Measure Theory and Integration

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June 21, 2020

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Chapter 1

Positive Measures

1. Algebras of Sets

This section is intented to give the basic structures on sets, needed for the definition and properties of measures. We start with the following:

Preliminaries:

Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. If I is any nonempty set, a function $f : I \longrightarrow \mathcal{P}(X)$ defines a family $\{A_i,\, i \in I\}$ of subsets of X, with $A_i = f(i) \in \mathcal{P}(X)$. For such family we perform the union and the intersection by:

$\underset{i}{\cup} A_i = \{x : \exists i \in I,\, x \in A_i\}$

$\underset{i}{\cap} A_i = \{x : \forall i \in I,\, x \in A_i\}$

Let us recall the frequently used **De Morgan's Laws:**

$\left(\underset{i}{\cup} A_i\right)^c = \underset{i}{\cap} A_i^c, \quad \left(\underset{i}{\cap} A_i\right)^c = \underset{i}{\cup} A_i^c$

valid for any family $\{A_i,\, i \in I\}$, where A^c denotes the complement of the set A.

Definition 1.1.

Let $\mathcal{A}$ be a family of subsets of X.
We say that $\mathcal{A}$ is an algebra on X if:
(1) X, ϕ are in $\mathcal{A}$
(2) For every subset A in $\mathcal{A}$, the complement A^c of A is in $\mathcal{A}$
(3) For every subsets $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$

Example 1.2.

(a) For any X the power set $\mathcal{P}(X)$ is an algebra
(b) Let X be a set and let $\mathcal{A}$ be the family given by $\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ finite}\}$. It is not difficult to check that $\mathcal{A}$ is an algebra, using the De Morgan's Laws given in the Preliminaries
(c) If $\mathcal{A}$ is an algebra and if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$
(d) For any finite sequence $A_1, ..., A_n$ in $\mathcal{A}$ the union $\overset{n}{\underset{1}{\cup}} A_i$ and the intersection $\overset{n}{\underset{1}{\cap}} A_i$ are in $\mathcal{A}$.

Definition 1.3.

Let $\mathcal{F}$ be a family of subsets of X.
We say that $\mathcal{F}$ is a $\sigma-$field or $\sigma-$algebra on X if:
(1) X, ϕ are in $\mathcal{F}$
(2) For every subset A in $\mathcal{F}$, the complement A^c of A is in $\mathcal{F}$
(3) For every sequence (A_n) of subsets $A_n \in \mathcal{F}$, $\underset{n}{\cup} A_n \in \mathcal{F}$

The pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ a $\sigma-$field on X is called a measurable space and sets A in $\mathcal{F}$ are called measurable sets.

Examples 1.4.
(a) For any X the power set $\mathcal{P}(X)$ is a $\sigma-$field on X.
(b) Let X be an infinite set and let $\mathcal{F}$ be the family given by $\mathcal{F} = \{A \subset X : A \text{ or } A^c \text{ countable}\}$.
Then it is not difficult to prove that $\mathcal{F}$ is a $\sigma-$field on X
(use the De Morgan's Laws given in the Preliminaries).
(c) Every $\sigma-$field on X is an algebra, but the converse is not true as is shown by the following:
take $X = \mathbb{Z}$, the integers and the algebra $\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ finite}\}$,
put $A_n = \{n\}, n \geq 0$; then $A_n \in \mathcal{A}, \forall n \geq 0$, but $\underset{n\geq 0}{\cup} A_n \notin \mathcal{A}$.

Remark 1.5.
(a) If $\mathcal{F}$ is a $\sigma-$field on X, then for every sequence (A_n) in $\mathcal{F}$, $\underset{n}{\cap} A_n \in \mathcal{F}$.
(b) For every sequence (A_n) such that $A_i \cap A_j = \phi$, for $i \neq j$
we denote the set $\underset{n}{\cup} A_n$ by $\sum\limits_n A_n$.

2. Exercises

1. Prove that the family $\mathcal{F}$ is a $\sigma-$field on X, if and if the following conditions are satisfied:
(a) $\phi \in \mathcal{F}$
(b) For any finite sequence $A_1, ..., A_n$ in $\mathcal{F}$, $\overset{n}{\underset{1}{\cap}} A_i \in \mathcal{F}$
(c) For every sequence (A_n) such that $A_i \cap A_j = \phi$, for $i \neq j$. we have $\sum\limits_n A_n \in \mathcal{F}$

2. For every sequence (A_n), define the sequence (B_n) by the following recipe:
$B_1 = A_1,\ B_2 = A_2 \backslash A_1,\ B_3 = A_3 \backslash (A_1 \cup A_2), ...B_n \backslash \left(\underset{i<n}{\cup} A_i\right)$
Prove that $\underset{n}{\cup} A_n = \sum\limits_n B_n$.

3. Generations

Lemma 3.1.
Let $\mathcal{F}_i$, $i \in I$ be an arbitrary family of $\sigma-$fields
(resp. algebras). Then the family $\underset{i}{\cap} \mathcal{F}_i$ is a $\sigma-$field (resp. algebra).
Proof. Straightforward.■
Corollary 3.2.
Let $\mathcal{H}$ be a family of subsets of a set X
Then there exist a smallest $\sigma-$field on X containing $\mathcal{H}$, denoted by $\sigma(\mathcal{H})$.
Smallest is taken in the sens of the inclusion ordering.
$\sigma(\mathcal{H})$.is called the $\sigma-$field generated by $\mathcal{H}$.
Proof. Let $\mathfrak{I} = \{\mathcal{F}\text{: } \mathcal{F}\ \sigma - \text{field on } X, \text{ with } \mathcal{H} \subset \mathcal{F}\}$
then by **Lemma 3.1**, $\underset{\mathcal{F}\in\mathfrak{I}}{\cap} \mathcal{F}$ is a $\sigma-$field on X and it is clear that:
$\sigma(\mathcal{H}) = \underset{\mathcal{F}\in\mathfrak{I}}{\cap} \mathcal{F}$.■

Example 3.3.
(a) Let $\mathcal{H}$ be a family given by one subset A, $\mathcal{H} = \{A\}$
then $\sigma(\mathcal{H}) = \{A, A^c, \phi, X\}$.
(b) If $\mathcal{I}$ is the family of one point sets given by $\mathcal{I} = \{\{x\} : x \in X\}$
then we have $\sigma(\mathcal{I}) = \{A \subset X : A \text{ or } A^c \text{ countable}\}$(see **Example 1.4** ($b$))

Definition 3.4.(Product $\sigma-$field)
Let $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$ be measurable spaces. Consider on the product set $X_1 \times X_2$ the family $\mathcal{R} = \{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$.
The product $\sigma-$field on $X_1 \times X_2$ is defined by $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{R})$.
The measurable space $(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ is called the product of $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$.

Definition 3.5. (Borel $\sigma-$field)
Let X be a topological space. The Borel $\sigma-$field of X is the $\sigma-$field generated by the family of all the open sets of X.
It is denoted by $\mathcal{B}_X$. Sets in $\mathcal{B}_X$ are called Borel sets of X. One can see that $\mathcal{B}_X$ is also generated by the closed sets of X.

Proposition 3.6.
The Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$of $\mathbb{R}$ is generated by the open intervals of $\mathbb{R}$.
In fact $\mathcal{B}_{\mathbb{R}}$ is generated by the family $\{]-\infty, t[\,, t \in \mathbb{R}\}$.

Proof. Every open set of $\mathbb{R}$ is the union of a sequence of open intervals.■

Definition 3.7. (Monotone family)
Let $\mathcal{M}$ be a family of subsets of a set X. $\mathcal{M}$ is said to be monotone if:
(i) For any sequence (A_n) with $A_1 \subset A_2 \subset ... \subset A_n \subset ...$, we have $\underset{n}{\cup} A_n \in \mathcal{M}$
(ii) For any sequence (A_n) with $A_1 \supset A_2 \supset ... \supset A_n \supset ...$, we have $\underset{n}{\cap} A_n \in \mathcal{M}$

Example 3.8.
(a) Any $\sigma-$field is a monotone family
(b) Let $\mathcal{A}$ be an algebra, then $\mathcal{A}$ is a $\sigma-$field iff $\mathcal{A}$ is a monotone family.

Lemma 3.9.
Let $\mathcal{M}_i$, $i \in I$ be an arbitrary class of monotone families
Then the family $\underset{i}{\cap} \mathcal{M}_i$ is a monotone family.

Proof. Straightforward.■

Corollary 3.10.
Let $\mathcal{H}$ be a family of subsets of a set X
Then there exist a smallest monotone family on X containing $\mathcal{H}$, denoted by $\mathcal{M}(\mathcal{H})$. Smallest is taken in the sens of the inclusion ordering.
$\mathcal{M}(\mathcal{H})$.is called the monotone family generated by $\mathcal{H}$.

Proof. Let $\mathfrak{I} = \{\mathcal{M}$: $\mathcal{M}$ monotone family on X, with $\mathcal{H} \subset \mathcal{M}\}$
then by **Lemma 3.9**, $\underset{\mathcal{M} \in \mathfrak{I}}{\cap} \mathcal{M}$ is a monotone family on X and it is clear that:
$\mathcal{M}(\mathcal{H}) = \underset{\mathcal{M} \in \mathfrak{I}}{\cap} \mathcal{M}$.■

Theorem 3.11.
Let $\mathcal{A}$ be an algebra on the set X. Then the $\sigma-$field generated by $\mathcal{A}$ is identical to the monotone family generated by $\mathcal{A}$.

Proof. Put $\mathcal{M} = \mathcal{M}(\mathcal{A})$, $\mathcal{B} = \sigma(\mathcal{A})$. Then $\mathcal{M} \subset \mathcal{B}$ (**Example 3.8.** (a)).
To show that $\mathcal{B} \subset \mathcal{M}$ it is enough to prove that $\mathcal{M}$ is an algebra
(see **Example 3.8.** (b))
First we prove that $B \in \mathcal{M} \Longrightarrow B^c \in \mathcal{M}$. To this end let $\mathcal{M}^{'} = \{B \in \mathcal{M} : \quad B^c \in \mathcal{M}\}$
Then we have $\mathcal{A} \subset \mathcal{M}^{'} \subset \mathcal{M}$. Moreover $\mathcal{M}^{'}$ is monotone and so $\mathcal{M}^{'} = \mathcal{M}$.
It remains to prove that $\mathcal{M}$ is stable by intersection. For each $A \in \mathcal{M}$, consider the family $\mathcal{M}_A = \{B \in \mathcal{M} : \quad A \cap B \in \mathcal{M}\}$, then $\mathcal{M}_A$ is a monotone family with $\mathcal{M}_A \subset \mathcal{M}$. Moreover if $A \in \mathcal{A}$, we have $\mathcal{A} \subset \mathcal{M}_A$, so we deduce that $\mathcal{M}_A = \mathcal{M}$. On the other hand it is clear that $A \in \mathcal{M}_B$ iff $B \in \mathcal{M}_A$, therefore $A \in \mathcal{M}_B$ for every $A \in \mathcal{A}$ and $B \in \mathcal{M}$. Finally $\mathcal{M}_B = \mathcal{M}$, for all $B \in \mathcal{M}$. This proves that $\mathcal{M}$ is an algebra.$\blacksquare$

4. Exercises

3. Let $\mathcal{A}$ be a family of subsets of a set X. If E is any subset in X, we define the trace of $\mathcal{A}$ on E by the family $\mathcal{A} \cap E = \{A \cap E, A \in \mathcal{A}\}$.
Prove that $\sigma(\mathcal{A} \cap E) = \sigma(\mathcal{A}) \cap E$.
4. Let $\mathcal{S}$ be a family of subsets of a set X. We say that $\mathcal{S}$ is a semialgebra if it satisfies:

(a) ϕ, X are in $\mathcal{S}$
(b) If A, B are in $\mathcal{S}$ then $A \cap B$ is in $\mathcal{S}$
(c) If A is in $\mathcal{S}$ then $A^c = \sum_1^n A_k$, where the sets A_k are pairwise disjoint in $\mathcal{S}$.

Prove that the algebra generated by the semialgebra $\mathcal{S}$ is the family
$$\mathcal{A} = \left\{ A : A = \sum_1^n S_k, \text{ where the } S_k \text{ are pairwise disjoint in } \mathcal{S}. \right\}$$
5. Let $\mathbb{R}$ the set of real numbers equiped with the usual topology, prove that the family of all intervals is a semialgebra.
6. Let $\mathcal{S}_1$,$\mathcal{S}_2$ be semialgebras on the set X and consider the family $\mathcal{S} = \{S_1 \cap S_2, \quad S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}$.
Prove that $\mathcal{S}$ is a semialgebra and that the algebra generated by $\mathcal{S}$ is identical to the algebra generated by $\mathcal{S}_1$ and $\mathcal{S}_2$.
7. Let $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$ be measurable spaces. Prove that the family $\{A_1 \times A_2 : \quad A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$ is a semialgebra.on $X_1 \times X_2$, (see exercise **4.**).

