{D}(B^*)$ is dense in Z, then $B \subset B^{**}$.

Definition 1.2.7 (Invertible unbounded operators)
We say that an operator $A : \mathcal{D}(A) \subset Z \to Y$ is invertible if A is bijective and has an inverse $A^{-1} : Y \to \mathcal{D}(A) \subset Z$ bounded.

Definition 1.2.8
Let A be a closed operator. We say that the complex number λ is in the resolvent set $\rho(A)$ of A, if $\lambda I - A$ is bijective from $\mathcal{D}(A)$ into Z such that $(\lambda I - A)^{-1}$ is bounded.
If $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda I - A)^{-1}$ is called the resolvent or resolvent operator of A.
The spectrum of A is the set $\sigma(A) = \mathbb{C} \backslash \rho(A)$.

Remark 1.2.3
We note that in the case of bounded linear operators the spectrum is never empty and the spectrum is never equal to $\mathbb{C}$, but in the case of unbounded operators the spectrum can be empty as it can be the set $\mathbb{C}$.

Definition 1.2.9 (Symmetric and self adjoint operators)
- *We say that the operator $A : \mathcal{D}(A) \subset Z \to Y$ is symmetric if*
$\langle Ax, y\rangle = \langle x, Ay\rangle$, $\forall x, y \in \mathcal{D}(A)$ *and* $\mathcal{D}(A) \neq \mathcal{D}(A^*)$.

- *We say that the operator A is self adjoint if*
$\langle Ax, y\rangle = \langle x, Ay\rangle$, $\forall x, y \in \mathcal{D}(A)$ *and* $\mathcal{D}(A) = \mathcal{D}(A^*)$.

Example 1.2.1

Let $Z = L^2(0,1)$ *and* $Af = if'$ *an operator defined on* $\mathcal{D}(A) \subset Z$.

1. *Suppose that* $\mathcal{D}(A) = \{f \in Z : f$ *is absolutely continuous and* $f' \in Z\}$. *Calculate* A^* *the adjoint of* A.

   $$\begin{aligned}\langle Af, g\rangle &= \int_0^1 if'(t)\overline{g(t)}dt = [if(t)\overline{g(t)}]_0^1 - \int_0^1 if(t)\overline{g'(t)}dt \\ &= if(1)\overline{g(1)} - if(0)\overline{g(0)} + \int_0^1 f(t)\overline{ig'(t)}dt\end{aligned}.$$

   This expression can be written in the form $\langle f, A^*g\rangle$ *if and only if* g *is absolutely continuous,* $g(1) = g(0) = 0$ *and* $g' \in L^2(0,1)$. *Then* $\mathcal{D}(A^*) = \{g : g$ *is absolutely continuous and* $g' \in L^2(0,1)$ *and* $g(1) = g(0) = 0\}$, *and* $A^*g = ig'$.

   As $A = A^*$ *and* $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ *we can say that the operator* A *is symmetric.*

2. *If* $\mathcal{D}(A) = \{f \in Z$ *is absolutely continuous and* $f' \in Z$ *and* $f(0) = f(1)\}$. *Calculate the adjoint of* A.

   $$\begin{aligned}\langle Af, g\rangle &= \int_0^1 if'(t)\overline{g(t)}dt = [if(t)\overline{g(t)}]_0^1 - \int_0^1 if(t)\overline{g'(t)}dt \\ &= if(1)\overline{g(1)} - if(0)\overline{g(0)} + \int_0^1 f(t)\overline{ig'(t)}dt\end{aligned}.$$

   This expression can be written in the form $\langle f, A^*g\rangle$ *if and only if* g *is absolutely continuous,* $g(1) = g(0)$ *and* $g' \in L^2(0,1)$. *Then* $\mathcal{D}(A^*) = \{g : g$ *is absolutely ontinuous and* $g' \in L^2(0,1)$ *and* $g(1) = g(0)\}$, *and* $A^*g = ig'$.

   As $A = A^*$ *and* $\mathcal{D}(A) = \mathcal{D}(A^*)$ *we can say that the operator* A *is self adjoint.*

Definition 1.2.10

We say that the operator $A : \mathcal{D}(A) \subset Z \to Y$ *is dissipative if*

$$\forall x \in \mathcal{D}(A), \quad \lambda > 0, \quad \|\lambda x - Ax\| \geq \lambda\|x\|.$$

Theorem 1.2.2

An unbounded linear operator $(A, \mathcal{D}(A))$ *in* Z *is dissipative if and only if*

$$\forall x \in \mathcal{D}(A), \qquad (Ax, x) \leq 0.$$

In the case of a complex Hilbert space, the previous condition is remplaced by

$$\forall x \in \mathcal{D}(A), \qquad Re(Ax, x) \leq 0.$$

1.3 Semigroups of linear operators

Definition 1.3.1 *[7]*
We say that the familly of bounded linear operators $(T(t))_{t\geqslant 0}$ from Z to Z is a semigroup if

1. $T(0)=I$, *(where I is the identity operator).*
2. $\forall s,\ t\geqslant 0$, $T(t+s)=T(t)T(s)$.

Definition 1.3.2 *[7]*
We say that the semigroup $(T(t))_{t\geqslant 0}$ is

1. *Uniformly continuous if*
$$\lim_{t\to 0^+}\|T(t)-I\|=0.$$
2. *Strongly continuous if*
$$\lim_{t\to 0^+}T(t)x=x,\ \forall x\in Z.$$
A strongly continuous semigroup is said to be C_0-semigroup.

Definition 1.3.3 (Infinetisimal generator) *[7]*
We call infinitesimal generator of a semigroup $(T(t))_{t\geqslant 0}$, the unbounded linear operator A defined by
$$\begin{aligned} A:\ \mathcal{D}(A)\subset Z &\longrightarrow Z\\ x &\longrightarrow Ax=\lim_{t\to 0^+}\frac{T(t)x-x}{t}, \end{aligned}$$
where
$$\mathcal{D}(A)=\left\{x\in Z,\ \lim_{t\to 0}\frac{T(t)x-x}{t}\ \ \text{exists in}\ \ Z\right\}.$$

Example 1.3.1
Let $Z=l^2=\{x=(x_n)_{n\in\mathbb{N}^*};\ \ (\sum_{n=1}^{+\infty}|x_n|^2)<+\infty\}$ endowed with the norm
$$\|x\|_Z=\|(x_n)_{n\in\mathbb{N}^*}\|_Z=(\textstyle\sum_{n=1}^{+\infty}|x_n|^2)^{\frac{1}{2}}.$$
Let $(T(t))_{t\geq 0}$ be a family of linear operators defined by
$$T(t)x=T(t)(x_n)_{n\in\mathbb{N}^*}=(e^{-n^2t}x_n)_{n\in\mathbb{N}^*},\ \forall t\geqslant 0.$$

I. *We show that $(T(t))_{t\geq 0}$ is a C_0-semigroup on Z;*
1. $T(0)=Id$, *indeed, we have* $T(0)(x_n)_{n\in\mathbb{N}^*}=(e^{-n^2.0}x_n)_{n\in\mathbb{N}^*}=(e^0x_n)_{n\in\mathbb{N}^*}=(x_n)_{n\in\mathbb{N}^*}$, *then* $T(0)=I$.

2. We show that $T(t+s) = T(t)T(s),\ \ \forall t,\ s \geqslant 0,\ \ \forall t,\ s \geqslant 0,$ *we have*

$$T(t+s)(x_n)_{n\in\mathbb{N}^*} = (e^{-n^2(t+s)}x_n)_{n\in\mathbb{N}^*} = (e^{-n^2t-n^2s}x_n)_{n\in\mathbb{N}^*} = (e^{-n^2t}e^{-n^2s}x_n)_{n\in\mathbb{N}^*}, \tag{1.1}$$

On the other hand we have

$$T(t)T(s)(x_n)_{n\in\mathbb{N}^*} = T(t)(e^{-n^2s}x_n)_{n\in\mathbb{N}^*} = e^{-n^2t}(e^{-n^2s}x_n)_{n\in\mathbb{N}^*} = (e^{-n^2t}e^{-n^2s}x_n)_{n\in\mathbb{N}^*}. \tag{1.2}$$

From (1.1) *and* (1.2) *we get* $T(t+s) = T(t)T(s),\ \forall t,\ s \geqslant 0.$

3. Now we show that $\lim_{t\to 0^+} \|T(t)x - x\| = 0,\ \forall x \in Z.$
For all $x \in Z$*, we have*

$$\begin{aligned}
\|T(t)x - x\|_Z^2 &= \|T(t)(x_n)_{n\in\mathbb{N}^*} - (x_n)_{n\in\mathbb{N}^*}\|_Z^2 = \|(e^{-n^2t}x_n - x_n)_{n\in\mathbb{N}^*}\|_Z^2,\\
&= \|((e^{-n^2t} - 1)x_n)_{n\in\mathbb{N}^*}\|_Z^2,\\
&= \sum_{n=1}^{+\infty} |((e^{-n^2t} - 1)x_n|^2,\\
&\leqslant \left(\sum_{n=1}^{+\infty} |(e^{-n^2t} - 1)|^2\right).\left(\sum_{n=1}^{+\infty} |(x_n)|^2\right) = \left(\sum_{n=1}^{+\infty} |(e^{-n^2t} - 1)|^2\right)\|(x_n)\|_Z^2.
\end{aligned}$$

Then $\quad 0 \leqslant \lim\limits_{t\to 0^+} \|T(t)(x_n)_{n\in\mathbb{N}^*} - (x_n)_{n\in\mathbb{N}^*}\| \leqslant \lim\limits_{t\to 0^+} \left(\sum_{n=1}^{+\infty} |(e^{-n^2t} - 1)|^2\right)\|(x_n)\|_Z^2,$

such that $\quad \lim\limits_{t\to 0^+} \left(\sum_{n=1}^{+\infty} |(e^{-n^2t} - 1)|^2\right)\|(x_n)\|_Z^2 = 0.$

Then $\ \lim\limits_{t\to 0^+} \|T(t)(x_n)_{n\in\mathbb{N}^*} - (x_n)_{n\in\mathbb{N}^*}\| = 0,$
Thus $(T(t))_{t\geqslant 0}$ *where* $T(t)(x_n)_{n\in\mathbb{N}^*} = (e^{-n^2t}x_n)_{n\in\mathbb{N}^*},\ \ \forall t \geqslant 0$ *is a* C_0*-semigroup.*

II. Determine the infinitesimal generator of this C_0*-semigroup.*
By definition we have $Ax = \lim\limits_{t\to 0^+} \dfrac{T(t)x - x}{t}$
indeed,

$$T(t)x = (e^{-n^2t}x_n)_{n\in\mathbb{N}^*} = (e^{-t}x_1, e^{-4t}x_2, ..., e^{-n^2t}x_n, ...),$$

$$\text{then}\quad \frac{T(t)x - x}{t} = (\frac{e^{-t}x_1 - x_1}{t}, \frac{e^{-4t}x_2 - x_2}{t}, ..., \frac{e^{-n^2t}x_n - x_n}{t}, ...),$$

$$\begin{aligned}
\text{it follows that}\quad \lim_{t\to 0^+} \frac{T(t)x - x}{t} &= (\lim_{t\to 0^+} \frac{e^{-t}x_1 - x_1}{t}, \lim_{t\to 0^+} \frac{e^{-4t}x_2 - x_2}{t}, ..., \lim_{t\to 0^+} \frac{e^{-n^2t}x_n - x_n}{t}, ...),\\
&= (-x_1, -4x_2, ..., -n^2x_n, ...).
\end{aligned}$$

$$\text{Consequently}\quad Ax = (-x_1, -4x_2, ..., -n^2x_n, ...),$$

Thus $Ax = (Ax_n)_{n\in\mathbb{N}^*} = (-n^2x_n)_{n\in\mathbb{N}^*}$ *and* $\mathcal{D}(A) = \{(x_n)_{n\in\mathbb{N}^*} \in l^2 : (-n^2x_n)_{n\in\mathbb{N}^*} \in l^2\}.$

1.3.1 Properties of semigroups

Theorem 1.3.1.1
A linear operator A is the infitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.