5. Limsup and Liminf

Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. We assume that $\mathcal{P}(X)$ is endowed with the inclusion ordering $\subset$. then:

Definition 5.1.
For any sequence (A_n) in $\mathcal{P}(X)$, we define the sets $\limsup_n A_n$ and $\liminf_n A_n$ by:

$$\limsup_n A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k$$

$$\liminf_n A_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k$$

Similarly let $\mathbb{R}, \leq$ be the ordered real number system and:

Definition 5.2.
For any sequence (a_n) in $\mathbb{R}$, we define the numbers $\limsup_n a_n$ and $\liminf_n a_n$ in $\overline{\mathbb{R}} = [-\infty, \infty]$ by:

$$\limsup_n a_n = \inf_{n\geq 1} \sup_{k\geq n} a_k$$

$$\liminf_n a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k$$

Definition 5.3.
If $f_n : X \longrightarrow \mathbb{R}$ us a sequence of functions from a set X into $\mathbb{R}$, we define the functions $\limsup_n f_n$ and $\liminf_n f_n$ from X into $\overline{\mathbb{R}}$,by:

$$\left(\limsup_n f_n\right)(x) = \limsup_n \left(f_n(x)\right)$$

$$\left(\liminf_n f_n\right)(x) = \liminf_n \left(f_n(x)\right)$$

6. Exercises

8. Prove that for any sequence (A_n) in $\mathcal{P}(X)$ we have:

$$\liminf_n A_n \subset \limsup_n A_n$$

$$\left(\liminf_n A_n\right)^c = \limsup_n A_n^c$$

$$\left(\limsup_n A_n\right)^c = \liminf_n A_n^c$$

9. Let I_A be the indicator function of the set A, i.e $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.
Prove that for any sequence (A_n) in $\mathcal{P}(X)$ we have::

$$I_{\limsup_n A_n} = \limsup_n I_{A_n} \quad \text{and} \quad I_{\liminf_n A_n} = \liminf_n I_{A_n}$$

7. Positive Measures

Let $(X, \mathcal{F})$ be a measurable space.

Definition 7.1.
A positive measure μ on $\mathcal{F}$ is a set function
$\mu : \mathcal{F} \longrightarrow [0\ \infty]$ such that:

(i) $\mu(\phi) = 0$
(ii) For every pairwise disjoint sequence (A_n) in $\mathcal{F}$:
$\mu\left(\sum\limits_n A_n\right) = \sum\limits_n \mu(A_n)$ ($\sigma-$additivity of μ).

The triple $(X, \mathcal{F}, \mu)$ is called measure space.
Let us observe that for a finite pairwise disjoint sequence
$A_k, 1 \leq k \leq n$ in $\mathcal{F}$, we have: $\mu\left(\sum\limits_1^n A_k\right) = \sum\limits_1^n \mu(A_k)$.

Example 7.2.

(a) Let X be a set and fix $x_0 \in X$. Define μ on $\mathcal{P}(X)$ by:
$A \in \mathcal{P}(X)$, $\mu(A) = I_A(x_0)$ (see exercise **9** defining the function I_A). $I_{(\cdot)}(x_0)$ is called Dirac measure at x_0.
To prove the $\sigma-$additivity of μ, observe that $I_{\sum\limits_n A_n} = \sum\limits_n I_{A_n}$ for pairwise disjoint sequences (A_n).

(b) For $A \subset X$ put $\mu(A) = \infty$ if A is an infinite set and $\mu(A) = n$ if A is a finite set with n elements. This measure is called the cardinality measure on $\mathcal{P}(X)$.

Proposition 7.3.
Let $(X, \mathcal{F}, \mu)$ be a measure space and let A, B be in $\mathcal{F}$, then:
(a) $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$.
(b) $A \subset B$ and $\mu(A) < \infty \Longrightarrow \mu(B \backslash A) = \mu(B) - \mu(A)$.
($B \backslash A$ is the difference set $B \cap A^c$)
Proof. If $A \subset B$, then $B = (B - A) \cup A$ and $\mu(B) = \mu(B \backslash A) + \mu(A)$, by additivity; so $\mu(B) \geq \mu(A)$ .If moreover $\mu(A) < \infty$ we deduce that:
$\mu(B \backslash A) = \mu(B) - \mu(A)$.■

Proposition 7.4. Let $(X, \mathcal{F}, \mu)$ be a measure space. Then for any sequence (A_n) in $\mathcal{F}$ we have:
$\mu\left(\bigcup\limits_n A_n\right) \leq \sum\limits_n \mu(A_n)$ (sub $\sigma-$additivity of μ).
Proof. Define the sequence (B_n) by the following recipe: $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash (A_1 \cup A_2)$, ...$B_n \backslash \left(\bigcup\limits_{i<n} A_i\right)$, then $\bigcup\limits_n A_n = \sum\limits_n B_n$ and $B_n \subset A_n$, $\forall n$. So $\mu\left(\bigcup\limits_n A_n\right) = \mu\left(\sum\limits_n B_n\right) = \sum\limits_n \mu(B_n)$;by Proposition **7.3**$(a)$ $\mu(B_n) \leq \mu(A_n)$,$\forall n$.■

Proposition 7.5. (sequential continuity of a measure)
Let $(X, \mathcal{F}, \mu)$ be a measure space. If (A_n) is a sequence in $\mathcal{F}$, then we have
(a) if $A_1 \subset A_2 \subset ... \subset A_n \subset .... \subset A = \underset{n}{\cup} A_n$ then $\mu(A) = \underset{n}{Lim} \mu(A_n)$
(b) if $A_1 \supset A_2 \supset ... \supset A_n \supset .... \supset A = \underset{n}{\cap} A_n$ and if $\mu(A_{n_0}) < \infty$ for some n_0 then $\mu(A) = \underset{n}{Lim} \mu(A_n)$

Proof. (a) Define the sequence (B_n) by:
$B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash A_2, ..., B_n = A_n \backslash A_{n-1}$, so we have $A = \sum\limits_n B_n$ and $\mu(A) = \sum\limits_n \mu(B_n) = \sum\limits_n \mu(A_n \backslash A_{n-1}) = \underset{n}{Lim} \sum\limits_{k=1}^{n} \mu(A_k \backslash A_{k-1}) = \underset{n}{Lim} \mu\left(\sum\limits_{k=1}^{n} A_k \backslash A_{k-1}\right)$;
but $\sum\limits_{1}^{n} A_k \backslash A_{k-1} = A_n$ by construction and we deduce that $\mu(A) = \underset{n}{Lim} \mu(A_n)$.

(b) We can assume $n_0 = 1$, so $\mu(A_n) < \infty$ for all n. On the other hand we have $A_1 \backslash A_1 \subset A_1 \backslash A_2 \subset ... \subset A_1 \backslash A_n \subset ... \underset{n}{\cup} A_1 \backslash A_n = A_1 \backslash A$. By (a) we deduce $\mu(A_1 \backslash A) = \underset{n}{Lim} \mu(A_1 \backslash A_n)$. Since $\mu(A_n) < \infty$ for all n we get, by Proposition **7.3**(b), $\mu(A_1 \backslash A) = \mu(A_1) - \mu(A)$ and $\mu(A_1 \backslash A_n) = \mu(A_1) - \mu(A_n)$, whence $\mu(A) = \underset{n}{Lim} \mu(A_n)$.■

Example 7.6. The condition (b) above is essential as is shown by taking μ the counting measure on $\mathbb{N}$ and taking $A_n = \{p : \ p \geq n\}$; indeed we have $\underset{n}{\cap} A_n = \phi$, so $\mu(\phi) = 0$ but $\mu(A_n) = \infty$, for all n, and then $\underset{n}{Lim} \mu(A_n) = \infty$.■

Proposition 7.7. (Borel-Cantelli Lemma)
Let $(X, \mathcal{F}, \mu)$ be a measure space. Let (A_n) be a sequence in $\mathcal{F}$ such that:
$\sum\limits_n \mu(A_n) < \infty$, then: $\mu\left(\limsup\limits_n A_n\right) = 0$

Proof. Put $B_n = \underset{k \geq n}{\cup} A_k$, then B_n is decreasing and $\limsup\limits_n A_n = \underset{n \geq 1}{\cap} B_n$. Since $\mu(B_n) = \mu\left(\underset{k \geq n}{\cup} A_k\right) \leq \sum\limits_{k \geq n} \mu(A_n) \leq \sum\limits_n \mu(A_n) < \infty$ for all n, we deduce, from Proposition **7.5** (b), that $\mu\left(\limsup\limits_n A_n\right) = \underset{n}{Lim} \mu(B_n) \leq \underset{n}{Lim} \sum\limits_{k \geq n} \mu(A_n) = 0$, because $\sum\limits_{k \geq n} \mu(A_n)$ is the remainder of a convergent series.■

Proposition 7.7. (Borel-Cantelli Lemma)
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Proof. Put $B_n = \underset{k \geq n}{\cup} A_k$, then B_n is decreasing and $\limsup\limits_n A_n = \underset{n \geq 1}{\cap} B_n$. Since $\mu(B_n) = \mu\left(\underset{k \geq n}{\cup} A_k\right) \leq \sum\limits_{k \geq n} \mu(A_n) \leq \sum\limits_n \mu(A_n) < \infty$ for all n, we deduce, from Proposition **1.6.5** (b), that $\mu\left(\limsup\limits_n A_n\right) = \underset{n}{Lim} \mu(B_n) \leq \underset{n}{Lim} \sum\limits_{k \geq n} \mu(A_n) = 0$, because $\sum\limits_{k \geq n} \mu(A_n)$ is the remainder of a convergent series.■

8. Complete Measures

Definition 8.1.

Let $(X,\mathcal{F},\mu)$ be a measure space and let N be a subset of X, we say that N is a null set if there is $A\in\mathcal{F}$, with $\mu(A)=0$ such that $N\subset A$. Let $\mathcal{N}$ be the family of null subsets of X. The space $(X,\mathcal{F},\mu)$ is said to be complete if $\mathcal{N}\subset\mathcal{F}$ i.e every null set is mesurable.

Examples 8.2.

(a) The counting measure on any set X is complete since in this case ϕ is the only null set.

(b) If μ_s is the Dirac measure at s on $(X,\mathcal{F})$ (**Example 7.2.**(a)), every subset N not containing s is a null set

Lemma 8.3.

The family $\mathcal{N}$ is closed by countable union.

Proof. Let (N_k) be a sequence in $\mathcal{N}$, then for each k there is $A_k\in\mathcal{F}$, with $\mu(A_k)=0$ such that $N_k\subset A_k$. So $N=\underset{k}{\cup}N_k\subset\underset{k}{\cup}A_k$; by the sub $\sigma-$ additivity of μ we have $\mu\left(\underset{n}{\cup}A_n\right)\leq\sum\limits_n\mu(A_n)=0.\blacksquare$

It is possible to complete any measure space $(X,\mathcal{F},\mu)$ according to the following:

Theorem 8.4.

Let $(X,\mathcal{F},\mu)$ be a measure space and let $\mathcal{N}$ be the family of null subsets of X. Let us put:

$\mathcal{F}_0=\{E\subset X:\ \ E=F\cup N,\ F\in\mathcal{F},\ N\in\mathcal{N}\}$

$\mu_0(E)=\mu_0(F\cup N)=\mu(F)$, if $E=F\cup N$, $F\in\mathcal{F}$, $N\in\mathcal{N}$

Then: $\mathcal{F}_0$ is a $\sigma-$field on X containing $\mathcal{F}$, and $\mathcal{N}$

μ_0 is a well defined measure on $\mathcal{F}_0$ that coincides with μ on $\mathcal{F}$.

The measure space $(X,\mathcal{F}_0,\mu_0)$ is complete.

Proof. First $\mathcal{F}_0$ is a $\sigma-$field

it is clear that ϕ and X are in $\mathcal{F}_0$

let $E\in\mathcal{F}_0$ with $E=F\cup N$, $F\in\mathcal{F}$, $N\in\mathcal{N}$ and let $A\in\mathcal{F}$, such that $\mu(A)=0, N\subset A$; then we have $E^c=F^c\cap N^c=(F^c\cap N^c\cap A)+(F^c\cap N^c\cap A^c)=(F^c\cap N^c\cap A)+(F^c\cap A^c)$; since $F^c\cap N^c\cap A\in\mathcal{N}$ and $F^c\cap A^c\in\mathcal{F}$

we have $E^c\in\mathcal{F}_0$. Finally $\mathcal{F}_0$ is closed by countable union and this comes from the same property for the family $\mathcal{N}$ (**Lemma 8.3**).

To finish the proof, we consider the set function μ_0. First it is well defined, indeed suppose the set $E\in\mathcal{F}_0$ can be written as $E=F_1\cup N_1=F_2\cup N_2$, then $F_1\cap F_2^c\subset N_1\cup N_2$ and $F_2\cap F_1^c\subset N_1\cup N_2$ which gives $\mu(F_1\cap F_2^c)=\mu(F_2\cap F_1^c)=0$, so $\mu(F_1)=\mu(F_2)$ and $\mu_0(E)=\mu_0(F\cup N)=\mu(F)$ is well defined.

To prove the $\sigma-$additivity of μ_0, let (E_n) be a pairwise disjoint sequence in $\mathcal{F}_0$, and write $E_k=F_k\cup N_k$, $k\geq 1$, with $F_k\in\mathcal{F}$, $N_k\in\mathcal{N}$.

Then we have $\sum\limits_k E_k=\sum\limits_k F_k\cup\sum\limits_k N_k$, with $\sum\limits_k N_k\in\mathcal{N}$ (**Lemma 8.3**).

and $\mu_0\left(\sum\limits_k E_k\right)=\mu\left(\sum\limits_k F_k\right)=\sum\limits_k\mu(F_k)=\sum\limits_k\mu_0(E_k)$, since μ is $\sigma-$additive.

Finally we prove that $(X,\mathcal{F}_0,\mu_0)$ is complete. Let M_0 be a μ_0 null set in X, so there is $E_0 \in \mathcal{F}_0$ with $\mu_0(E_0) = 0$ and $M_0 \subset E_0$; write $E_0 = F \cup N$, $F \in \mathcal{F}$, $N \in \mathcal{N}$ with $\mu_0(E_0) = \mu(F) = 0$ and $N \subset A \in \mathcal{F}, \mu(A) = 0$, so $M_0 \subset F \cup A$, with $\mu(F \cup A) = 0$; this proves that $M_0 \in \mathcal{N} \subset \mathcal{F}_0$ and M_0 is $\mathcal{F}_0$ measurable.$\blacksquare$

9. Exercises

10. A family σ of subsets of X is $\sigma-$additive if:

(1) ϕ and X are in σ

(2) If (A_n) is an increasing sequence in σ then $\underset{n}{\cup} A_n \in \sigma$

(3) For any A, B in σ we have:

$A \subset B \Longrightarrow B \cap A^c \in \sigma$

$A \cap B = \phi \Longrightarrow A + B \in \sigma$

(a) prove that any $\sigma-$field is a $\sigma-$additive family

(b) let μ, λ be two measures on the same measurable space $(X, \mathcal{F})$ such that $\mu(X) = \lambda(X) < \infty$.

Prove that the family $\sigma = \{A \in \mathcal{F}\colon\ \mu(A) = \lambda(A)\}$ is $\sigma-$additive.

Let $\boldsymbol{C}$ be a family of subsets of X then there exists a smallest $\sigma-$additive family on X containing $\boldsymbol{C}$ called the $\sigma-$additive family generated by $\boldsymbol{C}$.

11. Let $\Im$ be a family of subsets of X closed by finite intersection

Prove that the $\sigma-$field generated by $\Im$ coincides with the $\sigma-$additive family generated by $\Im$.

Chapter 2

Outer measures Extension of measures
1. Outer measures

Definition 1.1.

An outer measure on a set X is a set function
$\lambda : \mathcal{P}(X) \longrightarrow [0\ \infty]$ such that:
(1) $\lambda(\phi) = 0$
(2) if $A \subset B$ then $\lambda(A) \leq \lambda(B)$
(3) if (E_n) is any sequence in $\mathcal{P}(X)$ then $\lambda\left(\underset{n}{\cup} E_n\right) \leq \sum\limits_n \lambda(E_n)$

Remark.1.2.
It is not difficult to see that if λ is additive then λ is a positive measure on $\mathcal{P}(X)$.

Example.1.3.

(a) Any positive measure on $\mathcal{P}(X)$ is an outer measure.

(b) Define λ on $\mathcal{P}(X)$ by $\lambda(\phi) = 0$ and $\lambda(E) = 1$ if $E \neq \phi$; if X has more than one point then λ is an outer measure but not a measure.

We can say that the notion of outer measure is a natural generalization of that of positive measure. We will see below that an outer measure acts as a true measure on a some specific family of subsets of X. Let us start with the following:

Definition 1.4.

Let λ be an outer measure on X. A subset $E \subset X$ is said to be outer measurable or $\lambda-$measurable if we have:

$$\text{for every } A \subset X, \quad \lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c)$$

Example.1.5.
(a) A subset $E \subset X$ with $\lambda(E) = 0$ is $\lambda-$measurable.
(b) X, ϕ are $\lambda-$measurable for every outer measure λ.
(c) .Let λ be defined on X by $\lambda(\phi) = 0$, $\lambda(X) = 2$, $\lambda(E) = 1$ for $E \neq \phi, X$.
Then λ is an outer measure and ϕ, X are the only $\lambda-$measurable sets.

Now we go to the important assertion:

Theorem.1.6.
Let λ be an outer measure on X
and let $\mathcal{F}$ be the family of the $\lambda-$measurable sets.
Then $\mathcal{F}$ is a $\sigma-$field and the restriction of λ to $\mathcal{F}$ is a positive measure.
Proof. see [7].

2. Exercises

12. Let λ be an outer measure on X and let H be a $\lambda-$measurable set. Let λ_0 be the restriction of λ to $\mathcal{P}(H)$, prove that:

(a) λ_0 is an outer measure on $\mathcal{P}(H)$.

(b) $A \subset H$ is λ_0-measurable iff A is $\lambda-$measurable.

13. Let λ be an outer measure on X and let A be a $\lambda-$measurable set. If $B \subset X$ is a subset with $\lambda(B) < \infty$, prove that:

$\lambda(A \cup B) = \lambda(A) + \lambda(B) - \lambda(A \cap B)$

3. Extension of Measures

We start this section with the construction of an outer measure from a measure defined on an algebra of sets.

Definition 3.1.

Let $\mathcal{A}$ be an algebra on X. A positive measure on $\mathcal{A}$ is a set function $\mu : \mathcal{A} \longrightarrow [0\ \infty]$ such that:

(i) $\mu(\phi) = 0$

(ii) For every pairwise disjoint sequence (A_n) in $\mathcal{A}$ with $\underset{n}{\cup} A_n \in \mathcal{A}$:

$$\mu\left(\sum_n A_n\right) = \sum_n \mu(A_n) \quad (\sigma-\text{additivity of } \mu).$$

Any measure on an algebra $\mathcal{A}$ gives rise toan outer measure according to:

Theorem.3.2.

Let μ be a measure on an algebra $\mathcal{A}$.

For each subset $E \subset X$ define $\lambda(E)$ by the recipe:

$$\lambda(E) = \inf\left\{\sum_n \mu(A_n) : E \subset \underset{n}{\cup} A_n, \quad (A_n) \subset \mathcal{A}\right\}$$

the lower bound being taken over all sequences $(A_n) \subset \mathcal{A}$.

Then λ is an outer measure whose restriction to $\mathcal{A}$ coincides with μ.

Moreover the sets of $\mathcal{A}$ are $\lambda-$measurable.

Proof. see [7].

Definition 3.3. **($\sigma-$finite measures)**

Let $(X, \mathcal{F}, \mu)$ be a measure space. We say that the measure μ is $\sigma-$finite if there is a sequence (A_n) in $\mathcal{F}$, such that $\underset{n}{\cup} A_n = X$ and $\mu(A_n) < \infty$, $\forall n$.

A measure μ on an algebra $\mathcal{A}$ is $\sigma-$finite if there is a sequence (A_n) in $\mathcal{A}$ such that $\underset{n}{\cup} A_n = X$ and $\mu(A_n) < \infty$, $\forall n$.

Example 3.4.

(a) Any finite measure μ, i.e $\mu(X) < \infty$, is $\sigma-$finite

(b) The counting measure on $\mathbb{N}$ or on any infinite countable set is $\sigma-$finite but not finite.

(c) we will see later that the Lebesgue measure on $\mathbb{R}$ is a non trivial $\sigma-$finite measure.

Now we give the main extension theorem:

Theorem 3.5.

Let μ be a measure on an algebra $\mathcal{A}$ of subsets of X.
Then μ can be extended to a measure $\overline{\mu}$ on the $\sigma-$field $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. Moreover if μ is $\sigma-$finite on $\mathcal{A}$ the extension $\overline{\mu}$ is unique.

Proof.
Let μ^* be the outer measure given by Theorem**.** **3.2** and let $\mathcal{F}$ be the $\sigma-$field of μ^*-measurable sets. By the same theorem we have $\mathcal{A} \subset \mathcal{F}$ and μ^* coincides with μ on $\mathcal{A}$. So we have $\sigma(\mathcal{A}) \subset \mathcal{F}$. By Theorem**.** **1.6** μ^* acts as a true measure on $\mathcal{F}$. Then it is enough to take $\overline{\mu}$ as the restriction of μ^* to $\sigma(\mathcal{A})$. We prove the uniqueness in the case μ finite. Suppose the existence of two extensions μ_1, μ_2 for μ and consider the family $\mathcal{M} = \{A \in \sigma(\mathcal{A}) : \ \mu_1(A) = \mu_2(A)\}$. It is not difficult to prove that $\mathcal{M}$ is a monotone class which contains $\mathcal{A}$ (use the finiteness of the measures) So we have $\mathcal{A} \subset \mathcal{M} \subset \sigma(\mathcal{A})$ and since $\mathcal{A}$ is an algebra the monotone class generated by $\mathcal{A}$ is idendical to the $\sigma-$field generated by $\mathcal{A}$ (Theorem **3.10,** Chap. **1**) We deduce that $\mathcal{M} = \sigma(\mathcal{A})$. We leave the $\sigma-$finiteness case to the reader.■

Theorem 3.6.
Let μ be a $\sigma-$finite measure on an algebra $\mathcal{A}$ of subsets of X.
Let $\overline{\mu}$ be the unique extension of μ to the $\sigma-$field $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. If $B \in \sigma(\mathcal{A})$ with $\overline{\mu}(B) < \infty$, then:

$\forall \epsilon > 0$ there is $A_\epsilon \in \mathcal{A}$ such that $\overline{\mu}(B \triangle A_\epsilon) < \epsilon$

where $B \triangle A_\epsilon$ is the symmetric difference $(B \cap A_\epsilon^c) \cup (A_\epsilon \cap B^c)$.

Proof. By Theorems **3.2** and **3.5** the unique extension $\overline{\mu}$ has the form:

$$\overline{\mu}(B) = \inf \left\{ \sum_n \mu(A_n) : B \subset \underset{n}{\cup} A_n, \quad (A_n) \subset \mathcal{A} \right\}$$

If $B \in \sigma(\mathcal{A})$ with $\overline{\mu}(B) < \infty$, $\forall \epsilon > 0 \ \exists (A_n) \subset \mathcal{A}$ such that $B \subset \underset{n}{\cup} A_n$ and $\sum_n \mu(A_n) < \overline{\mu}(B) + \frac{\epsilon}{2}$.then use the fact that $\underset{n}{\cup} A_n = \lim_N \overset{N}{\underset{1}{\cup}} A_n$ and $\overset{N}{\underset{1}{\cup}} A_n \in \mathcal{A}$; put $B_N = \overset{N}{\underset{1}{\cup}} A_n$ then $\overline{\mu}\left(\underset{n}{\cup} A_n\right) = \lim_N \overline{\mu}(B_N) = \lim_N \mu(B_N)$.
So for some N_0 we have $\overline{\mu}\left(\underset{n}{\cup} A_n\right) < \overline{\mu}(B_{N_0}) + \frac{\epsilon}{2}$, then the set $A_\epsilon = B_{N_0}$ is in $\mathcal{A}$ and works.■

4. Exercises

14. An outer measure μ^* on X is regular if for any $A \subset X$ there is a μ^*-measurable set E such that $A \subset E$ and $\mu^*(A) = \mu^*(E)$.
(a) If μ^* is regular then for any sequence (A_n) of subsets of X we have

$$\mu^*\left(\liminf_n A_n\right) \leq \liminf_n \mu^*(A_n).$$

(b) If moreover the sequence (A_n) is increasing then $\mu^*\left(\lim_n A_n\right) = \lim_n \mu^*(A_n)$.

15. Let $(X, \mathcal{F}, \mu)$ be a measure space. Define μ^* on $\mathcal{P}(X)$ by the recipe:

$\mu^*(E) = \inf\{\mu(A) : A \in \mathcal{F} \quad E \subset A\}$

(a) Prove that μ^* is an outer measure.

(b) Prove that $\forall E \subset X \quad \exists A \in \mathcal{F}$ such that $E \subset A$ and $\mu^*(E) = \mu(A)$.

(c) Let us define μ^* on $\mathcal{P}(X)$ by the recipe:

$\mu_*(E) = \sup\{\mu(A) : A \in \mathcal{F} \quad E \subset A\}$

Prove that $\forall E \subset X$, in either case $\mu_*(E) < \infty$ or $\mu_*(E) = \infty$, there is$A \in \mathcal{F}$ such that $E \subset A$ and $\mu_*(E) = \mu(A)$.

(d) Prove that $\mu_*(E) \leq \mu^*(E), \forall E \subset X$ and if E is μ^*-measurable then $\mu_*(E) = \mu^*(E)$ .If $\mu_*(E) = \mu^*(E) < \infty$ then E is μ^*-measurable.

5. Lebesgue Measure on $\mathbb{R}$

Measure on the Algebra generated by the semialgebra of intervals

Let us recall that a family $\mathcal{S}$ of subsets of a set X is a semialgebra if it satisfies:

(a) ϕ, X are in $\mathcal{S}$

(b) If A, B are in $\mathcal{S}$ then $A \cap B$ is in $\mathcal{S}$

(c) If A is in $\mathcal{S}$ then $A^c = \sum_1^n A_k$, where the sets A_k are pairwise disjoint in $\mathcal{S}$

(see Chapter 1 exercise 4)

We recall also that the algebra generated by the semialgebra $\mathcal{S}$ is the family

$$\left\{A : A = \sum_1^n S_k, \text{ where the } S_k \text{ are pairwise disjoint in } \mathcal{S}.\right\}$$

It is easy to prove that the family $\mathcal{I}$ of all intervals of $\mathbb{R}$ is a semialgebra. Let $\mathcal{A}$ be the algebra generated by $\mathcal{I}$. It is well known that the borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$ is generated by $\mathcal{A}$ or simply by $\mathcal{I}$. Now if $A \in \mathcal{A}$ has the form $A = \sum_1^n I_k$, where the I_k are pairwise disjoint in $\mathcal{I}$, put $\mu(A) = \sum_1^n \lambda(I_k)$, where $\lambda(I)$ is the lengh of the interval I. Then μ is unambiguously defined on $\mathcal{A}$. Moreover μ is a $\sigma-$finite measure on the algebra $\mathcal{A}$. By Theorems **3.2** and **3.5** the unique extension $\overline{\mu}$ of μ to the $\sigma-$field $\sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$ generated by $\mathcal{A}$ has the form:

$$\overline{\mu}(B) = \inf\left\{\sum_n \mu(A_n) : B \subset \bigcup_n A_n, \quad (A_n) \subset \mathcal{A}\right\}$$

The completion of the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \overline{\mu})$ is the Lebesgue space $(\mathbb{R}, \mathcal{L}_{\mathbb{R}}, \overline{\mu}^*)$ (see Theorem **8.4,** Chap.**1**). In fact each set $E \in \mathcal{L}_{\mathbb{R}}$ has the form $E = B \cup N$, where $B \in \mathcal{B}_{\mathbb{R}}$ and N is a $\overline{\mu}-$null set. Let us note the following approximation result:

Theorem 5.1.