Proof
See [7, page 2] ■

Theorem 1.3.1.2
Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup, then there exists $\omega \geq 0$ et $M \geq 1$, such that :

$$\|T(t)\| \leq M\mathrm{e}^{\omega t}, \quad \forall t \geq 0.$$

Proof
See [7, page 4] ■

Theorem 1.3.1.3 *[1]*
Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup with generator infinitesimal A then

1. *If $x \in \mathcal{D}(A)$ then, $T(t)x \in \mathcal{D}(A)$ for all $t \geq 0$,*
2. *For all $x \in Z$, we have : $\lim \frac{1}{t}\int_0^t T(s)xds = x$,*
3. *For all $x \in Z$, $\int_0^t T(s)x\ ds \in \mathcal{D}(A)$, and $A\int_0^t T(s)x\ ds = T(t)x - x$,*
4. *For all $x \in \mathcal{D}(A)$, $T(t)x \in \mathcal{D}(A)$, and $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$,*
5. *For all $x \in \mathcal{D}(A)$, $T(t)x - T(s)x = \int_s^t T(\tau)Ax\ d\tau = \int_s^t AT(\tau)x\ d\tau$,*
6. *$T(nt) = T(t)^n$, for all $t \geq 0$ and $n \in \mathbb{N}$,*
7. *If $\omega_0 = \inf_{t>0}\left(\frac{1}{t}\log\|T(t)\|\right)$, then $\omega_0 = \lim_{t\to\infty}\left(\frac{1}{t}\log\|T(t)\|\right) < \infty$,*
8. *$\forall \omega > \omega_0$, there exists a constant M_ω such that $\forall t \geq 0$, $\|T(t)\| \leq M_\omega \mathrm{e}^{\omega t}$.*
   This constant ω_0 is called the growth bound of the semigroup.

Proof
1) Let $x \in \mathcal{D}(A)$, show that $T(t)x \in \mathcal{D}(A)$, for all $t \geq 0$. We have

$$\begin{aligned}\lim_{s\to 0^+}\frac{1}{s}(T(s)T(t)x - T(t)x) &= \lim_{s\to 0^+}\frac{1}{s}(T(t+s)x - T(t)x)\\ &= T(t)\lim_{s\to 0^+}\frac{1}{s}(T(s)x - x)\\ &= T(t)Ax.\end{aligned}$$

Then we deduce that $T(t)x \in \mathcal{D}(A)$ and $A(T(t)x) = T(t)Ax$.

2) Let $x \in Z$ and $\varepsilon > 0$, as the semigroup $(T(t))_{t\geq 0}$ is strongly continuous, we can choose a constant $\tau > 0$ such that

$$\|T(s)x - x\| \leq \varepsilon \quad \forall s \in [0, \tau].$$

for $t \in [0, \tau]$ we have

$$\begin{aligned}\|\frac{1}{t}\int_0^t T(s)x\ ds - x\| &= \|\frac{1}{t}\int_0^t [T(s)x - x]\ ds\| \\ &\leq \frac{1}{t}\int_0^t \|[T(s)x - x]\|\ ds \leq \frac{1}{t}\int_0^t \varepsilon\ ds = \varepsilon.\end{aligned}$$

Then

$$\|\frac{1}{t}\int_0^t T(s)x\ ds - x\| \leq \varepsilon.$$

Thus

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t T(s)x\ ds = x.$$

3) Let $x \in Z$; $t > 0$, show that $\lim_{s\to 0} \frac{1}{s}(T(s).\int_0^t T(r)x\ dr - \int_0^t T(r)x\ dr)$ exists. We have

$$\begin{aligned}\frac{1}{s}(T(s).\textstyle\int_0^t T(r)x\ dr - \int_0^t T(r)x\ dr) &= \frac{1}{s}(\textstyle\int_0^t T(s)T(r)x\ dr - \int_0^t T(r)x\ dr) \\ &= \frac{1}{s}(\textstyle\int_0^t T(s+r)x\ dr - \int_0^t T(r)x\ dr) \\ &= \frac{1}{s}(\textstyle\int_s^{s+t} T(\grave{s})x\ d\grave{s} - \int_0^t T(r)x\ dr) \quad \textit{tel que}\ \ \grave{s} = s + r \\ &= \frac{1}{s}(\textstyle\int_s^t T(\grave{s})x\ d\grave{s} + \int_t^{s+t} T(\grave{s})x\ d\grave{s} - \int_0^s T(r)x\ dr - \int_s^t T(r)x\ dr) \\ &= \frac{1}{s}(\textstyle\int_t^{s+t} T(\grave{s})x\ d\grave{s} - \int_0^s T(r)x\ dr) \\ &= \frac{1}{s}(\textstyle\int_0^s T(\bar{s}+t)x\ d\bar{s} - \int_0^s T(r)x\ dr) \quad \textit{tel que}\ \ \bar{s} = \grave{s} - t \\ &= \frac{1}{s}(\textstyle\int_0^s T(t)T(\bar{s})x\ d\bar{s} - \int_0^s T(r)x\ dr) \\ &= \frac{1}{s}(T(t)\textstyle\int_0^s T(\bar{s})x\ d\bar{s} - \int_0^s T(r)x\ dr) \\ &= \frac{1}{s}(T(t) - I)\textstyle\int_0^s T(r)x\ dr\end{aligned}$$

Then

$$\begin{aligned}\lim_{s\to 0^+} \frac{1}{s}(T(s).\textstyle\int_0^t T(r)x\ dr - \int_0^t T(r)x\ dr) &= \displaystyle\lim_{s\to 0^+} \frac{1}{s}(T(t) - I)\textstyle\int_0^s T(r)x\ dr \\ &= (T(t) - I)\displaystyle\lim_{s\to 0^+} \frac{1}{s}\textstyle\int_0^s T(r)x\ dr \\ &= (T(t) - I)x.\end{aligned}$$

Thus

$$\int_0^t T(t)x\ ds \in \mathcal{D}(A \textit{ and } A\int_0^t T(t)x\ ds = T(t)x - x.$$

4) Let $x \in \mathcal{D}(A)$, show that the right derivative of $T(t)x$ exists. We have

$$\begin{aligned}
\frac{d}{dt}T(t)x &= \lim_{\tau\to 0^+} \frac{T(t+\tau)-T(t)}{\tau}x \\
&= \lim_{\tau\to 0^+} \frac{T(t)T(\tau)-T(t)}{\tau}x \\
&= \lim_{\tau\to 0^+} T(t)\frac{(T(\tau)-I)x}{\tau}x \\
&= T(t)\lim_{\tau\to 0^+} \frac{(T(\tau)-I)x}{\tau}x = T(t)Ax.
\end{aligned}$$

Then the right derivative exists, we have also

$$\begin{aligned}
\frac{d}{dt}T(t)x &= \lim_{\tau\to 0^+} \frac{T(t)-T(t-\tau)}{\tau}x \\
&= \lim_{\tau\to 0^+} \frac{T(t-\tau)T(\tau)-T(t-\tau)}{\tau}x \\
&= \lim_{\tau\to 0^+} T(t-\tau)\frac{(T(\tau)-I)}{\tau}x \\
&= \lim_{\tau\to 0^+} T(t-\tau).\lim_{\tau\to 0} \frac{(T(\tau)-I)}{\tau}x \\
&= T(t)Ax.
\end{aligned}$$

Then the left derivative exists. Thus

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

5) Let $x \in \mathcal{D}(A)$, we integrate (3) we obtain

$$\begin{aligned}
T(t)x - T(s)x &= \textstyle\int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau}T(\tau)x \ \mathrm{d}\tau \\
&= \textstyle\int_s^t AT(\tau)x \ \mathrm{d}\tau \\
&= \textstyle\int_s^t T(\tau)Ax \ \mathrm{d}\tau
\end{aligned}$$

6) Easy to check by recurrence.

7) Let $t_0 > 0$ be a fixed number and $M = \sup_{t\in[0,t_0]} \|T(t)\|$, then for every $t \geq t_0$, there exists $n \in \mathbb{N}$ such that $nt_0 \leq t \leq (n+1)t_0$. Consequently

$$\begin{aligned}
\frac{\log\|T(t)\|}{t} &= \frac{\log\|T^n(t_0)T(t-nt_0)\|}{t} \\
&\leq \frac{n\log\|T(t_0)\|}{t} + \frac{\log M}{t} \\
&= \frac{\log\|T(t_0)\|}{t}.\frac{nt_0}{t} + \frac{\log M}{t}.
\end{aligned}$$

The latter is similar or equal to $\dfrac{\log\|T(t_0)\|}{t}+\dfrac{\log M}{t}$, if $\log\|T(t_0)\|$ is positive.
and it is smaller than or equal to $\dfrac{\log\|T(t_0)\|}{t}.\dfrac{t-t_0}{t}+\dfrac{\log M}{t}$, if $\log\|T(t_0)\|$ is negative.
Thus

$$\limsup_{t\to+\infty}\frac{\log\|T(t)\|}{t}\leq\frac{\log\|T(t_0)\|}{t_0}<+\infty.$$

and since t_0 is arbitrary, we have that

$$\limsup_{t\to+\infty}\frac{\log\|T(t)\|}{t}\leq\inf_{t>0}\frac{\log\|T(t)\|}{t}\leq\liminf_{t\to+\infty}\frac{\log\|T(t)\|}{t}.$$

Thus

$$\omega_0=\inf_{t>0}\frac{\log\|T(t)\|}{t}=\lim_{t\to+\infty}\frac{\log\|T(t)\|}{t}<+\infty.$$

8) If $\omega>\omega_0$, there exists a t_0 such that $\dfrac{\log\|T(t)\|}{t}<\omega \quad$ for $t\geq t_0$ that is,

$$\|T(t)\|\leq e^{\omega t}\qquad \text{for } t\geq t_0$$

But

$$\|T(t)\|\leq M_0\qquad \text{for } 0\leq t\leq t_0$$

and so with $M_\omega=M_0$, for the case that $\omega>0$,
and $M_\omega=e^{-\omega t_0}M_0$ for the case that $\omega<0$,
we obtain the stated result. ■

Proposition 1.3.1.1 *[7]*
Let $(T(t))_{t\geqslant0}$ be a C_0-semigroup, then its infinitesimal generator A is closed and $\mathcal{D}(A)$ is dense in Z.