Let $E \in \mathcal{L}_{\mathbb{R}}$, then we have:

$\forall \epsilon > 0$ there is a closed set F and an open set G such that:

$F \subset E \subset G$ and $\overline{\mu}^*(G \diagdown F) < \epsilon$

Chapter 3

Measurable Functions

1. Preliminaries

Definition.1.1.

Let X, Y be non empty sets.
To each function $f : X \longrightarrow Y$ it corresponds the preimage function $f^{-1} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ defined by: $B \in \mathcal{P}(Y)$, $f^{-1}(B) = \{x \in X : f(x) \in B\}$.
Also if $\Im$ is any subfamily of $\mathcal{P}(Y)$ put $f^{-1}(\Im) = \left\{f^{-1}(B), B \in \Im\right\}$.

Proposition.1.2.

The preimage function has the following properties:
(a) $f^{-1}\left(\underset{i}{\cup} B_i\right) = \underset{i}{\cup} f^{-1}(B_i)$ and $f^{-1}\left(\underset{i}{\cap} B_i\right) = \underset{i}{\cap} f^{-1}(B_i)$
for any family $(B_i) \subset \mathcal{P}(Y)$
(b) $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$, for any $B \in \mathcal{P}(Y)$
(c) $B \subset C \Longrightarrow f^{-1}(B) \subset f^{-1}(C)$ for any B, C in $\mathcal{P}(Y)$.

Proof. straightforward.■

Proposition.1.3.

Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measure spaces and $f : X \longrightarrow Y$ a function. Define the families:

$\Re_f = \left\{f^{-1}(G) : G \in \mathcal{G}\right\} = f^{-1}(\mathcal{G})$

$\mathcal{B}_f = \left\{B \subset Y : f^{-1}(B) \in \mathcal{F}\right\}$

Then $\Re_f$ is a $\sigma-$field on X and $\mathcal{B}_f$ a $\sigma-$field on Y
Moreover we have $f^{-1}(\mathcal{B}_f) \subset \mathcal{F}$.

Proof. We prove first that $\Re_f$ is a $\sigma-$field on X.
$X \in \Re_f$ since $X = f^{-1}(Y)$ and $Y \in \mathcal{G}$.
Let $A \in \Re_f$ with $A = f^{-1}(G)$ for some $G \in \mathcal{G}$, then $A^c = f^{-1}(G^c)$ since $G^c \in \mathcal{G}$, we deduce that $A^c \in \Re_f$.
Let (A_n) be a sequence in $\Re_f$ with $A_n = f^{-1}(G_n)$ for some $G_n \in \mathcal{G}$;
by Proposition. **1.2** (a) we have $\underset{n}{\cup} A_n = \underset{n}{\cup} f^{-1}(G_n) = f^{-1}\left(\underset{n}{\cup} G_n\right)$
since $\underset{n}{\cup} G_n \in \mathcal{G}$, we deduce that $\underset{n}{\cup} A_n \in \Re_f$. So $\Re_f$ is a $\sigma-$field on X.
The reader can do the remains by the same way.■

2. Measurable Functions Properties

Definition.2.1.

Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be measure spaces and $f : X \longrightarrow Y$ a function. We say that f is measurable if $f^{-1}(\mathcal{G}) \subset \mathcal{F}$.This means that:
$f^{-1}(G) \in \mathcal{F}$ for every $G \in \mathcal{G}$.

Theorem.2.2.
Let $f : X \longrightarrow Y$ be a function and $\Im$ a family of subsets of Y.
Then we have $\sigma\left(f^{-1}(\Im)\right) = f^{-1}(\sigma(\Im))$.
This means that: the $\sigma-$field $\sigma\left(f^{-1}(\Im)\right)$ generated by $f^{-1}(\Im)$ coincides with the preimage of the $\sigma-$field $\sigma(\Im)$.

Proof. $\Im \subset \sigma(\Im) \Longrightarrow f^{-1}(\Im) \subset f^{-1}(\sigma(\Im))$ and $f^{-1}(\sigma(\Im))$ is a $\sigma-$field, since the preimage of a $\sigma-$field is a $\sigma-$field by Proposition.**1.3.**
So we deduce that $\sigma\left(f^{-1}(\Im)\right) \subset f^{-1}(\sigma(\Im))$. Now consider the $\sigma-$field
$\mathcal{B}_f = \left\{B \subset Y : f^{-1}(B) \in \sigma\left(f^{-1}(\Im)\right)\right\}$. If $B \in \mathcal{B}_f$, then $f^{-1}(B) \subset \sigma\left(f^{-1}(\Im)\right)$, so $f^{-1}(\mathcal{B}_f) \subset \sigma\left(f^{-1}(\Im)\right)$. But $\Im \subset \mathcal{B}_f$, and then $\sigma(\Im) \subset \mathcal{B}_f$,
so we get $f^{-1}(\sigma(\Im)) \subset f^{-1}(\mathcal{B}_f) \subset \sigma\left(f^{-1}(\Im)\right)$.■

Proposition.2.3.
Let $(X,\mathcal{F}), (Y,\mathcal{G})$ be measurable spaces and $f : X \longrightarrow Y$ a function. Suppose there is a family $\Im$ of subsets of Y with $\sigma(\Im) = \mathcal{G}$ and satsfying $f^{-1}(\Im) \subset \mathcal{F}$.Then f is measurable with respect to $(X,\mathcal{F}), (Y,\mathcal{G})$.

Proof. Since $f^{-1}(\Im) \subset \mathcal{F}$ we have $\sigma\left(f^{-1}(\Im)\right) \subset \mathcal{F}$.
By Theorem.**2.2** $\sigma\left(f^{-1}(\Im)\right) = f^{-1}(\sigma(\Im))$, but $\sigma(\Im) = \mathcal{G}$
and so $f^{-1}(\mathcal{G}) \subset \mathcal{F}$.■

Examples.2.4.
(a) Let $f : X \longrightarrow \mathbb{R}$ be a function from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$.The Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$ is defined in Proposition **3.6**, chap.**1.** For f to be measurable it is enough that $f^{-1}(]-\infty,t[) \in \mathcal{F}$ (the intervals $]-\infty,t[$ generates $\mathcal{B}_{\mathbb{R}}$)

(b) Let X be a topological space with a countable base (U_n), endowed with its Borel $\sigma-$field $\mathcal{B}_Y$. It is well known that $\mathcal{B}_Y$ is generated by the family (U_n) and any open set is the union of a subfamily of (U_n). So for a function from $(X,\mathcal{F})$ into $(Y,\mathcal{B}_Y)$ to be measurable it is enough that $f^{-1}(U_n) \in \mathcal{F}$ for every n.

(c) Let X,Y be topological spaces endowed with their Borel $\sigma-$fields $\mathcal{B}_X, \mathcal{B}_Y$. A function $f : X \longrightarrow Y$ is measurable with respect to $\mathcal{B}_X, \mathcal{B}_Y$ iff $f^{-1}(G) \in \mathcal{B}_X$ for every open set $G \subset Y$. In particular any continuous function is measurable.

(d) Let $I_A : X \longrightarrow \mathbb{R}$ be the indicator function of the set A, i.e $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$. We have $I_A^{-1}(\mathcal{B}_{\mathbb{R}}) = \{A, A^c, X, \phi\}$, then I_A is measurable from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ iff $A \in \mathcal{F}$.

Now we state some important properties of measurable functions.

Proposition.2.5.
Let $(X,\mathcal{F}), (Y,\mathcal{G}), (Z,\mathcal{H})$ be measurable spaces and
$f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ measurable functions. Then the composition function $g \circ f : X \longrightarrow Z$ is measurable from $(X,\mathcal{F})$ into $(Z,\mathcal{H})$.

Proof. We have $(g \circ f)^{-1}(\mathcal{H}) = \left(f^{-1} \circ g^{-1}\right)(\mathcal{H}) = f^{-1}\left(g^{-1}(\mathcal{H})\right)$
Since g is measurable $g^{-1}(\mathcal{H}) \subset \mathcal{G}$, so $f^{-1}\left(g^{-1}(\mathcal{H})\right) \subset f^{-1}(\mathcal{G})$. But f is measurable then $f^{-1}(\mathcal{G}) \subset \mathcal{F}$. We deduce that $(g \circ f)^{-1}(\mathcal{H}) \subset \mathcal{F}$ and $g \circ f$ is measurable.■

Proposition.2.6.

Let $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ be the product of the measurable spaces $(X, \mathcal{F}), (Y, \mathcal{G})$ (see Definition **3.4.** Chap.**1**). Then the projection $\pi_1(x, y) = x$ is measurable from $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ into $(X, \mathcal{F})$. Similarly the projection $\pi_2(x, y) = y$ is measurable from $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ into $(Y, \mathcal{G})$.

Proof. By Definition **3.4** Chap.**1** the $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ contains the family $\{A \times B : \; A \in \mathcal{F}, \;\; B \in \mathcal{G}\}$. We get $\pi_1^{-1}(A) = A \times Y \in \mathcal{F} \otimes \mathcal{G}$ for every $A \in \mathcal{F}$ and $\pi_2^{-1}(B) = X \times B \in \mathcal{F} \otimes \mathcal{G}$ for every $B \in \mathcal{G}$. So π_1 and π_2 are measurable.$\blacksquare$

Proposition.2.7.

Let $(Z, \mathcal{H})$ be a measurable space and let $f : Z \longrightarrow X \times Y$ be a function with $f_1 = \pi_1 \circ f : Z \longrightarrow X$ and $f_2 = \pi_2 \circ f : Z \longrightarrow Y$. Then f is measurable from $(Z, \mathcal{H})$ into $(X \times Y, \mathcal{F} \otimes \mathcal{G})$ if and only if f_1 is measurable from $(Z, \mathcal{H})$ into $(X, \mathcal{F})$ and f_2 is measurable from $(Z, \mathcal{H})$ into $(Y, \mathcal{G})$.

Proof. The <if> part comes from the measurability of π_1 and π_2 (Proposition **2.6**) and the measurability of the composition function (Proposition **2.5**).
We prove the <only if> part:. Since the family $\{A \times B : \; A \in \mathcal{F}, \;\; B \in \mathcal{G}\}$ generates the product $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ it is enough to prove that $f^{-1}(A \times B) \in \mathcal{H}$ (Proposition **2.3**). Since f_1 and f_2 are measurable we have

$f_1^{-1}(A) = (\pi_1 \circ f)^{-1}(A) = f^{-1}(A \times Y) \in \mathcal{H}$

and $f_2^{-1}(A) = (\pi_2 \circ f)^{-1}(B) = f^{-1}(X \times B) \in \mathcal{H}$

$f^{-1}(A \times Y) \cap f^{-1}(X \times B) = f^{-1}((A \times Y) \cap (X \times B)) = f^{-1}(A \times B) \in \mathcal{H}$.$\blacksquare$

Remark. 2.8.

Let Let X be a topological space. Let us recall that the Borel $\sigma-$field of X is the $\sigma-$field generated by the family of all the open sets of X.
It is denoted by $\mathcal{B}_X$. Sets in $\mathcal{B}_X$ are called Borel sets of X. If X, Y are topological spaces whose product $X \times Y$ is endowed with the product topology then on the space $X \times Y$ one may put two $\sigma-$fields that are $\mathcal{B}_X \otimes \mathcal{B}_Y$ and $\mathcal{B}_{X \otimes Y}$. An interesting question is when do we have $\mathcal{B}_{X \otimes Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. It is known that if X and Y are separable metric spaces then $\mathcal{B}_{X \otimes Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. This result is of particular importance when $X = Y = \mathbb{R}$:

Theorem.2.9.

The space $\mathbb{R}$ is separable, since the countable set $\mathbb{Q}$ of rational numbers is dense. So the set $\mathbb{R}^2$ with the product topology is separable and we have $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

As a consequence of this Theorem we have:

Proposition. 2.10.

Let $f, g : X \longrightarrow \mathbb{R}$ be measurable functions from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then the following functions $f + g$, $f.g$, $\sup(f, g)$, $\inf(f, g)$ are measurable.

Proof. Since the functions f, g are measurable, the function $\varphi : X \longrightarrow \mathbb{R}^2$ defined by $\varphi(x) = (f(x), g(x))$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}_{\mathbb{R}^2}$(Proposition.**2.7**). On the other hand the functions $S, P, M, m : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by: $S(u, v) = u+v$, $P(u, v) = uv$, $M(u, v) = \sup(u, v)$, $m(u, v) = \inf(u, v)$ are continuous and so measurable with respect to $\mathcal{B}_{\mathbb{R}^2}$ and $\mathcal{B}_{\mathbb{R}}$. Now we have $S \circ \varphi = f + g$, $P \circ \varphi = fg$, $M \circ \varphi = \sup(f, g)$, $m \circ \varphi = \inf(f, g)$; the conclusion comes from Proposition.**2.5.**$\blacksquare$

Corollary. The family $\mathcal{M}(X,\mathbb{R})$ of measurable functions from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ is a vector space on the field $\mathbb{R}$ and even an algebra of functions.