Proof
1) Let $x\in Z$, show that there exists a sequence $(x_\varepsilon)_{\varepsilon>0}$ such that $x_\varepsilon\in\mathcal{D}(A),\forall\varepsilon>0$ and $\lim\limits_{\varepsilon\to0}x_\varepsilon=x$.
By Theorem 1.3.1.3 we have

$$\int_0^s T(t)x\;dt\in\mathcal{D}(A);s>0.$$

Let $x_\varepsilon=\varepsilon^{-1}\int_0^\varepsilon T(t)x\;dt$, then $x_\varepsilon\in D(A)$.
On the other hand we have

$$\begin{aligned}\lim_{\varepsilon\to0}x_\varepsilon &=\lim_{\varepsilon\to0}\;\varepsilon^{-1}\textstyle\int_0^\varepsilon T(t)x\;dt,\\ &=\lim_{\varepsilon\to0}\;\frac{1}{\varepsilon}(\textstyle\int_0^\varepsilon T(t)x\;dt-\int_0^0 T(t)x\;dt),\\ &=T(0)x,\\ &=x.\end{aligned}$$

Then $\lim_{\varepsilon\to 0} x_\varepsilon = x \in Z$, so there exists $(x_\varepsilon)_{\varepsilon>0} \in \mathcal{D}(A)$ such that $\lim_{\varepsilon\to 0} x_\varepsilon = x$. Thus $\overline{\mathcal{D}(A)} = Z$.

2) Show that A is closed. Let (x_n) be a sequence in $\mathcal{D}(A)$ which converges to $x \in Z$ such that $\lim_{n\to 0} Ax_n = y$, let us show that $x \in \mathcal{D}(A)$ et $Ax = y$. By Theorem 1.3.1.2, there exists $\omega \geq 0$ and $M \geq 1$ such that, for all $t \geq 0$,

$$\|T(t)Ax_n - T(t)y\| \leq Me^{\omega t}\|Ax_n - y\|,$$

and

$$\lim_{n\to+\infty} Ax_n = y \quad \Rightarrow \quad \lim_{n\to+\infty} T(t)Ax_n = T(t)y.$$

As $(x_n) \in \mathcal{D}(A)$ we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n \ \ ds.$$

Then we obtain

$$\lim_{n\to+\infty}(T(t)x_n - x_n) = \lim_{n\to+\infty}\int_0^t T(s)Ax_n \ \ ds = \int_0^t T(t)y \ \ ds,$$

$$\textit{and} \quad \lim_{t\to 0^+}\frac{T(t)x - x}{t} = \lim_{t\to 0^+}\frac{1}{t}\int_0^t T(s)y \ \ ds = y.$$

Consequently $x \in \mathcal{D}(A)$ and $Ax = y$. Then A is a closed operator. ∎

Lemma 1.3.1.1 *[7]*

If $(T(t))_{t\geqslant 0}$ is a C_0-semigroup of invertible operators, then $(T^{-1}(t))_{t\geqslant 0}$ is also a C_0-semigroup. Moreover if A is the infinitesimal generator of $(T(t))_{t\geqslant 0}$ then $-A$ is the generator infinitesimal of $(T^{-1}(t))_{t\geqslant 0}$.

Proof

Let $(T(t))_{t\geq 0}$ be an invertible C_0-semigroup, $(T^{-1}(t))_{t\geq 0}$ exists. We assume $(S(t))_{t\geq 0} = (T^{-1}(t))_{t\geq 0}$,

Let us show that $(S(t))$ is a semigroup

$$S(t+s) = T^{-1}(t+s) = (T(t)T(s))^{-1} = T^{-1}(s)T^{-1}(t) = S(s)S(t).$$

Let us show that $(S(t))_{t\geq 0}$ is strongly continuous. For $s > 0$, $ImT = Z$ (surjective).

Let $x \in Z$ and $s > 1$, there exists $y \in X$ such that $T(s)y = x$, so for $t < 1$ we have

$$\begin{aligned} \|T^{-1}(t)x - x\| &= \|T^{-1}(t)T(t)T(s-t)y - T(s)y\| \\ &= \|T(s-t)y - T(s)y\| \longrightarrow 0 \quad \textit{when} \quad t \longrightarrow 0. \end{aligned}$$

Then, $(S(t))_{t\geq 0}$ is strongly continuous. Finally, for $x \in \mathcal{D}(A)$ we have

$$\lim_{t\to 0}\frac{T^{-1}(t)x - x}{t} = \lim_{t\to 0} T(t)\frac{T^{-1}(t)x - x}{t} = \lim_{t\to 0}\frac{x - T(t)x}{t} = -Ax.$$

Thus $-A$ is the infinitesimal generator of $(T^{-1}(t))_{t\geq 0}$. ∎

Theorem 1.3.1.4 *[1]*
If $(T(t))_{t\geq 0}$ is a C_0-semigroup with infinitesimal generator A on a Hilbert space Z, then $(T^*(t))_{t\geq 0}$ is the C_0-semigroup with infinitesimal generator A^* on Z.

Lemma 1.3.1 *[1]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with infinitesimal generator A and with growth bound ω_0. If $\Re(\lambda) > \omega > \omega_0$, then $\lambda \in \rho(A)$, and for all $z \in Z$ the following result hold

$$R(\lambda, A)z = (\lambda I - A)^{-1}z = \int_0^{\infty} e^{-\lambda t}T(t)z\, dt \quad and \quad \|R(\lambda, A)\| \leq \frac{M}{\sigma - \omega}; \sigma = \Re(\lambda)$$

.

1.3.2 Stability of semigroups

Definition 1.3.2.1 *[7]*
We say that the semigroup $(T(t))_{t\geq 0}$ is

1. *Exponentially stable if there exists a constant $M \geq 1$ and $\alpha > 0$ such that*

$$\|T(t)\| \leq Me^{-\alpha t} \; for \; all \quad t \geq 0,$$

   equivalently if

$$\|T(t)x\| \leq Me^{-\alpha t}\|x\| \; for \; all \quad t \geq 0 \; and \; for \; all \; x \in Z.$$

2. *Uniformly stable if*

$$\|T(t)\| \to 0 \; when \; t \to +\infty$$

3. *Strongly stable if*

$$\|T(t)x\| \to 0 \; when \; t \to +\infty, \; for \; all \; x \in Z.$$

Remark 1.3.2.1
The uniform stability is equivalent to exponential stability and uniform stability implies the strong stability, but the reciprocal in general is not verified in infinite dimension.

Lemma 1.3.2.1 (Datko lemma) *[1]*
Let Z be a Hilbert space, the semigroup $(T(t))_{t\geq 0}$ is exponentially stable if and only if for each $y \in Z$ we have

$$\int_0^{+\infty} \|T(t)y\|^2\, dt < +\infty$$

Theorem 1.3.2.1
Suppose that A is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq}$ on Z.
Then the following properties are equivalent

(i). $(T(t))_{t\geq 0}$ *is exponentially stable;*

(ii). There exists positive operator $W \in \mathcal{B}(Z)$ *such that*
$\langle Ax, Wx\rangle + \langle Wx, Ax\rangle = -\,\langle x, x\rangle \quad ,\forall x \in \mathcal{D}(A);$

(iii). There exists a positive operator $W \in \mathcal{B}(Z)$ *such that*
$\langle Ax, Wx\rangle + \langle Wx, Ax\rangle \leq -\,\langle x, x\rangle \quad ,\forall x \in \mathcal{D}(A).$

Proof

(1) $\Longrightarrow$ (2). As $(T(t))_{t\geq 0}$ is exponentially stable, let the operator W given by $Wx = \int_0^{+\infty} T^*(t)T(t)x dt$, W is well defined. Indeed, we have

$$\begin{aligned} \|Wx\| &= \Big\| \int_0^{+\infty} T^*(t)T(t)x dt \Big\| \\ &\leq \int_0^{+\infty} \|T^*(t)T(t)x\| dt \\ &\leq \int_0^{+\infty} \|T^*(t)\|\|T(t)\|\|x\| dt \leq \int_0^{+\infty} \|T(t)\|^2\|x\| dt \\ &\leq \left(M^2 \int_0^{+\infty} e^{-2\alpha t} dt \right) \|x\| \leq cst\|x\|, \end{aligned}$$

then W is bounded, moreover

$$\begin{aligned} \langle x, Wx\rangle &= \left\langle x, \int_0^{+\infty} T^*(t)T(t)x dt \right\rangle \\ &= \int_0^{+\infty} \langle x, T^*(t)T(t)x\rangle\, dt \\ &= \int_0^{+\infty} \langle T(t)x, T(t)x\rangle\, dt \\ &= \int_0^{+\infty} \|T(t)x\|^2 dt \geq 0 \end{aligned}$$

$\langle x, Wx\rangle = 0$, then $\|T(t)x\| = 0$ almost everywhere on $[0, +\infty[$, and since $(T(t))_{t\geq 0}$ is strongly continuous, then $\|T(t)x\| = 0$ and therfore $x = 0$.

Let us show that W verifies the Lyapunov equation.

$$\begin{aligned} \langle Ax, Wx\rangle + \langle Wx, Ax\rangle &= \left\langle Ax, \int_0^{+\infty} T^*(t)T(t)xdt \right\rangle + \left\langle \int_0^{+\infty} T^*(t)T(t)xdt, Ax \right\rangle \\ &= \int_0^{+\infty} \langle Ax, T^*(t)T(t)xdt\rangle + \int_0^{+\infty} \langle T^*(t)T(t)x, Axdt\rangle \\ &= \int_0^{+\infty} \mid \langle T(t)Ax, T(t)x\rangle + \langle T(t)x, T(t)Ax\rangle \mid dt \end{aligned}$$

If $x \in \mathcal{D}(A)$, then $T(t)Ax = AT(t)x$. Thus

$$\langle Ax, Wx\rangle + \langle Wx, Ax\rangle = \int_0^{+\infty} \mid \langle AT(t)x, T(t)x\rangle + \langle T(t)x, AT(t)x\rangle \mid dt.$$

If $x \in \mathcal{D}(A)$, then $\dfrac{d}{dx}T(t)x = AT(t)x$. Thus

$$\begin{aligned} \langle Ax, Wx\rangle + \langle Wx, Ax\rangle &= \int_0^{+\infty} \left| \left\langle \tfrac{d}{dx}T(t)x, T(t)x \right\rangle + \left\langle T(t)x, \tfrac{d}{dx}T(t)x \right\rangle \right| dt \\ &= \int_0^{+\infty} \tfrac{d}{dx}(\langle T(t)x, T(t)x\rangle)dt. \end{aligned}$$

we have

$$\lim_{\tau\to\infty} \int_0^{\tau} \frac{d}{dt}(\langle T(t)x, T(t)x\rangle)dt = -\|x\|^2,$$

then

$$\langle Ax, Wx\rangle + \langle Wx, Ax\rangle = -\|x\|^2 = -\langle x, x\rangle .$$

(2) $\Longrightarrow$ (3) obvious.

(3) $\Longrightarrow$ (1). Let W be a positive symmetric solution of the Lyapunov equation and let

$$V(t,x) = \langle WT(t)x, T(t)x\rangle .$$

Since $W \geq 0$, then $V(t,x) \geq 0$ for $t \geq 0$.

For $x \in \mathcal{D}(A)$, then $V(.,x)$ is differentiable with respect to t and

$$\begin{aligned} \frac{d}{dt}V(t,x) &= \left\langle \tfrac{d}{dt}WT(t)x, T(t)x \right\rangle + \left\langle WT(t)x, \tfrac{d}{dt}T(t)x \right\rangle \\ &= \left\langle W\tfrac{d}{dt}T(t)x, T(t)x \right\rangle + \left\langle WT(t)x, \tfrac{d}{dt}T(t)x \right\rangle \\ &= \langle WAT(t)x, T(t)x\rangle + \langle WT(t)x, AT(t)x\rangle \\ &= \langle AT(t)x, WT(t)x\rangle + \langle WT(t)x, AT(t)x\rangle \leq -\|T(t)x\|^2. \end{aligned}$$

then we obtain

$$\frac{d}{dt}V(t,x) \leq -\|T(t)x\|^2.$$

Integrate the two sides of the above inequality from 0 to $s \quad (s > 0)$

$$V(s,x) - V(0,x) \leq -\int_0^s \|T(t)x\|^2 dt.$$

Where

$$0 \leq V(s,x) \leq V(0,x) - \int_0^s \|T(t)x\|^2 dt.$$

Thus

$$\int_0^s \|T(t)x\|^2 dt \leq V(0,x).$$

i.e.,

$$\int_0^s \|T(t)x\|^2 dt \leq \langle x, Wx \rangle, \quad \forall x \in \mathcal{D}(A)$$

Since the domain $\mathcal{D}(A)$ is dense then $\forall x \in Z$, $\exists\ (x_n)_{n\geq 1} \in \mathcal{D}(A)$ such that $\lim_{n\to\infty}(x_n) = x$,

then we obtain $\lim_{n\to+\infty} \int_0^s \|T(t)x_n\|^2 dt \leq \lim_{n\to+\infty} \langle x_n, Wx_n \rangle, \ \forall x \in Z$

implies

$$\int_0^s \|T(t)x\|^2 dt \leq \langle x, Wx \rangle, \quad \forall x \in Z,$$

then

$$\int_0^s \|T(t)x\|^2 dt < +\infty.$$

By Datko's lemma $(T(t))_{t\geq 0}$ is exponentially stable. ■

1.3.3 Contraction and isometric semigroups

Definition 1.3.3.1 (Contraction semigroup) *[1]*
$(T(t))_{t\geq 0}$ is a contraction semigroup on a Hilbert space Z if it is a C_0-semigroup that satisfies an estimate $\| T(t) \| \leq 1$ for all $t \geq 0$, equivalently if $\|T(t)x\| \leq \|x\|$ for all $t \geq 0$ and for all $x \in Z$.

We shall now give necessary and sufficient conditions for a closed, densely defined operators to be the infinitesimal generator of a contraction semigroup.

Theorem 1.3.3.1 *[1]*
Let A be a closed, densely defined operator with domain $\mathcal{D}(A)$ on a Hilbert space Z. Then $A - \omega I$ is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on Z if and only if the following conditions hold for all real $\alpha > \omega$

$$\|(\alpha I - A)z\| \geq (\alpha - \omega)\|z\| \ \textit{for}\ z \in \mathcal{D}(A);$$

$$\|(\alpha I - A^*)z\| \geq (\alpha - \omega)\|z\| \text{ for } z \in \mathcal{D}(A^*).$$

Corollary 1.3.3.1 *[1]*
Necessary and sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of a C_0-semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ are

$$\Re(\langle Az, z\rangle) \leq \omega\|z\|^2 \qquad \text{for } z \in \mathcal{D}(A)$$
$$\Re(\langle A^*z, z\rangle) \leq \omega\|z\|^2 \qquad \text{for } z \in \mathcal{D}(A^*)$$

Lemma 1.3.3.1 *[1]*
$(A-\omega I)$ is the infinitesimal generator of the contraction semigroup $(e^{-\omega t} \parallel T(t) \parallel)$ on the Hilbert space Z if and only if A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq e^{\omega t}$.

Definition 1.3.3.2 (Isometric Semigroup)
We say that the semigroup $(T(t))_{t\geq 0}$ is isometric if $\parallel T(t)x \parallel = \parallel x \parallel$ for all $t \geq 0$ and $\forall x \in Z$.

1.3.4 Perturbation of Semigroups

Theorem 1.3.4.1 (Bounded perturbation Theorem) *[3]*
Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space Z satisfying :

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all} \quad t \geq 0 \quad \text{and some} \quad \omega \in \mathbb{R}, M \geq 1$$

.

If $B \in \mathcal{B}(Z)$, then $C = A+B$ with $\mathcal{D}(C) = \mathcal{D}(A)$ is the infinitesimal generator of a C_0-semigroup $(T_B(t))_{t\geq 0}$ satisfying

$$\|T_B(t)\| \leq Me^{(\omega + M\|B\|t)} \quad \text{for} \quad \text{all} \quad t \geq 0$$

.

Corollary 1.3.4.1 *[3]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator A on Z and $(T_B(t))_{t\geq 0}$ the semigroup with generator $A+B$ for $B \in \mathcal{B}(Z)$. The semigroup $(T_B(t))_{t\geq 0}$ is the unique solution of the equation

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xds \qquad \text{where} \quad x \in Z$$

in the class of strongly continuous operators on Z. This C_0-semigroup satisfies the following equation for every $x \in Z$

$$T_B(t)x = T(t)x + \int_0^t T(t-s)BT_B(s)xds. \tag{1.3}$$

1.4 Notions from control theory

Let Z, U and Y be a Hilbert spaces. We consider the following class of infinite-dimensional systems with input u and output y :

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), t \geq 0, \quad z(0) = z_0, \\ y(t) &= Cz(t). \end{aligned}$$

$\Sigma(A,B,C)$ denotes the state linear system, where A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space Z, the state space. B is a bounded linear operator from the input space U to Z, C is a bounded linear operator from Z to the output space Y.
We consider $\Sigma(A,B,C)$ for all initial states $z_0 \in Z$ and all inputs $u \in L_2([0,\tau];U)$. The state is the mild solution of

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad 0 \leq t \leq \tau$$

To avoid clutter, we shall use the notation $\Sigma(A,B,-)$ when the operator C do not play a role and $\Sigma(A,-,C)$ when B do not play a role.

Definition 1.4.1 (Exact controllability) *[1]*
For the state linear system $\Sigma(A,B,-)$ we define the following concepts

a). The controllability map of $\Sigma(A,B,-)$ on $[0,\tau]$ (for some finite $\tau > 0$) is the bounded linear map $\mathcal{B}^\tau : L_2([0,\tau];U) \to Z$ defined by

$$\mathcal{B}^\tau u := \int_0^\tau T(\tau - s)Bu(s)ds$$

b). $\Sigma(A,B,-)$ is exactly controllable on $[0,\tau]$ (for some finite $\tau > 0$) if all points in Z can be reached from the origin at time τ, i.e., if $\operatorname{Im}\mathcal{B}^\tau = Z$ ($\mathcal{B}^\tau$ is surjective).

Theorem 1.4.1 *[1]*
The state linear system $\Sigma(A,B,-)$ is exactly controllable on $[0,\tau]$ if and only if any one of the following conditions hold for some $\gamma > 0$ and all $z \in Z$

i. $\left\|\mathcal{B}^{\tau^*} z\right\|_2^2 := \int_0^\tau \left\|\left(\mathcal{B}^{\tau^*} z\right)(s)\right\|_U^2 ds \geq \gamma \|z\|_{Z^2}^2,$

ii. $\int_0^\tau \|B^* T^*(s) z\|_U^2 ds \geq \gamma \|z\|_Z^2.$

Definition 1.4.2 (Exact observability) *[1]*
For the state linear system $\Sigma(A,-,C)$, we define the following concepts

a). The observability map of $\Sigma(A,-,C)$ on $[0,\tau]$ (for some finite $\tau > 0$) is the bounded linear map $C^\tau : Z \to L_2([0,\tau];Y)$ defined by

$$C^\tau z = CT(\cdot)z$$

b). $\Sigma(A,-,C)$ is exactly observable on $[0,\tau]$ (for some finite $\tau > 0$) if the initial state can be uniquely and continuously constructed from the knowledge of the output in $L_2([0,\tau];Y)$, i.e., C^τ is injective and its inverse is bounded on the range of C^τ.

Theorem 1.4.2 *[1]*
$\Sigma(A,-,C)$ is exactly observable on $[0,\tau]$ if and only if any one of the following conditions hold for some $\gamma > 0$ and for all $z \in Z$

i. $\|C^\tau z\|_2^2 := \int_0^\tau \|(C^\tau z)(s)\|_Y^2 \, ds \geq \gamma \|z\|_Z^2$,

ii. $\int_0^\tau \|CT(s)z\|_Y^2 ds \geq \gamma \|z\|_Z^2$,

iii. $\ker C^\tau = \{0\}$ *and* C^τ *has closed range.*

Lemma 1.4.1 (Duality between conrollability and observability) *[1]*
For the state linear system $\Sigma(A,-,C)$, we have the following duality result
$\Sigma(A,-,C)$ is exactly observable on $[0,\tau]$ if and only if the dual system $\Sigma(A^*,C^*,-)$ is exatly controllable on $[0,\tau]$.

Definition 1.4.3 (Admissibility of the observation operator C) *[12]*
Let Z and Y be two Hilbert spaces and $(T(t))_{t\geq 0}$ be a C_0-semigroup on Z with generator A. $C \in \mathcal{B}(\mathcal{D}(A),Y)$ is said to be admissible for $(T(t))_{t\geq 0}$ if for some (and hence any) $t > 0$, there exists $K_t > 0$ such that

$$\int_0^t \|CT(s)x\|^2 ds \leq K_t^2 \|x\|^2, \quad \forall x \in \mathcal{D}(A)$$

.

An extension concept of admissibility is as follows.

Definition 1.4.4 *[12]*
Let Z and Y be two Hilbert spaces and $(T(t))_{t\geq 0}$ be a C_0-semigroup on Z with generator A. $C \in \mathcal{B}(\mathcal{D}(A),Y)$ is said to be infinite-time admissible for $(T(t))_{t\geq 0}$ if there exists a constant $K > 0$ such that

$$\int_0^\infty \|CT(s)x\|^2 ds \leq K^2 \|x\|^2, \quad \forall x \in \mathcal{D}(A).$$

Remark 1.4.1

If $(T(t))_{t\geq 0}$ *is expenentially stable, the notion of admissibility and infinite-time admissibility are equivalent. Let* $\|T(t)\| \leq M_1 e^{wt}, \ \ \forall t \geq 0$. *If* C *is admissible for* $(T(t))_{t\geq 0}$*, then there exists a constant* $M > 0$ *such that*

$$\|CR(\lambda, A)\| \leq \frac{M}{\sqrt{\Re\lambda}}, \quad \forall \ \ \Re\lambda > \omega. \tag{1.4}$$

In what follows in chapter 2, we say that C *satisfying* (1.4) *is a Weiss class operator.*

Chapter 2

Characteristic of left invertible semigroups and admissibility of observation operators

2.1 Introduction

In this chapter we discuss the characteristic properties of the left invertible semigroups on general Banach spaces and admissibility of the observation operators for such semigroups. We obtain a sufficient and necessary condition about their generators. In their paper [12] Gen Qi Xu and Ying Feng Shang showed that for the left invertible and exponentially stable semigroup in Hilbert space there is an equivalent norm under which it is contractive. Based on these results they proved that for any observation operator satisfying the resolvent condition is admissible for the left invertible semigroup if its range is finite-dimensional. In addition they gave a sufficient condition of exact observability of the left invertible semigroup.