Definition.2.11.

Let $\{f_i, i\in I\}$ be a family of functions defined on a set X such that each $f_i : X \longrightarrow E_i$ sends X into the measurable space $(E_i,\mathcal{F}_i)$. The $\sigma-$field generated by the family $\{f_i, i\in I\}$ is defined as the smallest $\sigma-$field $\mathcal{F}$ on X making each function f_i measurable from $(X,\mathcal{F})$ into the space $(E_i,\mathcal{F}_i)$. We denote this $\sigma-$field $\mathcal{F}$ by $\sigma\{f_i, i\in I\}$; in other words $\sigma\{f_i, i\in I\}$ is the smallest $\sigma-$field $\mathcal{F}$ on X containing all the families $f_i^{-1}(\mathcal{F}_i), i\in I$.

Examples.2.12.

(a) Let X be a set and take $\{f_i, i\in I\} = \{I_A, A\in\mathcal{P}(X)\}$ where I_A is the indicator function, then $\sigma\{I_A, A\in\mathcal{P}(X)\} = \mathcal{P}(X)$.

(b) Let X be a topological space. The Baire $\sigma-$field on X is defined as the $\sigma-$field $\mathcal{B}_0(X)$ generated by all continuous functions $f_i : X \longrightarrow \mathbb{R}$, that is the smallest $\sigma-$field on X making each continuous function $f_i : X \longrightarrow \mathbb{R}$ measurable with respect to $\mathcal{B}_0(X)$ and $\mathcal{B}_{\mathbb{R}}$.

(c) If in Example (b) the space X is a metric space whose topology is defined by the distance d then $\mathcal{B}_0(X)$ coincides with the Borel $\sigma-$field $\mathcal{B}_X$ on X.

Indeed we have $\mathcal{B}_0(X)\subset\mathcal{B}_X$ since $\mathcal{B}_X$ makes each continuous function measurable as easily may be seen. On the other hand let F be a closed set in X and consider the continuous function $f : X \longrightarrow \mathbb{R}$ given by $f(x) = d(x,F)$. Then we have $F = \{x\in X : f(x) = 0\} = f^{-1}(0)\in\mathcal{B}_0(X)$; so $\mathcal{B}_0(X)$ contains all the closed sets of X and then $\mathcal{B}_X\subset\mathcal{B}_0(X)$ since $\mathcal{B}_X$ is generated by the family of closed sets in X (see Definition **3.5** Chap.**1**).

(d) Let $(X\times Y,\mathcal{F}\otimes\mathcal{G})$ be the product of the measurable spaces $(X,\mathcal{F}),(Y,\mathcal{G})$. Then the projection $\pi_1(x,y) = x$ and the projection $\pi_2(x,y) = y$ are measurable on $(X\times Y,\mathcal{F}\otimes\mathcal{G})$ (Proposition.**2.6**). Then $\pi_1^{-1}(A) = A\times Y\in\mathcal{F}\otimes\mathcal{G}$ for every $A\in\mathcal{F}$ and $\pi_2^{-1}(B) = X\times B\in\mathcal{F}\otimes\mathcal{G}$ for every $B\in\mathcal{G}$.

We deduce that $\sigma\{\pi_1,\pi_2\}\subset\mathcal{F}\otimes\mathcal{G}$. On the other hand we have:

$\pi_1^{-1}(A)\cap\pi_2^{-1}(B) = (A\times Y)\cap(X\times B) = A\times B\in\mathcal{F}\otimes\mathcal{G}$. So every set of the form $A\times B$ with $A\in\mathcal{F}$ and $B\in\mathcal{G}$ is in $\sigma\{\pi_1,\pi_2\}$. But $\sigma\{A\times B : A\in\mathcal{F}, B\in\mathcal{G}\} = \mathcal{F}\otimes\mathcal{G}$, finally $\mathcal{F}\otimes\mathcal{G}\subset\sigma\{\pi_1,\pi_2\}$. Then $\mathcal{F}\otimes\mathcal{G} = \sigma\{\pi_1,\pi_2\}$.

3. Exercises

20. Let X be a non empty set. Determine the $\sigma-$field $\mathcal{F}$ generated by the constant functions $f : X \longrightarrow \mathbb{R}$. Let $\Im$ be the family of measurable functions from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, prove that $\Im$ is isomorphic to $\mathbb{R}$.
21. Let f be a measurable function from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, prove that $|f|$ is measurable. Let E be a set not Lebesgue measurable (see section 5 for the definition of Lebesgue measurable sets). Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = xI_{E^c} - xI_E$, prove that f is not Lebesgue measurable but $|f|$ is measurable.
22. Let $\{(X_i, \mathcal{F}_i), 1 \le i \le n\}$ be a finite family of measurable spaces and form the product set $X = \prod\limits_1^n X_i = X_1 \times X_2 \times \cdots \times X_n$. We denote by $p_i : X \longrightarrow X_i$ the projection from X onto X_i given by $p_i(x_1, x_2, \cdots, x_n) = x_i$. Consider the $\sigma-$field $\sigma\{p_i, 1 \le i \le n\}$ generated by the functions $\{p_i, 1 \le i \le n\}$ and denoted by $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_n = \overset{n}{\underset{1}{\otimes}} \mathcal{F}_i$. The space $\left(X, \overset{n}{\underset{1}{\otimes}} \mathcal{F}_i\right)$ is called the product of the spaces $(X_i, \mathcal{F}_i), 1 \le i \le n$.
(a) Prove that $\overset{n}{\underset{1}{\otimes}} \mathcal{F}_i$ is generated by the subsets of X of the form
$A = A_1 \times A_2 \times \cdots \times A_n,\ A_i \in \mathcal{F}_i\ 1 \le i \le n$.
(b) Let $(Y, \mathcal{G})$ be a measurable space and let $g : Y \longrightarrow \prod\limits_1^n X_i$ be a function, prove that g is measurable with respect to $(Y, \mathcal{G})$ and $\left(X, \overset{n}{\underset{1}{\otimes}} \mathcal{F}_i\right)$ if and only if $p_i \circ g$ is measurable from $(Y, \mathcal{G})$ into $(X_i, \mathcal{F}_i)$ for each $1 \le i \le n$.
23. Let X be a non empty set and let $\{f_i, i \in I\}$ be a family of functions defined on X such that each $f_i : X \longrightarrow E_i$ sends X into the measurable space $(E_i, \mathcal{B}_i)$. Suppose that X is endowed with the $\sigma-$field $\sigma\{f_i, 1 \le i \le n\}$ generated by the functions $\{f_i, 1 \le i \le n\}$ (see Definition **2.11**). Let $(Y, \mathcal{G})$ be a measurable space and let $g : Y \longrightarrow X$, prove that g is measurable with respect to $(Y, \mathcal{G})$ and $(X, \sigma\{f_i, 1 \le i \le n\})$ if and only if $f_i \circ g$ is measurable from $(Y, \mathcal{G})$ into $(E_i, \mathcal{B}_i)$ for each $1 \le i \le n$.

4. Measurable Functions with values in $\mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$

Definition.4.1
(a) The set $\mathbb{R}$ is the real numbers system endowed with the Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$.
(b) The set $\overline{\mathbb{R}}$ is defined as $\{\mathbb{R}, -\infty, +\infty\}$. The $\sigma-$field we need on $\overline{\mathbb{R}}$ is given by $\sigma\{\mathcal{B}_{\mathbb{R}}, -\infty, \infty\}$ and denoted by $\mathcal{B}_{\overline{\mathbb{R}}}$.
(c) It is well known that the set $\mathbb{C}$ of complex numbers can be identified with the product space $\mathbb{R} \times \mathbb{R}$; so we can identify the Borel $\sigma-$field $\mathcal{B}_{\mathbb{C}}$with $\mathcal{B}_{\mathbb{R}\times\mathbb{R}}$, which is $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ by Theorem.**2.9.**

Notations. 4.2.

Let $(X, \mathcal{F})$ be a measurable space. In the sequel.we will use the following notations:

$\mathcal{M}(X, \mathbb{R})$ is the family of measurable functions f from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.
$\mathcal{M}(X, \mathbb{C})$ is the family of measurable functions f from $(X, \mathcal{F})$ into $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$

We already have seen that $\mathcal{M}(X, \mathbb{R})$ is a vector space on the field $\mathbb{R}$ (see the Corollary of Proposition**2.10).**

It is not difficult to prove the same for $\mathcal{M}(X, \mathbb{C})$

Arithmetic in $\overline{\mathbb{R}}$. 4.3.

We will agree with the following conventions in $\overline{\mathbb{R}} = \{\mathbb{R}, -\infty, +\infty\}$:

$0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0$
$(+\infty) + (+\infty) = +\infty$
$(-\infty) + (-\infty) = -\infty$
$a \pm (\pm\infty) = \pm\infty, \forall a \in \mathbb{R}$
$(-1) \cdot (\pm\infty) = (\mp\infty)$

Definition. 4.4.

Let $(X, \mathcal{F})$ be a measurable space.

A function $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable from $(X, \mathcal{F})$ into $\left(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}}\right)$ if:

$f^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}_{\mathbb{R}}$, and $f^{-1}(+\infty) \in \mathcal{F}, f^{-1}(-\infty) \in \mathcal{F}$

this comes from the fact that $\mathcal{B}_{\overline{\mathbb{R}}} = \sigma\{\mathcal{B}_{\mathbb{R}}, -\infty, \infty\}$ and Proposition **2.3.**

We denote by $\mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ the the family of measurable functions f from $(X, \mathcal{F})$ into $\left(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}}\right)$.

Proposition. 4.5.

The $\sigma-$field $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by all the intervals of the form $[-\infty, t\, [$.

Proof. Use the fact that $\mathcal{B}_{\mathbb{R}}$ is generated by all the open intervals by Proposition **3.6.**Chap.**1**■

Corollary.

A function $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable from $(X, \mathcal{F})$ into $\left(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}}\right)$ if:

$f^{-1}([-\infty, t\, [) \in \mathcal{F}, \forall t \in \mathbb{R}$.

Definition. 4.6.

Let $(X,\mathcal{F}),(Y,\mathcal{G})$ be measurable spaces and $E\subset X$ a subset of X.
If $f:X\longrightarrow Y$ is a function. We say that f is measurable on E if the restriction of f to E considered as a function from $(E,E\cap\mathcal{F})$ into $(Y,\mathcal{G})$ is measurable.

Example. 4.7.
If f,g are in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$, then the function $f+g$ is measurable on the set E with:
$E^c=(\{f=\infty\}\cap\{g=-\infty\})\cup(\{f=-\infty\}\cap\{g=\infty\})$
Let φ be the restriction of $f+g$ to E then we have
φ is well defined on E and $\{\varphi<t\}=\{f+g<t\}\cap E\in E\cap\mathcal{F}$.

5. Sequences of Measurable Functions

Definition. 5.1. (simple function)

Let $f:X\longrightarrow\mathbb{R}$ be a function from X into $\mathbb{R}$. The function f is simple if it takes a finite number of values, that is, f is simple if the set $f(X)$ is a finite subset of $\mathbb{R}$. So if $f(X)=\{a_1,a_2,...,a_n\}$ and $A_i=\{x:f(x)=a_i\},i=1,2,...,n$, then $\{A_1,A_2,...,A_n\}$ is a partition of X and the function f can be written as $f(\cdot)=\sum\limits_1^n a_i.I_{A_i}(\cdot)$, where I_{A_i} is the indicateur function of the set $A_i,i=1,2,...,n$.

Proposition. 5.2

A simple function $f(\cdot)=\sum\limits_{i=1}^n a_i.I_{A_i}(\cdot)$ is measurable from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ iff $A_i\in\mathcal{F},i=1,2,...,n$.

Proof. We have $f^{-1}\{a_i\}=A_i\in\mathcal{F},i=1,2,...,n$; so if $B\in\mathcal{B}_{\mathbb{R}}$ and $n_B=\{i:a_i\in B\}$, we deduce that $f^{-1}(B)=\underset{i\in n_B}{\cup}A_i\in\mathcal{F}$.■

Notation. 5.3. We denote by $\mathcal{E}$ the family of measurable simple functions from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$

Proposition. 5.4.

Let s,t be in $\mathcal{E}$ and $\lambda\in\mathbb{R}$, then:
the functions $s+t$, $s\cdot t$, $\lambda\cdot s$, $\sup(s,t)$, $\inf(s,t)$ are in $\mathcal{E}$.

Proof. Write $s(\cdot)=\sum\limits_1^n a_i.I_{A_i}(\cdot)$, $t(\cdot)=\sum\limits_1^m b_j.I_{Bj}(\cdot)$, then we have:

$$s+t=\sum_{i=1}^n\sum_{j=1}^m(a_i+b_j).I_{A_i\cap B_j}$$

$$s\cdot t=\sum_{i=1}^n\sum_{j=1}^m(a_ib_j).I_{A_i\cap B_j},\ \lambda\cdot s=\sum_1^n(\lambda a_i).I_{A_i}$$

(so the family $\mathcal{E}$ is an algebra on $\mathbb{R}$.)

$$\sup(s,t)=\sum_{i=1}^n\sum_{j=1}^m\sup(a_i,b_j).I_{A_i\cap B_j},\ \inf(s,t)=\sum_{i=1}^n\sum_{j=1}^m\inf(a_i,b_j).I_{A_i\cap B_j}$$

Since $\{A_i\cap B_j,\ \ 1\le i\le n,\ 1\le j\le m\}$ is a partition of X we get the result.■

Proposition. 5.5.