Moreover, from [6] we have illustrated the relation between the exact controllability and the right inverse of a C_0-semigroup, then by duality we deduce some results about the left inverse of a C_0-semigroup.

2.2 Characteristic property of the left invertible semigroups

In this section we shall investigate the characteristic property of generators of left invertible semigroups. We shall give necessary and sufficient conditions for a C_0-semigroup in Banach space to be left invertible. For the sake of completeness, we start by defining the left invertible semigroup.

Definition 2.2.1 *[12]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on Banach space X. If there exist some $t_0 > 0$ and a constant

$c > 0$ such that

$$\|T(t_0)x\| \geq c\|x\|, \quad \forall x \in X$$

then $(T(t))_{t\geq 0}$ is said to be left invertible semigroup.

Theorem 2.2.1 *[12]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X and A be its generator. Then the following statements are equivalent.

1. *$(T(t))_{t\geq 0}$,is a left invertible semigroup;*
2. *There exist two constants $\alpha > 0$ and $c > 0$ such that*

$$\|T(t)x\| \geq ce^{-\alpha t}\|x\|, \quad \forall x \in X, \ t > 0;$$

3. *There exists a constant $t_0 > 0$ such that*

$$\inf_{\|x\|=1, x\in X} \|T(t_0)x\| > 0.$$

Theorem 2.2.2 *[12]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a complex Banach space X and A be its generator. Then the following statements are equivalent.

1. *$(T(t))_{t\geq 0}$ is a left invertible semigroup;*
2. *There exists an equivalent norm $\|.\|_*$ on X such that, for some real number α, $-(A+\alpha I)$ is dissipative on $(X, \|.\|_*)$.*

Remark 2.2.1
Note that in a Banach space X we can always define an equivalent norm on X such that a uniformly bounded semigroup becomes a contraction semigroup, but for a Hilbert space it is not. However, if $(T(t))_{t\geq 0}$ is an exponentially stable and left invertible semigroup on a Hilbert space, we can do it. For a precise description see Theorem 2.3.1 in Section 3.

Corollary 2.2.1 *[12]*
Let $(T(t))_{t\geq 0}$ be a left invertible semigroup on Banach space X with generator A. If $\sigma_r(A) = \emptyset$, then $(T(t))_{t\geq 0}$ can be embedded in a C_0-group.

As consequence of Theorem 2.2.2, we have the corollary.

Corollary 2.2.2 *[12]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on Banach space X with the generator A. Then the following statements are equivalent.

1. $(T(t))_{t\geq 0}$ *is a isometric semigroup;*

2. *A and* $-A$ *are both dissipative operators and* $Im(I-A)=X$*.*

As a direct result of Corollaries 2.2.1 and 2.2.2 we have the following corollary.

Corollary 2.2.3 *[12]*
Let $(T(t))_{t\in\mathbb{R}}$ *be a* C_0*-group on Banach space* X *with the generator* A*. Then the following two assertions are equivalent.*

1. $(T(t))_{t\in\mathbb{R}}$ *is an isometric group;*

2. *A and* $-A$ *are both dissipative operators and* $Im(I\pm A)=X$*.*

The following theorem gives invariability of left invertible semigroups on Banach spaces under the bounded perturbation.

Theorem 2.2.3 *[12]*
Let $(T(t))_{t\geq 0}$ *be a left invertible semigroup on Banach space* X *with generator* A*. If* B *is a bounded linear operator on* X*, then the semigroup* $(T_B(t))_{t\geq 0}$ *generated by* $A+B$ *is also a left invertible semigroup.*

Proof
Let A be the generator of C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\|\leq Me^{\omega t}$ and B be a linear operator. By the perturbation theory of C_0-semigroups (see Theorem 1.3.4.1 Chapter 1), $A+B$ generates a C_0-semigroup $(T_B(t))_{t\geq 0}$ satisfying $\|T_B(t)\|\leq Me^{(\omega+M\|B\|)t}$, $t\geq 0$. Moreover, $(T_B(t))_{t\geq 0}$ satisfies the integral equation (1.3) (see Corollary 1.3.4.1 Chapter 1).
Using the integral equation (1.3) we get the estimate

$$\left\|\int_0^t T(t-s)BT_B(s)z_0\right\|\leq tM^2\|B\|e^{(2\omega+M\|B\|)t}\|z_0\|,\quad \forall z_0\in X,\quad t\geq 0.$$

Since $(T(t))_{t\geq 0}$ is a left invertible semigroup, according to Theorem 2.2.1, there exist constants $c>0$ and $\alpha>0$ such that

$$\|T(t)z_0\|\geq ce^{-\alpha t}\|z_0\|,\quad \forall z_0\in X, t>0.$$

Let $M_1=M^2\|B\|\mathrm{e}^{(2\omega+M\|B\|)}$ and $t_0<\min\left\{1,\dfrac{c}{2M_1}\mathrm{e}^{-\alpha}\right\}$, $z_0\in X$ then we have

$$\begin{aligned}\|T_B(t_0)z_0\|&\geq\|T(t_0)x\|-\left\|\int_0^t T(t-s)BT_B(s)\right\|\\&\geq\left(c\mathrm{e}^{-\alpha t_0}-t_0M_1\right)\|z_0\|\geq\left(c\mathrm{e}^{-\alpha}-t_0M_1\right)\|z_0\|\\&\geq\frac{1}{2}c\mathrm{e}^{-\alpha}\|z_0\|,\quad \forall z_0\in \mathrm{X},\end{aligned}$$

which means that $(T_B(t))_{t\geq 0}$, is a left invertible semigroup. The proof is then complete. ∎

Corollary 2.2.4 *[12]*
Let $(T(t))_{t\geq 0}$ be a left invertible semigroup on Banach space X with generator A. If $\sigma_r(A)=\emptyset$ and B is a linear bounded operator, then $A+B$ generates a C_0-semigroup which can be embedded in a C_0-group.

2.3 Admissibility of observation operator

In this section we shall discuss the admissibility of observation operators for left invertible semigroups. In the existing results on the admissibility, a result shows that if $B\in\mathcal{B}(U,\mathcal{D}(A))$ satisfies the condition

$$\|R(\lambda,A)B\|\leq\frac{M}{\sqrt{\Re\lambda}},\quad \Re\lambda>\omega,$$

then the control operator B is admissible for $(T(t))_{t\geq 0}$. There is no result on the observation operator for the left invertible semigroups. However, the following proposition shows that the left invertible semigroup can become a contraction semigroup.

Theorem 2.3.1 *[12]*
Let $(T(t))_{t\geq 0}$ be an exponentially stable and left invertible C_0-semigroup on Hilbert space Z. Then there exists an equivalent inner product on Z such that $(T(t))_{t\geq 0}$ is an exponentially stable and contraction semigroup.

Proof
Let $(T(t))_{t\geq 0}$ be an exponentially stable and left invertible C_0-semigroup. Then there exist positive constant M, c, α and δ such that $c\mathrm{e}^{-\alpha t}\|x\|\leq\|\mathrm{T(t)}x\|\leq M\mathrm{e}^{-\delta t}\|x\|,\quad \forall\mathrm{t}\geq 0, x\in Z$. Define an inner product on Z by

$$\langle x,y\rangle_1=\int_0^\infty\langle T(t)x,T(t)y\rangle\mathrm{d}t,\quad \forall x,y\in Z$$

then we have

$$\|x\|_1^2=\int_0^\infty\|T(t)x\|^2\ \mathrm{d}t$$

Clearly, $\|x\|_1$ is an equivalent norm on Z. In the sense of this norm we have

$$\begin{aligned}\|T(t)x\|_1^2&=\int_0^\infty\|T(s)T(t)x\|^2\ \mathrm{d}s=\int_t^\infty\|T(s)x\|^2\mathrm{ds}\\&\leq\int_0^\infty\|T(s)x\|^2\mathrm{ds}=\|x\|_1^2\end{aligned}$$

That is $(T(t))_{t\geq 0}$is an exponentially stable and contraction semigroup on Z. ■

For the contraction semigroups, the following result is due to Jacob and Partington, which is probably one of the most important results in the area of admissibility.

Lemma 2.3.1 *[12]*
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contraction on a separable Hilbert space with generator A and let $C \in \mathcal{B}(\mathcal{D}(A), \mathbb{C})$. Then C is infinite-time admissible if and only if C satisfies the condition that there exists a constant $M > 0$ such that

$$\|CR(\lambda, A)\| \leq \frac{M}{\sqrt{\Re\lambda}}, \quad \Re\lambda > 0,$$

Let A be the generator of a left invertible C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq M_1 e^{\omega t}$. We can choose a real $\alpha > \omega$ such that $T_\alpha(t) = e^{-\alpha t}T(t)$ is exponentially stable and contraction semigroup.
As a direct result of this lemma , we have the following result.

Theorem 2.3.2 *[12]*
Let $(T(t))_{t\geq 0}$ be the left invertible C_0-semigroup on Hilbert space Z. Let $C \in \mathcal{B}(\mathcal{D}(A), \mathbb{C}^n)$. Then C is admissible if and only if C satisfies the condition that, for positive contants M and ω,

$$\|CR(\lambda, A)\| \leq \frac{M}{\sqrt{\Re\lambda}}, \quad \Re\lambda > \omega,$$

This theorem shows that the Weiss class operators are admissible for the left invertible semigroups if Y is a finite-dimentional Hilbert space. However if $Im\ C$ and hence Y is infinite-dimentional, then the question of admissibility becomes more complicated. As a result the resolvent condition (1.4) is not a sufficient condition of admissibility for the left invertible semigroup.

Finally we close this section by giving a sufficient condition of exact observability for the left invertible semigroup.

Theorem 2.3.3 *[12]*
Let $(T(t))_{t\geq 0}$ be the left invertible semigroup on Hilbert space Z. Let $C \in \mathcal{B}(\mathcal{D}(A), Y)$ be a weiss class operator. If there exists a $\tau > 0$ such that

$$\|CT(\tau)x\| \geq \delta\|x\|, \quad \forall x \in \mathcal{D}(A)$$

then the system $\Sigma(A, -, C)$ is exactly observable in finite time.