Let (f_n) be a sequence of functions in $\mathcal{M}(X,\mathbb{R})$ or either in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ then: the functions $\sup\limits_n f_n$ and $\inf\limits_n f_n$ are in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$.

Proof. For any $t \in \mathbb{R}$ we have $\left\{\sup\limits_n f_n \leq t\right\} = \bigcap\limits_n \{f_n \leq t\}$ whence the mesurability of $\sup\limits_n f_n$. Since $\inf\limits_n f_n = -\sup\limits_n - f_n$ we deduce the mesurability of $\inf\limits_n f_n$.■

Corollary. 1.

Let (f_n) be a sequence of functions in $\mathcal{M}(X,\mathbb{R})$ or either in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ then: the functions $\limsup\limits_n f_n$ and $\liminf\limits_n f_n$ are measurable

Proof. Comes directly from the proposition above since $\limsup\limits_n f_n = \inf\limits_{n\geq 1}\sup\limits_{k\geq n} f_k$ and $\liminf\limits_n f_n = \sup\limits_{n\geq 1}\inf\limits_{k\geq n} f_k$.■

Corollary. 2.

Let (f_n) be a sequence of functions in $\mathcal{M}(X,\mathbb{R})$ or either in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ then: The set $C = \left\{x : \limsup\limits_n f_n(x) = \liminf\limits_n f_n(x)\right\}$ belongs to $\mathcal{F}$.

Proof. Observe that C is the convergence set of the sequence (f_n). Put :

$$C_1 = \left(\left\{x : \limsup_n f_n(x) = \infty\right\} \cap \left\{x : \liminf_n f_n(x) = \infty\right\}\right)$$
$$C_2 = \left(\left\{x : \limsup_n f_n(x) = -\infty\right\} \cap \left\{x : \liminf_n f_n(x) = -\infty\right\}\right)$$
$$C_3 = \left\{x : \limsup_n f_n(x) \in \mathbb{R}\right\} \cap \left\{x : \limsup_n f_n(x) = \liminf_n f_n(x)\right\}$$

Then C_1 and C_2 and C_3 are in $\mathcal{F}$ and $C = C_1 \cup C_2 \cup C_3$.■.

Corollary. 3.

Let (f_n) be a sequence of functions in $\mathcal{M}(X,\mathbb{R})$ or either in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$
Suppose that: $\lim\limits_n f_n(x) = f(x) \in \overline{\mathbb{R}}$ exists for each $x \in X$. Then $f \in \mathcal{M}\left(X,\overline{\mathbb{R}}\right)$.

Proof. The convergence set $C = \left\{x : \limsup\limits_n f_n(x) = \liminf\limits_n f_n(x)\right\}$ given in Corollary 2 is equal to X here.
So the function $f(x)$ is equal to $\limsup\limits_n f_n(x) = \liminf\limits_n f_n(x), \forall x \in X$. Then f is measurable by Corollary 1.■

The following theorem is fundamental and will be used in the construction of the integral of a measurable function.

Theorem. 5.6.

Let $f \in \mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ be such that $f(x) \in [0,\infty]$, $\forall x \in X$.Then:
there exists a sequence $(s_n)$of positive measurable simple functions from $(X,\mathcal{F})$ into $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ with:

(i) $0 \leq s_n \leq s_{n+1}$
(ii) $\lim\limits_n s_n(x) = f(x)$, $\forall x \in X$.

Proof. For each $n \geq 1$ and each $x \in X$, define s_n by:

$s_n(x) = \dfrac{i-1}{2^n}$ if $\dfrac{i-1}{2^n} \leq f(x) < \dfrac{i}{2^n}$, $i = 1, 2, ..., n2^n$

$s_n(x) = n$ if $f(x) \geq n$

we can use a consolidated form for s_n:

$$s_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} I_{\left\{\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\right\}} + n\, I_{\{f(x) \geq n\}}$$

recall that I_A is the function defined by $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

Then (s_n) is an increasing sequence of positive simple functions (check it!).

Let us prove that $\lim_n s_n(x) = f(x)$, $\forall x \in X$:

if $f(x) < \infty$ then for every $n > f(x)$ we have $0 < f(x) - s_n(x) < \dfrac{1}{2^n}$, so $\lim_n s_n(x) = f(x)$

if $f(x) = \infty$ then $f(x) \geq n$ for every n and so we have $s_n(x) = n$ for all n whence $\lim_n s_n(x) = \infty$.■

Definition. 5.7.

Let $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$. Define the positive measurable functions f^+, f^- by:

$f^+ = \sup(f, 0)$, $f^- = -\inf(f, 0)$

Remark. 5.8.

It is easy to check that:

$f = f^+ - f^-$

$|f| = f^+ + f^-$

Proposition. 5.9.

Let $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$. Then there exists a sequence (s_n) of measurable simple functions from $(X, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with $\lim_n s_n(x) = f(x)$, $\forall x \in X$.

Proof. We have $f = f^+ - f^-$ where f^+, f^- are simple positives.

By Theorem**. 5.6**.there exist simple positive functions s'_n, s''_n such that: $\lim_n s'_n(x) = f^+(x)$, $\forall x \in X$ and $\lim_n s''_n(x) = f^-(x)$, $\forall x \in X$. Then $s_n = s'_n - s''_n$ is measurable simple and $\lim_n s_n(x) = f^+(x) - f^-(x) = f(x)$, $\forall x \in X$.■

Corollary.

Let $f \in \mathcal{M}(X, \mathbb{R})$ and suppose f bounded. Then there is a sequence (s_n) of measurable simple functions converging uniformly to f on X.

Proof. By the Proposition above it is enough to consider the case f positive. Since f is bounded there is n such that $n > f(x)$ for every $x \in X$. So there exists a sequence (s_n) of positive measurable simple functions

with $0 \leq f(x) - s_m(x) < \dfrac{1}{2^m}, \forall x \in X, \forall m > n$, from which we deduce the uniform convergence of s_n to f on X.■

6. Convergence of Measurable Functions

Let us recall that if $(X, \mathcal{F}, \mu)$ is a measure space, a subset N of X is a null set if there is $A \in \mathcal{F}$, with $\mu(A) = 0$ such that $N \subset A$.
In this section we describe different type of convergence of measurable functions and the relations between them.

Definition. 6.1.

Let $\mathcal{P}$ be a property depending on a variable $x \in X$, that is $\mathcal{P}$ may be true or false according to x. We say that $\mathcal{P}$ is true almost every where if there is a null subset N of X such that $\mathcal{P}$ is true for any x outside N.

Examples. 6.2.

(a) A function $f : X \longrightarrow \overline{\mathbb{R}}$ is said to be finite almost every where if there is a null subset N of X such that $f(x) \in \mathbb{R}\ \forall x \in X \backslash N$. If moreover $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ then $\{f = \pm\infty\} \in \mathcal{F}$ and the condition of finiteness almost every where may be written simply as $\mu\{f = \pm\infty\} = 0$.
(b) .A function $f : X \longrightarrow \mathbb{R}$ is said to be bounded almost every where if there is a constant $M > 0$ and a null subset N such that $|f(x)| \leq M, \forall x \in X \backslash N$. If moreover $f \in \mathcal{M}(X, \mathbb{R})$ then $\{|f| > M\} \in \mathcal{F}$ and the condition of boundedness almost every where may be written simply as $\mu\{|f| > M\} = 0$.

(c). Let $f, g : X \longrightarrow \overline{\mathbb{R}}$ be functions. We say that $f = g$ almost every where if there is a null subset N such that $f(x) = g(x), \forall x \in X \backslash N$. If moreover $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$, the condition may be written as $\mu\{f \neq g\} = 0$.

Abbreviation. almost every where with respect to μ is abbreviated to: $\mu - a.e$

Definition. 6.3.

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. We say that f_n converges $\mu - a.e$ if the set $N = \left\{\limsup_n f_n \neq \liminf_n f_n\right\}$ is a null set. In other words f_n converges $\mu - a.e$ if for each $x \in X \backslash N$ the real sequence $f_n(x)$ converge to the real number $f(x)$, that is: $\forall \epsilon > 0, \exists m(\epsilon, x) \geq 1$ such that $\forall n \geq m(\epsilon, x), |f_n(x) - f(x)| < \epsilon$.

Definition. 6.4.

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. We say that f_n is a Cauchy sequence $\mu - a.e$ if there is a null subset N such that for each $x \in X \backslash N$ the real sequence $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, that is satisfies the following condition:

$$\forall \epsilon > 0, \exists M(\epsilon, x) \geq 1 \text{ such that } \forall n, m \geq M(\epsilon, x), |f_n(x) - f_m(x)| < \epsilon$$

Proposition. 6.5.

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. The following conditions are equivalent:
(a) The sequence f_n converges to $\mu - a.e$ to a function $f : X \longrightarrow \mathbb{R}$
(b) f_n is a Cauchy sequence $\mu - a.e$

Proof. For each x outside of a null set $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, so the Proposition results from the validity of the same properties in $\mathbb{R}$.■

Now let us come to the convergence of measurable functions.

Proposition. 6.6.

Let f_n be a sequence of functions in $\mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ converging $\mu - a.e$ on X. Then there is $f \in \mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ such that f_n converges $\mu - a.e$ to f.
Conversely if there is $f : X \longrightarrow \overline{\mathbb{R}}$ such that f_n converges $\mu - a.e$ to f, then f is measurable on a set E with $\mu\left(E^c\right) = 0$.

Proof. Take $E = \left\{x : \limsup_n f_n(x) = \liminf_n f_n(x)\right\}$ and take f defined by:

$f(x) = \liminf_n f_n(x)$ for $x \in E$ and $f(x) = 0$ for $x \in E^c$

(see Definition **4.6** for the measurability of f on E).■

Definition. 6.7. (uniform convergence $\mu - a.e$)

Let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of functions. We say that f_n converges uniformly $\mu - a.e$ to the function $f : X \longrightarrow \mathbb{R}$ if there is a null set N such that f_n converges uniformly to f on $X \backslash N$, that is:

$\forall \epsilon > 0, \exists M(\epsilon) \geq 1$ such that $\forall n \geq M(\epsilon), \left|f_n(x) - f(x)\right| < \epsilon, \forall x \in X \backslash N$

We say that f_n is a Cauchy sequence for the uniform convergence $\mu - a.e$ if there is a null set N such that:

$\forall \epsilon > 0, \exists M(\epsilon) \geq 1$ such that $\forall n, m \geq M(\epsilon), \left|f_n(x) - f_m(x)\right| < \epsilon, \forall x \in X \backslash N$

let us observe that the integer $M(\epsilon)$ does not depend on x.

Remark. 6.8.

In most of our discussion, especially in integration theory, we frequently use a complete measure space $(X, \mathcal{F}, \mu)$ as our basic space.
So in this case every null set is in $\mathcal{F}$ and this avoids some cumbersome measurability character of functions.

The following Theorem localizes the points of the space X where the convergence of a sequence fails to be uniform. Let us start with an example:

Example. 6.9.

Consider the space $X = [0, 1]$ endowed with the Lebesgue measure μ and let $f_n : X \longrightarrow \mathbb{R}$ be the sequence of functions given by $f_n(x) = x^n, x \in [0, 1]$. The sequence converges pointwise to the function f given by $f(x) = 0$ for $0 \leq x < 1$, and $f(x) = 1$ for $x = 1$, but the convergence is not uniform (why?). However for $\epsilon > 0$, we see that the sequence f_n converges uniformly on the interval $\left[0, 1 - \frac{\epsilon}{2}\right]$; intuitevely the points where the uniform convergence fails are localized in the set $B = \left[1 - \frac{\epsilon}{2}, 1\right]$ and $\mu(B) < \epsilon$.

Theorem. 6.10. (Egorov)

Let $(X, \mathcal{F}, \mu)$ be a measure space, with $\mu(X) < \infty$. Let $f_n, f \in \mathcal{M}\left(X,\overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e$.
Suppose that the sequence f_n converges $\mu - a.e$ to f on X. Then we have:

For every $\epsilon > 0$ there is $B \in \mathcal{F}$ such that $\mu(B) < \epsilon$
and f_n converges uniformly to f on $X \backslash B$.

Proof. Without losing general hypothesis, we can assume that:
f_n, f take values in $\mathbb{R}$ and f_n converges everywhere to f on X.