Proof
Suppose that a Weiss class operator $C \in B(D(A), Y)$ satisfies the condition that there exists a constant $\delta > 0$ such that

$$\|CT(\tau)x\| \geq \delta\|x\|, \quad \forall x \in \mathcal{D}(A).$$

Then for any $x \in \mathcal{D}(A)$, it holds that

$$\delta^2 \int_0^\tau \|T(t)x\|^2 \, \mathrm{d}t \leq \int_0^\tau \|CT(\tau)T(t)x\|^2 \mathrm{d}t$$
$$\leq \int_0^\infty \|CT(\tau)T(t)x\| \leq \frac{\mathrm{e}^{4\alpha\tau}}{(1-\mathrm{e}^{-\tau})^2} M^2 K^2 \|x\|^2$$

Note that the semigroup $(T(t))_{t\geq 0}$ is left invertible. Denote

$$\varepsilon = \inf_{t\in[0,\tau]} \inf_{\|x\|=1} \|T(t)x\|.$$

Then we get

$$\delta^2\varepsilon^2\tau\|x\|^2 \leq \int_0^\tau \|T(t)x\|^2 \, \mathrm{d}t \leq \int_0^\tau \|CT(\tau)T(t)x\|^2 \, \mathrm{d}t$$
$$= \int_\tau^{2\tau} \|CT(t)x\|^2 \, \mathrm{d}t$$

The proof is then complete. ■

2.4 Exact controllability and right inverse of a C_0-semigroup

In this section we discuss the relation between the exact controllability and the right inverse of a C_0-semigroup, then by duality we pass to the left inverse of a C_0-semigroup using Lyapunov's equation defined under the following Theorem .

Theorem 2.4.1 *[6]*
Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z. Then the following conditions are equivalent :

(i) There exists a Hilbert space U and an operator $B \in \mathcal{B}(U, Z)$ such that the pair (A, B) is exactly controllable;

(ii) $(T(t))_{t\geq 0}$ admits a right-inverse C_0-semigroup $(S(t))_{t\geq 0}$ on Z, i.e., $T(t)S(t) = I$ (the identity on Z) for all $t \geq 0$;

(iii) $(T(t))_{t\geq 0}$ is surjective for all $t \geq 0$; and

(iv) There exists $t_0 > 0$ such that $T(t_0)$ is surjective.

From Theorem 2.4.1 we obtain some new information about the unique self-adjoint solution $K \in \mathcal{B}(Z)$ of the Lyapunov equation

$$2\Re\langle Ax, Kx\rangle = -\|x\|^2, \quad \forall x \in \mathcal{D}(A), \tag{2.1}$$

where A generates an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$. It is well known that K is defined by

$$Kx = \int_0^{+\infty} T^*(t)T(t)xdt, \quad \forall x \in Z$$

We may use the relation between exact controllability and Lyapunov equations to derive from Theorem 2.4.1 the following corollary

Corollary 2.4.1 *[6]*
Assume that A generates an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z. Then the following conditions are equivalent :

(i) The unique self-adjoint solution K of the Lyapunov equation (2.1) is coercive;

(ii) $(T(t))_{t\geq 0}$ admits a left-inverse C_0-semigroup on Z;

(iii) There exists t_0 such that $T(t_0)$ admits a bounded linear left-inverse.

Chapter 3

Left-invertible semigroups on Hilbert spaces

3.1 Introduction

In [6] Louis and Wexler showed that if a strongly continuous semigroup on a Hilbert space is left invertible for one (or equivalently all) positive time instants, then there exists a left inverse which is also a strongly continuous semigroup. Their proof uses optimal control and Riccati equations. In this chapter we present a shorter proof which uses Lyapunov equations. Furthermore, using this Lyapunov equation, we can show that any left-invertible semigroup is a bounded perturbation of an isometric semigroup, see Theorem 3.2.3. Moreover, we show that a C_0-semigroup is left invertible if and only if minus its generator can be extended to an infinitesimal generator of a C_0-semigroup.

3.2 Left Invertible Semigroups

Definition 3.2.1 (Left Invertible Semigroups)
We say that the C_0-semigroup $(T(t))_{t\geq 0}$ is left invertible on the Hilbert space Z if there exists a C_0-semigroup $(S(t))_{t\geq 0}$ such that $S(t)T(t) = I$ for all $t \geq 0$.

Definition 3.2.2 (Left Invertible Semigroups)
The C_0-semigroup $(T(t))_{t\geq 0}$ is left invertible if there exists a function $t \mapsto m(t)$ such that $m(t) > 0$ and for all $z_0 \in Z$ there holds

$$m(t)\,\|z_0\| \leq \|T(t)z_0\|, \quad t \geq 0 \tag{3.1}$$

Proposition 3.2.1
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z. Then tho following are equivalent

1. *$(T(t))_{t\geq 0}$ is left invertible,*

2. *There exists a $t_0 > 0$ such that $T(t_0)$ is left invertible, i.e., there exists a m_0 such that for all $z_0 \in Z$ there holds $m_0 \|z_0\| \leq \|T(t_0) z_0\|$.*

Proof

1.($\Longrightarrow$)

$(T(t))_{t\geq 0}$ is left invertible semigroup, then there exists a function $m(t)$ such that $\forall z_0 \in Z$ there holds

$$m(t) \|z_0\| \leq \|T(t)z_0\|, \forall t \geq 0,$$

it follows that there exists $t_0 > 0$ and m_0, such that $\forall z_0 \in Z$ there holds $m_0\|z_0\| \leq \|T(t)z_0\|$.

2.($\Longleftarrow$)

There exists $t_0 > 0$ such that $T(t_0)$ is left invertible, then by Theorem 2.2.1 there exists two constants $\alpha, c > 0$ such that $\|T(t)z_0\| \geq ce^{-\alpha t}\|z_0\|$, $\forall z_0 \in Z, t > 0$, then there exists a function $m(t) = ce^{-\alpha t} > 0$, such that $\forall z_0 \in Z$ we have $\|T(t)z_0\| \geq m(t)\|z_0\|$, then $(T(t))_{t\geq 0}$ is left invertible. ∎

Before giving the main result, we need the following Lemma.

Lemma 3.2.1

Let A_1, A_2 be the infinitesimal generators of the C_0-semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$, respectively. Then $X \in \mathcal{B}(Z)$ satisfies the Sylvester equation

$$\langle A_1 z_1, X z_2\rangle + \langle z_1, X A_2 z_2\rangle = 0, \quad z_1 \in \mathcal{D}(A_1), z_2 \in \mathcal{D}(A_2) \tag{3.2}$$

if and only if

$$T_1^*(t) X T_2(t) = X, \quad \text{for all } t \geq 0 \tag{3.3}$$

Moreover, if X is (boundedly) invertible, then

$$X^{-1} T_1^*(t) X T_2(t) = I, \quad \text{for all } t \geq 0$$

Thus, $\left(X^{-1}T_1^*(t)X\right)_{t\geq 0}$ is the left inverse of $(T_2(t))_{t\geq 0}$.

Proof

($\Longrightarrow$)We see that (3.3) is equivalent to $\langle T_1(t)z_1, XT_2(t)z_2\rangle = \langle z_1, Xz_2\rangle$, for all $z_1,\ z_2 \in Z$.
If (3.2) holds, then for $z_1 \in \mathcal{D}(A_1)$ and $z_2 \in \mathcal{D}(A_2)$ we have

$$\frac{d}{dt}\langle T_1(t)z_1, XT_2(t)z_2\rangle = \langle A_1T_1(t)z_1, XT_2(t)z_2\rangle + \langle T_1(t)z_1, XA_2T_2(t)z_2\rangle = 0$$

then we have

$$\int_0^t \frac{d}{dt}\langle T_1(s)z_1, XT_2(s)z_2\rangle = \langle T_1(t)z_1, XT_2(t)z_2\rangle - \langle T_1(0)z_1, XT_2(0)z_2\rangle = 0, \forall z_1 \in \mathcal{D}(A_1),\ z_2 \in \mathcal{D}(A_2).$$

hence

$$\langle T_1(t)z_1, XT_2(t)z_2\rangle = \langle z_1, Xz_2\rangle \forall z_1 \in \mathcal{D}(A_1),\ z_2 \in \mathcal{D}(A_2). \tag{3.4}$$

Now since the domains $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ are dense it follows that
$\forall z_1 \in Z$, there exists a sequence $(\omega_n) \in \mathcal{D}(A_1)$ *such that* $\lim_{n\to\infty} \omega_n = z_1$. and,
$\forall z_2 \in Z$,there exists a sequence $(\zeta_n) \in \mathcal{D}(A_2)$ *such that* $\lim_{n\to\infty} \zeta_n = z_2$.

We remark that (3.4) is true for $(\omega_n) \in \mathcal{D}(A_1)$ and $(\zeta_n) \in \mathcal{D}(A_2)$ i.e., $\langle T_1(t)\omega_n, XT_2(t)\zeta_n\rangle = \langle \omega_n, X\zeta_n\rangle\ \forall n \in \mathbb{N}$.
Passing to the limit we get

$$\lim_{n\to\infty}\langle T_1(t)\omega_n, XT_2(t)\zeta_n\rangle = \lim_{n\to\infty}\langle \omega_n, X\zeta_n\rangle \Rightarrow \langle T_1(t)z_1, XT_2(t)z_2\rangle = \langle z_1, Xz_2\rangle, \quad \forall z_1,\ z_2 \in Z.$$

So we conclude that (3.2) holds.

($\Longleftarrow$) If(3.3) holds, we show that $\langle A_1z_1, Xz_2\rangle + \langle z_1, XA_2z_2\rangle = 0, \quad z_1 \in \mathcal{D}(A_1),\ z_2 \in \mathcal{D}(A_2)$
We substitute the value of X into equation (3.2), we get

$$\begin{aligned}
\langle A_1z_1, Xz_2\rangle + \langle z_1, XA_2z_2\rangle &= \langle A_1z_1, T_1^*(t)XT_2(t)z_2\rangle + \langle z_1, T_1^*(t)XT_2(t)A_2z_2\rangle, \forall z_1 \in \mathcal{D}(A_1),\ \forall z_2 \in \mathcal{D}(A_2)\\
&= \langle T_1(t)A_1z_1, XT_2(t)z_2\rangle + \langle T_1(t)z_1, XT_2(t)A_2z_2\rangle, \forall z_1 \in \mathcal{D}(A_1),\ \forall z_2 \in \mathcal{D}(A_2)\\
&= \langle A_1T_1(t)z_1, XT_2(t)z_2\rangle + \langle T_1(t)z_1, XA_2T_2(t)z_2\rangle, \forall z_1 \in \mathcal{D}(A_1),\ \forall z_2 \in \mathcal{D}(A_2)\\
&= \frac{d}{dt}\langle T_1(t)z_1, XT_2(t)z_2\rangle, \forall z_1 \in \mathcal{D}(A_1),\ z_2 \in \mathcal{D}(A_2)\\
&= \frac{d}{dt}\langle z_1, Xz_2\rangle, \forall z_1 \in \mathcal{D}(A_1),\ \forall z_2 \in \mathcal{D}(A_2).
\end{aligned}$$

But

$$\frac{d}{dt}\langle z_1, Xz_2\rangle = 0,\ \forall z_1 \in \mathcal{D}(A_1),\ z_2 \in \mathcal{D}(A_2)$$

Thus

$$\langle A_1z_1, Xz_2\rangle + \langle z_1, XA_2z_2\rangle = 0, \quad z_1 \in \mathcal{D}(A_1),\ \forall z_2 \in \mathcal{D}(A_2)$$

So we conclude that (3.3) is true.
Moreover

$$\begin{aligned}
\text{if X is boundedly invertible} &\Longrightarrow X^{-1}T_1^*(t)XT_2(t) = X^{-1}X\\
&\Longrightarrow X^{-1}T_1^*(t)XT_2(t) = I
\end{aligned}$$

Thus $X^{-1}T_1^*(t)X$ is the left inverse of $(T_2(t))_{t\geq 0}$. ■

Now we can formulate and prove the main result of this chapter.