Let $E_n^m = \underset{j\geq n}{\cap} \left\{|f_j - f| < \frac{1}{m}\right\}$, since f_n, f are measurable we get $E_n^m \in \mathcal{F}$,$\forall n, m$. Moreover it is clear that $E_n^m \subset E_{n+1}^m \subset .... \subset \underset{n\geq 1}{\cup} E_n^m$. Since f_n converges everywhere to f on X, we have $\underset{n\geq 1}{\cup} E_n^m = X, \forall m \geq 1$.
So $X \backslash E_n^m \supset X \backslash E_{n+1}^m \supset .... \supset \underset{n\geq 1}{\cap} (X \backslash E_n^m) = \phi$ for each $m \geq 1$. Since $\mu(X) < \infty$ we deduce that $\lim_n \mu(X \backslash E_n^m) = 0$; so for each $m \geq 1$ there is $n(m) \geq 1$ such that $\mu\left(X \backslash E_{n(m)}^m\right) < \dfrac{\epsilon}{2^m}$. Now put $B = \underset{m\geq 1}{\cup} X \backslash E_{n(m)}^m$; then we have:
$\mu(B) \leq \sum_{m\geq 1} \mu\left(X \backslash E_{n(m)}^m\right) < \sum_{m\geq 1} \dfrac{\epsilon}{2^m} = \epsilon$. So $\mu(B) < \epsilon$ and $X \backslash B = \underset{m\geq 1}{\cap} E_{n(m)}^m$, therefore $|f_n(x) - f(x)| < \frac{1}{m}, \forall x \in X \backslash B, \forall n > n(m)$ and then the uniform convergence of f_n to f on $X \backslash B$.■

Remark. 6.11.

Egorov'Theorem is not valid in the case μ infinite as is shown by the following:
Take for $(X, \mathcal{F}, \mu)$ the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with μ the counting measure;
if $f_n = I_{\{1,2,\ldots,n\}}$ then $f_n(k)$ converges to 1 for each $k \in \mathbb{N}$; nevertheless there is no $F \subset \mathbb{N}$ such that $\mu(F) < \epsilon$ and f_n converges uniformly to 1 on $X \backslash F$ (indeed take $0 < \epsilon < 1$).

Remark. 6.12.

It is not difficult to prove the equivalence of the following assertions:
(a) f_n converges almost uniformly
(b) f_n is a Cauchy sequence for the almost uniform convergence.

Definition. 6.13.

Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f_n, f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$
(a) the sequence f_n converges almost uniformly if:
$\forall \epsilon > 0\ \exists B \in \mathcal{F}$ such that $\mu(B) < \epsilon$ and f_n converges uniformly to f on $X \backslash B$.
(b) the sequence f_n is a Cauchy sequence for the almost uniform convergence if:
$\forall \epsilon > 0\ \exists B \in \mathcal{F}$ such that $\mu(B) < \epsilon$ and f_n is a Cauchy sequence for the uniform convergence on $X \backslash B$.

Here is a specific type of convergence of measurable functions:

Definition. 6.14.

Let $f_n, f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$.
We say that the sequence (f_n) converges in measure to f if:

$\forall \epsilon > 0$, $\lim_n \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$

Notation: $f_n \xrightarrow{\mu} f$

Proposition. 6.15.

The almost uniform convergence implies:
(a) The convergence $\mu - a.e$
(b) The convergence in measure

Proof. By almost uniform convergence we have:

$\forall k \geq 1, \exists F_k \in \mathcal{F}$, with $\mu(F_k) < \frac{1}{k}$, and f_n converges uniformly on $X \backslash F_k$.
Take $F = \underset{k}{\cap} F_k$ then $F \in \mathcal{F}$, $\mu(F) = 0$. If $x \in X \backslash F$, there is k such that $x \in X \backslash F_k$, so $\lim\limits_n f_n(x) = f(x)$ and proves (a).
By almost uniform convergence we have:
$\forall \delta > 0, \exists F_\delta \in \mathcal{F}$, with $\mu(F_\delta) < \delta$, and f_n converges uniformly on $X \backslash F_\delta$.
Put $E_n(\epsilon) = \{x : |f_n(x) - f(x)| > \epsilon\}$, then $E_n(\epsilon) = E_n(\epsilon) \cap F_\delta + E_n(\epsilon) \cap X \backslash F_\delta$; we deduce that $\mu(E_n(\epsilon)) < \delta + \mu(E_n(\epsilon) \cap X \backslash F_\delta)$. Now since f_n converges uniformly on $X \backslash F_\delta$ there is $N(\epsilon, \delta) \geq 1$ such that for $n \geq N(\epsilon, \delta)$, $\mu(E_n(\epsilon) \cap X \backslash F_\delta) = 0$. This proves that $\forall \epsilon > 0, \lim\limits_n \mu(E_n(\epsilon)) = 0$ whence $f_n \xrightarrow{\mu} f$.■

Proposition. 6.16.

Let $(X, \mathcal{F}, \mu)$ be a measure space, with $\mu(X) < \infty$. Then:
The convergence $\mu - a.e$ implies the convergence in measure.

Proof. By Egorov Theorem (**6.10**) convergence $\mu - a.e$ implies almost uniform convergence from which the convergence in measure comes by Proposition **6.15.**■

Proposition. 6.17.

If $f_n \xrightarrow{\mu} f$ then f_n is a Cauchy sequence for the convergence in measure that is:

$$\forall \epsilon > 0, \lim_{n,m} \mu\{x : |f_n(x) - f_m(x)| > \epsilon\} = 0$$

Moreover if also $f_n \xrightarrow{\mu} g$ then $f = g \ \mu - a.e.$

Proof. Since $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$, we deduce that:

$$\{x : |f_n(x) - f_m(x)| > \epsilon\} \subset \left\{x : |f_n(x) - f(x)| > \tfrac{\epsilon}{2}\right\} \cup \left\{x : |f_m(x) - f(x)| > \tfrac{\epsilon}{2}\right\}$$

and we have:

$$\mu\{x : |f_n(x) - f_m(x)| > \epsilon\} \leq \mu\left\{x : |f_n(x) - f(x)| > \tfrac{\epsilon}{2}\right\} + \mu\left\{x : |f_m(x) - f(x)| > \tfrac{\epsilon}{2}\right\}$$

so
$$\lim_{n,m} \mu\{x : |f_n(x) - f_m(x)| > \epsilon\} \leq \lim_n \mu\left\{x : |f_n(x) - f(x)| > \tfrac{\epsilon}{2}\right\} + \lim_m \mu\left\{x : |f_m(x) - f(x)| > \tfrac{\epsilon}{2}\right\} = 0$$

now suppose $f_n \xrightarrow{\mu} g$; it is clear that

$$\{x : |f(x) - g(x)| > 0\} = \bigcup_n \left\{x : |f(x) - g(x)| > \tfrac{1}{n}\right\}$$

and
$$\left\{x : |f(x) - g(x)| > \tfrac{1}{n}\right\} \subset \left\{x : |f(x) - f_k(x)| > \tfrac{1}{2n}\right\} \cup \left\{x : |f_k(x) - g(x)| > \tfrac{1}{2n}\right\}, \forall k, n;$$
then
$$\mu\left\{x : |f(x) - g(x)| > \tfrac{1}{n}\right\} \leq \mu\left\{x : |f(x) - f_k(x)| > \tfrac{1}{2n}\right\} + \mu\left\{x : |f_k(x) - g(x)| > \tfrac{1}{2n}\right\}$$

the right side goes to 0 as $k \longrightarrow \infty$, for each n since $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, so $\mu\left\{x : |f(x) - g(x)| > \frac{1}{n}\right\} = 0$ for all n and then $\mu\{x : |f(x) - g(x)| > 0\} = 0$ whence $f = g \ \mu - a.e.$■

Lemma. 6.18.

Every Cauchy sequence in measure f_n contains a subsequence f_{n_k} satisfying Cauchy condition for the almost uniform convergence (Definition **6.13**(b)).

Proof. Left to the reader.■

Theorem. 6.19.

Every Cauchy sequence in measure f_n converges in measure to a measurable function f

Proof. By Lemma **6.18**, f_n contains a subsequence f_{n_k} satisfying the Cauchy condition for the almost uniform convergence. So from Remark**.6.12** the subsequence f_{n_k} converges almost uniformly to some measurable function f and then f_{n_k} converges in measure to f by Proposition**. 6.15** (b)**.** But f_n itself converges in measure to f, indeed we have:

$\{x : |f_n(x) - f(x)| > \epsilon\} \subset \left\{x : |f_n(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} \cup \left\{x : |f(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\}$

and $\mu\{x : |f_n(x) - f(x)| > \epsilon\} \leq$

$\mu\left\{x : |f_n(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} + \mu\left\{x : |f(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\}$

so if $n, k \longrightarrow \infty, \mu\left\{x : |f_n(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} \longrightarrow 0$, since f_n is Cauchy sequence in measure and $\mu\left\{x : |f(x) - f_{n_k}(x)| > \frac{\epsilon}{2}\right\} \longrightarrow 0$ because f_{n_k} converges in measure to f .■

7. Exercises

24. (a) Prove that in any measure space the uniform convergence implies the convergence in measure.

(b) In the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ the uniform convergence is equivalent to the convergence in measure.

25. In the space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ consider the sequence of indicator functions $f_n = I_{\{1,2,\ldots,n\}}$; prove that f_n converges $\mu - a.e$ but does not converge in measure.What do we deduce about Proposition**. 6.16.**

26. Let $f_n, f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ be functions finite $\mu - a.e.$. Suppose f_n converges pointwise to f and there is a positive measurable function g satisfying $\lim\limits_n \mu\{g > \epsilon_n\} = 0$ for some sequence of positive numbers ϵ_n with $\lim\limits_n \epsilon_n = 0$. Then if $|f_n| \leq g, \forall n$, prove that f_n converges in measure to f.

27. Let $f : X \longrightarrow \mathbb{R}$ be measurable in the space $(X, \mathcal{F}, \mu)$ and put:

$M(f) = \inf\{\alpha \geq 0 : \quad \mu\{|f| > \alpha\} = 0\}$,Prove that $|f| \leq M(f) \quad \mu - a.e.$

Prove that $\lim\limits_n M(f_n - f) = 0$ iff $\lim\limits_n f_n = f$ uniformly $\mu - a.e.$

28 Let $f_n, f : X \longrightarrow \mathbb{R}$ be measurable functions in the space $(X, \mathcal{F}, \mu)$ and suppose that f_n converges in measure to f ; if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a uniformly continuous function prove that the sequence $g \circ f_n$ converges in measure to $g \circ f$ ■.

Chapter 4

INTEGRATION

1. Preliminaries

Introduction.

Let $(X, \mathcal{F}, \mu)$ be a measure space. This chapter concerns the Lebesgue integration process $\int\limits_X f.d\mu$ of numerical measurable functions on X with respect to the measure μ. Such classes of functions have been introduced with their convergence properties in sections **1-3** of chapter **3**.

If X is the closed interval $[a, b]$ in the real system $\mathbb{R}$, it is also possible to define the Riemann integral $\int\limits_a^b f.dx$ of some function $f : [a, b] \longrightarrow \mathbb{R}$ (e.g continuous function).

If the Lebesgue integration process is applied to a sequence of Riemann integrable functions, it leads to a kind of convergence properties less restrictive and easier in applications than those needed in the Riemann process framework. Let us recall:

Classes of functions.1.1. (see sections 1-3 of chapter 3.)

$\mathcal{E} = \{s : X \longrightarrow \mathbb{R},\ s \text{ simple measurable}\}$

$\mathcal{E}_+ = \{s \in \mathcal{E} :\ s \text{ positive}\}$

$\mathcal{M}_+ = \{f : X \longrightarrow [0, \infty],\ \ f \ \text{ measurable}\}$

$\mathcal{M}(\mathbb{R}) = \{f : X \longrightarrow \mathbb{R},\ \ f \ \text{ measurable}\}$

$\mathcal{M}(\mathbb{C}) = \{f : X \longrightarrow \mathbb{C},\ \ f \ \text{ measurable}\}$

Let us recall that if $f \in \mathcal{M}_+$, there is an increasing sequence s_n in $\mathcal{E}_+$ with: $\lim\limits_n s_n(x) = f(x)$, $\forall x \in X$.

2. Integration in $\mathcal{E}_+$

Definition.2.1.

Let $s \in \mathcal{E}_+$ with $s(\cdot) = \sum\limits_1^n a_i.I_{A_i}(\cdot)$, where I_A is the Dirac function of the set A, and the sets $A_i, 1 \leq i \leq n$ form a partition of X in $\mathcal{F}$.

The integral of s with respect to μ is defined by:

$$\int\limits_X s.d\mu = \sum_1^n a_i.\mu(A_i)$$

with the convention $0 \cdot \infty = 0$.

Remark.2.2.

Suppose $s \in \mathcal{E}_+$ with $s(\cdot) = \sum\limits_{i=1}^n a_i.I_{A_i}(\cdot) = \sum\limits_{j=1}^m b_j.I_{B_j}(\cdot)$, where $\{A_i, 1 \leq i \leq n\}$ and $\{B_j, 1 \leq j \leq m\}$ are partitions of X. Then we have:

$A_i = \{x \in X : \ s(x) = a_i\}$ and $B_j = \{x \in X : \ s(x) = b_j\}$
so $a_i.I_{A_i \cap B_j}(\cdot) = b_j.I_{A_i \cap B_j}(\cdot)$ for $1 \leq i \leq n, 1 \leq j \leq m$.
$a_i.I_{A_i}(\cdot) = \sum_{j=1}^{m} a_i.I_{A_i \cap B_j}(\cdot)$ and $\sum_{i=1}^{n} a_i.I_{A_i}(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i.I_{A_i \cap B_j}(\cdot)$
likewise $\sum_{j=1}^{m} b_j.I_{B_j}(\cdot) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j.I_{A_i \cap B_j}(\cdot)$ and the terms in the two double sums are equivalent so $\sum_{i=1}^{n} a_i.\mu(A_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i.\mu(A_i \cap B_j)$
and $\sum_{j=1}^{m} b_j.\mu(B_j) = \sum_{j=1}^{m}\sum_{i=1}^{n} b_j.\mu(A_i \cap B_j)$ then $\sum_{i=1}^{n} a_i.\mu(A_i) = \sum_{j=1}^{m} b_j.\mu(B_j)$
we deduce that the integral $\int\limits_X s.d\mu = \sum_{1}^{n} a_i.\mu(A_i)$ is well defined.