Theorem 3.2.1

Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z. Then $(T(t))_{t\geq 0}$ is left invertible if and only if $-A$ can be extended to an infinitesimal generator of a C_0-semigroup.

Proof

(1$\Longrightarrow$2)

Assume that $(T(t))_{t\geq 0}$ is left invertible, then there exists a C_0-semigroup $(S(t))_{t\geq 0}$ such that $S(t)T(t) = I$. Let A_2 with domain $\mathcal{D}(A_2)$ be the infinitisimal generator of $(S(t))_{t\geq 0}$. For $z \in Z$, we have that

$$S(t)z - z = S(t)z - S(t)T(t)z = S(t)(z - T(t)z)$$

For $z \in \mathcal{D}(A) = \mathcal{D}(-A)$, we obtain, also using the strong continuity of $(T(t))_{t\geq 0}$,

$$\begin{aligned} -Az = \lim_{t\to 0} - \left\{ \frac{T(t)z - z}{t} \right\} = \lim_{t\to 0} \frac{z - T(t)}{t} = \lim_{t\to 0} \frac{S(t)T(t)z - T(t)z}{t} &= \lim_{t\to 0} \frac{(S(t)-I)T(t)z}{t} \\ &= \lim_{t\to 0} \frac{(S(t)-I)z}{t} \\ &= \lim_{t\to 0} \frac{S(t)z - z}{t} \\ &= A_2 z, \end{aligned}$$

Then $\forall z \in \mathcal{D}(A) = \mathcal{D}(-A) \Rightarrow z \in \mathcal{D}(A_2)$,i.e., $\mathcal{D}(-A) \subset \mathcal{D}(A_2)$,
and $\forall z \in \mathcal{D}(A),\ -Az = A_2 z \Rightarrow -A \subset A_2$, then we have $A_2^* \subset -A^*$.

(2$\Longrightarrow$1)

Assume now that A_2 is the infinitesimal generator of the C_0-semigroup $(S(t))_{t\geq 0}$ and that A_2 is an extension of $-A$,i.e., $\mathcal{D}(-A) = \mathcal{D}(A) \subset \mathcal{D}(A_2)$ and $\forall z \in \mathcal{D}(A),\ -Az = A_2 z$, then we deduce from the definition of the adjiont that $-A^*$ is the extension of A_2^* ($A_2^* \subset -A^*$) and $\mathcal{D}(A_2^*) \subset \mathcal{D}(A^*)$. For $z_1 \in \mathcal{D}(A_2^*)$ and $z_2 \in \mathcal{D}(A)$, we have

$$\begin{aligned} \langle A_2^* z_1, z_2 \rangle + \langle z_1, A z_2 \rangle &= \langle A_2^* z_1, z_2 \rangle + \langle A^* z_1, z_2 \rangle \\ &= \langle A_2^* z_1, z_2 \rangle + \langle -A_2^* z_1, z_2 \rangle = 0. \end{aligned}$$

By Lemma 3.2.1, we conclude that $S(t)T(t) = I$, for all $t \geq 0$. ∎

Theorem 3.2.2

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z with infinitesimal generator A. Then the following are equivalent.

1. *$(T(t))_{t\geq 0}$ is left invertible;*

2. *The system* $\Sigma(A,-,I)$ *is exactly observable in finite-time;*

3. *There exists a Hilbert space* Y *and an operator* $C \in \mathcal{B}(Z,Y)$ *such that* $\Sigma(A,-,C)$ *is exactly observable in finite-time;*

4. *There exists an* $\omega \in \mathbb{R}$*, a Hilbert space* Y*, an operator* $C \in \mathcal{B}(Z,Y)$*, and a positive symmetric operator* $X \in \mathcal{B}(Z)$ *such that* X *is (boundedly) invertible and satisfies the Lyapunov equation*

$$\langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle = -\langle Cz_1, Cz_2\rangle, \tag{3.5}$$

for all $z_1, z_2 \in \mathcal{D}(A)$*;*

5. *There exists a* C_0*-semigroup* $(S(t))_{t\geq 0}$ *such that* $S(t)T(t) = I$ *for all* $t \geq 0$*.*

Proof

$(1 \implies 2)$

Since (3.1) holds $\Leftrightarrow$ There exists a function $m(t) > 0, \forall z_0 \in Z, m(t)\|z_0\| \leq \|T(t)z_0\|, t \geq 0$;

$$\Rightarrow \forall t_0 > 0, \forall z_0 \in Z, \int_0^{t_0} m^2(t)\|z_0\|^2 dt \leq \int_0^{t_0} \|T(t)z_0\|^2 dt;$$
$$\Rightarrow \forall t_0 > 0, \forall z_0 \in Z, \int_0^{t_0} m^2(t)dt\|z_0\|^2 \leq \int_0^{t_0} \|T(t)z_0\|^2 dt;$$
$$\Rightarrow \forall t_0 > 0, \exists m > 0, \forall z_0 \in Z, \int_0^{t_0} m^2(t)dt\|z_0\|^2 = m\|z_0\|^2 \leq \int_0^{t_0} \|T(t)z_0\|^2 dt;$$

$\Rightarrow$ The system$\Sigma(A,-,I)$ is exactly observable in finite-time.

$(3 \implies 4)$

First, the infinitesimal generator $A - \omega I = B$ is generated by the semigroup $e^{-\omega t}T(t) = S(t)$, therefore $B + \omega I = A$ is generated by the semigroup $e^{\omega t}S(t) = T(t)$.

Choose $\omega \in \mathbb{R}$ larger than the growth bound of $(T(t))_{t\geq 0}$. Then we have that the semigroup $(e^{-\omega t}T(t))$ is exponentially stable, i.e., there exists a constant $K \geq 1$ and $\alpha > 0$, such that

$$\|(e^{-\omega t}T(t))\| \leq Ke^{-\alpha t}, \ \forall t \geq 0.$$

Now for $z \in Z$ we have

$$\begin{aligned} m\|z\|^2 \leq \int_0^{t_0} \|CT(t)z\|^2 dt = \int_0^{t_0} \|Ce^{\omega t}S(t)z\|^2 dt = \int_0^{t_0} e^{2\omega t}\|CS(t)z\|^2 dt \\ &\leq m_1 \int_0^{t_0} \|Ce^{-\omega t}T(t)z\|^2 dt, \\ &\leq m_1 \int_0^{t_0} e^{-2\omega t}\|CT(t)z\|^2 dt, \\ &\leq m_1 \int_0^{\infty} e^{-2\omega t}\|CT(t)z\|^2 dt, \\ &= m_1 \int_0^{\infty} \|Ce^{-\omega t}T(t)z\|^2 dt, \\ &\leq m_1 \int_0^{\infty} \|C\|^2\|e^{-\omega t}T(t)z\|^2 dt, \\ &\leq m_1 \int_0^{\infty} \|C\|^2 K^2 e^{-2\alpha t}\|z\|^2 dt, \\ &= m_1\|C\|^2 \int_0^{\infty} e^{-2\alpha t} dt \|z\|^2, \\ &= m_1 M\|z\|^2. \end{aligned}$$

Define $\langle z_1, Xz_2\rangle = \int_0^\infty e^{-2\omega t}\langle CT(t)z_1, CT(t)z_2\rangle dt$, then we have
a)X is symmetric since in one hand we have

$$\langle z_1, Xz_2\rangle = \int_0^\infty e^{-2\omega t}\langle CT(t)z_1, CT(t)z_2\rangle dt = \overline{\langle Xz_2, z_1\rangle}, \ \forall z_1,\ z_2 \in Z. \tag{3.6}$$

on the other hand we have

$$\overline{\langle z_1, Xz_2\rangle} = \int_0^\infty e^{-2\omega t}\langle CT(t)z_2, CT(t)z_1\rangle dt = \langle z_2, Xz_1\rangle, \ \forall z_1,\ z_2 \in Z. \tag{3.7}$$

From(3.6) and (3.7) we obtain $\langle Xz_2, z_1\rangle = \langle z_2, Xz_1\rangle, \ \forall z_1,\ z_2 \in Z$.
b) By the above relation

$$m\|z\|^2 \leq m_1 \int_0^\infty e^{-2\omega t}\|CT(t)z\|^2 dt \leq m_1 M\|z\|^2, \ \forall z \in Z$$

equivalently

$$m\langle z, z\rangle \leq m_1\langle z, Xz\rangle \leq m_1 M\langle z, z\rangle, \ \forall z \in Z$$

then

$$\frac{m}{m_1}\langle z, z\rangle \leq \langle z, Xz\rangle \leq M\langle z, z\rangle, \ \forall z \in Z \tag{3.8}$$

which (3.8) is equivalent to

$$\frac{m}{m_1} \leq X \leq MI$$

Thus X is a bounded operator with bounded inverse.

c) Now we can show that X is a solution to the Lyapunov equation

$$\langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle = -\langle Cz_1, Cz_2\rangle\,,\ \forall z_1,\ z_2 \in \mathcal{D}(A)$$

We have

$$\begin{aligned}\langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle &= \int_0^\infty e^{-2\omega t}\langle CT(t)(A-\omega I)z_1, CT(t)z_2\rangle dt \\ &+ \int_0^\infty e^{-2\omega t}\langle CT(t)z_1, CT(t)(A-\omega I)z_2\rangle dt \\ &= \int_0^\infty \langle Ce^{-\omega t}T(t)(A-\omega I)z_1, Ce^{-\omega t}T(t)z_2\rangle dt \\ &+ \int_0^\infty \langle Ce^{-\omega t}T(t)z_1, Ce^{-\omega t}T(t)(A-\omega I)z_2\rangle dt \\ &= \int_0^\infty \frac{d}{dt}\langle Ce^{-\omega t}T(t)z_1, Ce^{-\omega t}T(t)z_2\rangle dt \\ &= [\langle Ce^{-\omega t}T(t)z_1, Ce^{-\omega t}T(t)z_2\rangle dt]_0^{+\infty} \\ &= -\langle Cz_1, Cz_2\rangle.\end{aligned}$$

Thus X is a solution to the Lyapunov equation (3.5).
($4 \Longrightarrow 5$)
We rewrite the Lyapunov equation (3.5) to the Sylvester equation

$$\begin{aligned}\text{We have (3.5)} &\Leftrightarrow \langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle = -\langle Cz_1, Cz_2\rangle\,,\ \forall z_1,\ z_2 \in \mathcal{D}(A) \\ &\Rightarrow \langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle + \langle XX^{-1}C^*Cz_1, z_2\rangle = 0,\ \forall z_1,\ z_2 \in \mathcal{D}(A) \\ &\Rightarrow \langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle + \langle X^{-1}C^*Cz_1, Xz_2\rangle = 0,\ \forall z_1,\ z_2 \in \mathcal{D}(A) \\ &\Rightarrow \langle (A-\omega I + X^{-1}C^*C)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle = 0,\ \forall z_1,\ z_2 \in \mathcal{D}(A)\end{aligned}$$

Since this can be seen as (3.2) with $A_1 = A-\omega I + X^{-1}C^*C$ and $A_2 = A-\omega I$, and since a bounded perturbation of an infinitisimal generator is still an infinitesimal generator, we obtain by Lemma 3.2.1 that the semigroup generated by $A-\omega I$ is left invertible, and

$$X^{-1}T_1^* X T(t)e^{-\omega t} = I,$$

Where $(T_1(t))_{t\geq 0}$ is the semigroup generated by $A-\omega I + X^{-1}C^*C$. Thus, $S(t) = X^{-1}T_1^*Xe^{-\omega t}$ is the left inverse of $T(t)$.

($5 \Longrightarrow 1$) Is clear by definition. ∎

Remark 3.2.1

We remark that the Theorem 3.2.2 also holds for $C \in \mathcal{B}(Z,Y)$ for which $\Sigma(A,-,C)$ is exactly

observable in infinite-time, provided we have for all $z_0 \in Z$ *that* $\int_0^\infty \|CT(t)z_0\|^2 dt < \infty$, *or equivalently*
$\int_0^\infty \|CT(t)z_0\|^2 dt < m\|z_0\|^2$ *for some* m *independent of* z_0. *Just define* X *in part 4 as*

$$\langle z_1, Xz_2\rangle = \int_0^\infty \langle CT(t)z_0, CT(t)z_2\rangle dt$$

and take $\omega = 0$. *The condition* $\int_0^\infty \|CT(t)z_0\|^2 dt < \infty$ *for all* $z_0 \in Z$ *is known as infinite-time admissibility or output stability.*

In the proof of Theorem 3.2.2 $3 \Rightarrow 4$ we used that $e^{-\omega t}CT(t)z_0$ is square integrable on $[0,\infty)$. Even when $(T(t))_{t\geqslant 0}$ is exponentially stable, this does not imply that C is bounded. The class of operators $C : \mathcal{D}(A) \mapsto Y$ for which this holds is called admissible. Hence it may seem that if item 3 holds for an admissible C, then item 1 will hold as well. The following example shows that this does not hold for general admissible output operators.

Example 3.2.1
Consider the left-shift semigroup on $L^2(0,1)$, *i.e.,*

$$(T(t)f)(\eta) = \begin{cases} f(\eta+t) & \eta+t \in [0,1] \\ 0 & \eta+t \geq 1 \end{cases}$$

with the observation at $\eta = 0$, *i.e.,*

$$Cf = f(0)$$

We show that $(T(t))_{t\geq 0}$ *is not left invertible ;*
For $t \in [0,1]$ *we have*

$$\|T(t)f\|^2 = \langle T(t)f, T(t)f\rangle_{L^2(0,1)} = \int_0^1 |(T(t)f)(s)|^2 ds = \int_0^1 |f(t+s)|^2 ds = \|f(t)\|^2$$

then

$$\|T(t)f\| = \|f(t)\|$$

On the other hand we have

$$\begin{aligned} & \|T(t)f\| = \|f(t)\| \\ \textit{equivalently} \quad & \|e^{At}f\| = \|f(t)\| \\ \textit{then} \quad & \|f(t)\| = \|e^{At}f\| \leq \|e^{At}\|\|f\| \\ \textit{hence} \quad & \|T(t)f\| = \|f(t)\| \leq m_0\|f\|, \quad t \in [0,1] \quad \textit{with} \quad m_0 = \|e^{At}\| \end{aligned}$$

there exists $t \in [0,1]$ *such that* $\|T(t)f\| \leq m_0\|f\|$, *thus* $(T(t))_{t\geq 0}$ *is not left invertible,*
But since
$\int_0^1 |CT(t)f|^2 dt = \|f\|^2$, *then*
the operator C *is admissible and exactly observable in finite-time.*

It is well-known that the left inverse need not to be unique. However, since in item 5 the left inverse is a C_0-semigroup, it might be expected that this extra structure induces uniqueness. The following example shows that the left-inverse semigroup in item 5 of Theorem 3.2.2 need not to be unique.

Example 3.2.2

As Hilbert space Z we take $L^2(0,\infty)\bigoplus L^2(0,1)$. Furthermore, we take

$$A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\dot{f}_1 \\ -\dot{f}_2 \end{pmatrix}$$

with domain

$$\mathcal{D}(A_1) = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in Z \mid f_1 \in H^1(0,\infty), f_2 \in H^1(0,1) \text{ and } f_1(0) = \alpha f_2(0), f_2(1) = \sqrt{2} f_2(0) \right\},$$

where $H^1(\Omega)$ denotes the Sobolev space of $L^2(\Omega)$-functions whose (distributional) derivative lies in $L^2(\Omega)$. The operator A_2 is defined similarly,

$$A_2\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -\dot{g}_1 \\ -\dot{g}_2 \end{pmatrix}$$

with domain

$$\mathcal{D}(A_2) = \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in Z \mid g_1 \in H^1(0,\infty),\ g_2 \in H^1(0,1) \text{ and } g_1(0) = 0, \sqrt{2} g_2(1) = g_2(0) \right\},$$

It is not hard to show that these operators generate strongly continuous semigroups on Z. For $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{D}(A)$, there holds

$$\begin{aligned}
\left\langle A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle &= \int_0^\infty -\dot{f}_1(x)\overline{f_1(x)}dx + \int_0^\infty f_1(x)(-\overline{\dot{f}_1(x)})dx \\
&+ \int_0^1 -\dot{f}_2(x)\overline{f_2(x)}dx + \int_0^1 f_1(x)(-\overline{\dot{f}_2(x)})dx \\
&= -[|f_1(x)|^2]_0^\infty - [|f_2(x)|^2]_0^1 \\
&= -0 + |\alpha f_2(0)|^2 - 2|f_2(0)|^2 + |f_2(0)|^2 \leq 0,
\end{aligned}$$

for $0 \leq \alpha \leq 1$. Thus, for these values of α, A_1 generates a contraction semigroup.

For $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{D}(A)$*, there holds*

$$\begin{aligned}
\left\langle A_1\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, A_1\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right\rangle &= \int_0^\infty -\dot{g}_1(x)\overline{g_1(x)}dx + \int_0^\infty g_1(x)(-\overline{\dot{g}_1(x)})dx \\
&+ \int_0^1 -\dot{g}_2(x)\overline{g_2(x)}dx + \int_0^1 g_2(x)(-\overline{\dot{g}_2(x)})dx \\
&= -[|g_1(x)|^2]_0^\infty - [|g_2(x)|^2]_0^1 \\
&= -0 + |g_1(0)|^2 - |g_2(1)|^2 + |g_2(0)|^2 \\
&= -\left|\frac{1}{\sqrt{2}}g_2(0)\right|^2 + |g_2(0)|^2 \leq 0.
\end{aligned}$$

Next we show that (3.2) is satisfied for $X = I$ *for all* α*.*

$$\begin{aligned}
\left\langle A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, A_2\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right\rangle &= \int_0^\infty -\dot{f}_1(x)\overline{g_1(x)}dx + \int_0^\infty f_1(x)(-\overline{\dot{g}_1(x)})dx \\
&+ \int_0^1 -\dot{f}_2(x)\overline{g_2(x)}dx + \int_0^1 f_2(x)(-\overline{\dot{g}_2(x)})dx \\
&= -[f_1(x)\overline{g_1(x)}]_0^\infty - [f_2(x)\overline{g_2(x)}]_0^1 \\
&= -0 + 0 - f_2(1)\overline{g_2(1)} + f_2(0)\overline{g_2(0)} \\
&= -\sqrt{2}f_2(0)\frac{1}{\sqrt{2}}\overline{g_2(0)} + f_2(0)\overline{g_2(0)} = 0,
\end{aligned}$$

where we used the boundary conditions. Hence by Lemma 3.2.1 we obtain $T_1^*(t)T_2(t) = I$*, thus* A_1^* *generates the left inverse* $(T_1^*(t))_{t\geq 0}$ *of* $(T_2(t))_{t\geq 0}$*. but because of its dependance on* α *it is not unique.*

Theorem 3.2.3

Let $(T(t))_{t\geq 0}$ *be a* C_0*-semigroup on the Hilbet space* Z *with generator* A*. Then the following are equivalent*

1. $(T(t))_{t\geq 0}$ *is left invertible;*

2. *There exists a bounded operator* Q *and an equivalent inner product such that* $A + Q$ *generates an isometric semigroup in the norm.*

Proof

(1$\Longrightarrow$2)

Suppose that $(T(t))_{t\geq 0}$ is left invertible. By Theorem 3.2.2, $\exists\ \omega \in \mathbb{R}$, a Hilbert space Y, $C \in \mathcal{B}(Z, Y)$ and an operator $X \in Z$ boundedly invertible, such that

$$\langle (A - \omega I)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = -\langle z_1, z_2\rangle, \forall z_1, z_2 \in \mathcal{D}(A) \qquad (3.9)$$

Now we can write this equation as

$$\begin{aligned}
(3.9) &\Leftrightarrow \langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle = -\frac{1}{2}\langle z_1, z_2\rangle - \frac{1}{2}\langle z_1, z_2\rangle, \quad \forall z_1, z_2 \in \mathcal{D}(A) \\
&\Leftrightarrow \langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle + \frac{1}{2}\langle XX^{-1}z_1, z_2\rangle + \frac{1}{2}\langle z_1, XX^{-1}z_2\rangle = 0, \\
&\Leftrightarrow \langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle + \frac{1}{2}\langle X^{-1}z_1, Xz_2\rangle + \frac{1}{2}\langle z_1, XX^{-1}z_2\rangle = 0, \\
&\Leftrightarrow \langle (A-\omega I + \frac{1}{2}X^{-1})z_1, Xz_2\rangle + \langle z_1, X(A-\omega I + \frac{1}{2}X^{-1})z_2\rangle = 0, \quad \forall z_1, z_2 \in \mathcal{D}(A).
\end{aligned}$$

By defining $Q = -\omega I + \frac{1}{2}X^{-1}$, taking as new product $\langle z_1, z_2\rangle_{new} = \langle z_1, Xz_2\rangle$, then the equation

$$\langle (A-\omega I + \frac{1}{2}X^{-1})z_1, Xz_2\rangle + \langle z_1, X(A-\omega I + \frac{1}{2}X^{-1})z_2\rangle = 0, \quad \forall z_1, z_2 \in \mathcal{D}(A)$$

becomes in the following form

$$\langle (A-\omega I + \frac{1}{2}X^{-1})z_1, z_2\rangle_{new} + \langle z_1, (A-\omega I + \frac{1}{2}X^{-1})z_2\rangle_{new} = 0, \ \forall z_1, z_2 \in \mathcal{D}(A),$$

then using Lemma 3.2.1 we obtain $S^*(t)S(t) = I$.

Thus, the semigroup $(S(t))_{t\geq 0}$ generated by $A+Q$ is isometric.

■

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