Proposition.2.3.

Let s, t be in $\mathcal{E}_+$ and $c \geq 0$ then we have:

(1) $\int\limits_X (s+t).d\mu = \int\limits_X s.d\mu + \int\limits_X t.d\mu$

$\int\limits_X c.s.d\mu = c.\int\limits_X s.d\mu$

(2) If $s \leq t$ then $\int\limits_X s.d\mu \leq \int\limits_X t.d\mu$

(3) If $E \in \mathcal{F}$ and $s(\cdot) = \sum_{i=1}^{n} a_i.I_{A_i}(\cdot)$ we have $s.I_E = \sum_{i=1}^{n} a_i.I_{A_i \cap E}(\cdot)$ and

$\int\limits_X s.I_E.d\mu = \int\limits_E s.d\mu = \sum_{1}^{n} a_i.\mu(A_i \cap E)$

Proof. Put $s(\cdot) = \sum_{i=1}^{n} a_i.I_{A_i}(\cdot)$, $t(\cdot) = \sum_{j=1}^{m} b_j.I_{B_j}(\cdot)$, then

(1) $s + t = \sum_{i.j}.(a_i + b_j).I_{A_i \cap B_j}$, $c.s = \sum_{i=1}^{n} ca_i.I_{A_i}$

$\int\limits_X (s+t).d\mu = \sum_{i.j}.(a_i + b_j).\mu(A_i \cap B_j) = \sum_{i.j}.a_i.\mu(A_i \cap B_j) + \sum_{i.j}.b_j.\mu(A_i \cap B_j)$

but $\sum_{i=1}^{n} a_i.\sum_{j=1}^{m} \mu(A_i \cap B_j) = \sum_{i=1}^{n} a_i.\mu(A_i) = \int\limits_X s.d\mu$

and $\sum_{j=1}^{m} b_j.\sum_{i=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{m} b_j.\mu(B_j) = \int\limits_X t.d\mu$

so $\int\limits_X (s+t).d\mu = \int\limits_X s.d\mu + \int\limits_X t.d\mu$, similarly $\int\limits_X c.s.d\mu = c.\int\limits_X s.d\mu$
(2) If $s \leq t$, then $t - s \geq 0$ and $t = s + (t - s)$

so $\int\limits_X t.d\mu = \int\limits_X s.d\mu + \int\limits_X (t-s)\,.d\mu \geq \int\limits_X s.d\mu$. Point (3) is obvious.■

Theorem.2.4.

Let (s_n) be an increasing sequence in $\mathcal{E}_+$.
If $r \in \mathcal{E}_+$ is such that $r \leq \sup\limits_n .s_n$, then:

$$\int\limits_X r.d\mu \leq \sup\limits_n .\int\limits_X s_n.d\mu$$

Proof. Since s_n is increasing, the sequence $\int\limits_X s_n.d\mu$ is increasing in $[0,\infty]$ by Proposition **5.2.3**(2) so $\sup\limits_n .\int\limits_X s_n.d\mu$ exists in $[0,\infty]$. Let $0 < c < 1$ and put $E_n = \{s_n \geq cr\}$. Since $s_n \leq s_{n+1}$ we have $E_n \subset E_{n+1}$. On the other hand for $x \in X$ we have $c.r\,(x) < r\,(x) \leq \sup\limits_n .s_n\,(x)$, therefore there is n with $s_n\,(x) \geq c.r\,(x)$ and this gives $X = \bigcup\limits_n E_n$. Now put $r = \sum\limits_i \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int\limits_X s_n.d\mu \geq \int\limits_X c.r.I_{E_n}.d\mu$ (since $s_n \geq c.r.I_{E_n}$ on X), then $\int\limits_X s_n.d\mu \geq c.\sum\limits_i \alpha_i.\mu\,(A_i \cap E_n)\,, \forall n$. This implies $\sup\limits_n .\int\limits_X s_n.d\mu \geq \lim\limits_n .\left(c.\sum\limits_i \alpha_i.\mu\,(A_i \cap E_n)\right) = c.\sum\limits_i \alpha_i.\mu\,(A_i) = c.\int\limits_X r.d\mu$, because $\mu\,(A_i \cap E_n)$ goes to $\mu\,(A_i)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get the proof.■

Corollary.

Let s_n, t_n be two increasing sequences in $\mathcal{E}_+$ such that $\sup\limits_n .s_n = \sup\limits_n .t_n$ then $\sup\limits_n .\int\limits_X s_n.d\mu = \sup\limits_n .\int\limits_X t_n.d\mu$

Proof. We have $\sup\limits_n .s_n = \sup\limits_n .t_n \Longrightarrow s_k \leq \sup\limits_n .t_n$, $\forall k$; from the Theorem we get $\int\limits_X s_k.d\mu \leq \sup\limits_n .\int\limits_X t_n.d\mu$, this gives $\sup\limits_k .\int\limits_X s_k.d\mu \leq \sup\limits_n .\int\limits_X t_n.d\mu$. By the same way we prove the reverse inequality.■

Now we are in a position to extend the integration process from the class $\mathcal{E}_+$ to the class $\mathcal{M}_+ = \{f : X \longrightarrow [0,\infty]\,,\ f \ \text{ measurable}\}$.

3. Integration in $\mathcal{M}_+$

Definition.3.1.

Let $f \in \mathcal{M}_+$, we know by Theorem**. 5.6.** that for some increasing sequence s_n in $\mathcal{E}_+$ we have $\lim\limits_n . s_n(x) = f(x)$, $\forall x \in X$.

We define the integral of f with respect to μ by $\int\limits_X f.d\mu = \sup\limits_n . \int\limits_X s_n.d\mu$.

This integral is well defined, that is, it does not depend on the sequence s_n in $\mathcal{E}_+$ converging to f (corollary of Theorem**.2.4.**).

Definition.3.2.

Let $f \in \mathcal{M}_+$ and $E \in \mathcal{F}$. We define the integral of f over E by:

$$\int\limits_E f.d\mu = \int\limits_X f.I_E.d\mu$$

where $(f.I_E)(x) = f(x)$ for $x \in E$ and $(f.I_E)(x) = 0$ for $x \in E^c$

Proposition.3.3.

The integral in $\mathcal{M}_+$ has the following properties:

If $f, g \in \mathcal{M}_+$, $c \geq 0$, and $E, F \in \mathcal{F}$, then:

(1) $\int\limits_X (f+g).d\mu = \int\limits_X f.d\mu + \int\limits_X g.d\mu$

$\int\limits_X c.f.d\mu = c.\int\limits_X f.d\mu$

(2) If $f \leq g$ then $\int\limits_X f.d\mu \leq \int\limits_X g.d\mu$ and $\int\limits_E f.d\mu \leq \int\limits_E g.d\mu$

(3) $E \subset F \Longrightarrow \int\limits_E f.d\mu \leq \int\limits_F f.d\mu$

(4) If $f = 0$ on E then $\int\limits_E f.d\mu = 0$ even if $\mu(E) = \infty$.

(5) If $\mu(E) = 0$ then $\int\limits_E f.d\mu = 0$ even if $f = \infty$ on E.

Proof. All properties are consequence of Definitions **3.1-3.2.**$\blacksquare$

Theorem.3.4.

Let $f \in \mathcal{M}_+$ then we have:

$$\int\limits_X f.d\mu = \sup.\left\{\int\limits_X s.d\mu : \ s \in \mathcal{E}_+ \text{ and } s \leq f\right\}$$

Proof. If $s \in \mathcal{E}_+$ and $s \leq f$ then $\int\limits_X s.d\mu \leq \int\limits_X f.d\mu$

so $\sup\limits_{n}.\left\{\int\limits_X s.d\mu : \ s \in \mathcal{E}_+ \text{ and } s \leq f\right\} \leq \int\limits_X f.d\mu.$

But by Definition **5.3.1.** we have $\int\limits_X f.d\mu = \sup\limits_{n}.\left\{\int\limits_X s_n.d\mu, \ s_n \in \mathcal{E}_+ \text{ and } s_n \leq f\right\}$ from this we deduce the proof of the Theorem.■

Theorem.3.5. (Beppo-Levy monotone convergence Theorem)

Let (f_n) be an increasing sequence in $\mathcal{M}_+$, then:

$\lim\limits_{n} f_n = f \in \mathcal{M}_+$ and $\int\limits_X f.d\mu = \lim\limits_{n} \int\limits_X f_n\, d\mu$, in other words:

$$\lim_{n} \int\limits_X f_n\, d\mu = \int\limits_X \lim_{n} f_n\, d\mu$$

Proof. We know that $\lim\limits_{n} f_n = f \in \mathcal{M}_+$ (see chapter 4, section 2) and since (f_n)

is increasing we have $\int\limits_X f_n\, d\mu \leq \int\limits_X f_{n+1}\, d\mu \leq \int\limits_X f.d\mu$, $\forall n$. So $a = \lim\limits_{n} \int\limits_X f_n\, d\mu$ exists

and $a \leq \int\limits_X f.d\mu$. Let $s \in \mathcal{E}_+$ with $s \leq f$ and for $0 < c < 1$ put $E_n = \{f_n \geq c.s\}$.

We have $E_n \subset E_{n+1}$ since $f_n \leq f_{n+1}$ and $\bigcup\limits_{n} E_n = X$ because $c.s < f = \sup\limits_{n} f_n$.

On the other hand $f_n \geq 0 \Longrightarrow f_n \geq c.s.I_{E_n}, \forall n$.

Now put $s = \sum\limits_{i} \alpha_i.I_{A_i}$ and taking integrals, we obtain $\int\limits_X f_n.d\mu \geq \int\limits_X c.s.I_{E_n}.d\mu$

(since $f_n \geq c.s.I_{E_n}$ on X), then $\int\limits_X f_n.d\mu \geq c.\sum\limits_{i} \alpha_i.\mu(A_i \cap E_n), \forall n$. This implies

$a = \lim\limits_{n}.\int\limits_X f_n.d\mu \geq \lim\limits_{n}.\left(c.\sum\limits_{i} \alpha_i.\mu(A_i \cap E_n)\right) = c.\sum\limits_{i} \alpha_i.\mu(A_i) = c.\int\limits_X s\, d\mu$, because $\mu(A_i \cap E_n)$ goes to $\mu(A_i)$ since E_n is increasing to X. Making $c \longrightarrow 1$ we get $a \geq \int\limits_X s\, d\mu$ for all $s \in \mathcal{E}_+$ with $s \leq f$, so $a \geq \sup\left\{\int\limits_X s\, d\mu, s \in \mathcal{E}_+, s \leq f\right\} =$

$\int\limits_X f.d\mu$ by Theorem.5.3.4, then $a = \int\limits_X f.d\mu$.■

Remark. Theorem.**3.5.** is not valid in general for decreasing sequences (f_n) as is shown by the following example: let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ be the Borel measure space and $f_n = I_{]n,\infty[}$, then f_n decreases to 0 but $\lim\limits_{n}.\int\limits_X f_n.d\mu = \infty$.■

Lemma 3.6. (Fatou Lemma)

Let (f_n) be any sequence in $\mathcal{M}_+$, then:

$$\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$$

Proof. Put $F_k = \inf_{n \geq k} f_n$ then F_k is increasing in $\mathcal{M}_+$ to $\liminf_n f_n$,

so by Theorem.5.3.5, $\lim_k . \int_X F_k . d\mu = \int_X \liminf_n f_n \, d\mu$.

But $F_k \leq f_n, \forall n \geq k$, which implies $\int_X F_k . d\mu \leq \inf_{n \geq k} \int_X f_n \, d\mu$ and then

making $k \longrightarrow \infty$ we get $\lim_k . \int_X F_k . d\mu = \int_X \liminf_n f_n \, d\mu \leq \lim_k \inf_{n \geq k} \int_X f_n \, d\mu =$

$\liminf_n \int_X f_n \, d\mu . \blacksquare$

4. Exercises

29.(a) Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ be the counting measure on $\mathbb{N}$.

If $f : \mathbb{N} \longrightarrow [0, \infty[$ is given by $f(i) = a_i$ $i \in \mathbb{N}$ prove that:

$$\int_{\mathbb{N}} f . d\mu = \sum_i a_i$$

(b) Let $\mu = \delta_{x_0}$ be the Dirac measure on the power set $\mathcal{P}(X)$ of X.

then for any $f : X \longrightarrow [0, \infty[$, $\int_X f . d\mu = f(x_0)$.

30. Let (f_n) be any sequence in $\mathcal{M}_+$, prove that $\sum_n f_n \in \mathcal{M}_+$ and:

$$\int_X \sum_n f_n \, d\mu = \sum_n \int_X f_n . d\mu$$

(Hint $\sum_1^n f_i$ increases to $\sum_n f_n$ and use Theorem.**3.5**).

31. Let $f \in \mathcal{M}_+$

(a) Prove that the set function $\nu : A \longrightarrow \int_A f . d\mu$, defined on $\mathcal{F}$ is a positive measure

(b) If $g \in \mathcal{M}_+$ prove that $\int_X g . d\nu = \int_X f . g . d\mu$

(Hint: check (b) for $g \in \mathcal{E}_+$ and apply Theorem **3.5** for $g \in \mathcal{M}_+$)

32. Let (f_n) be a sequence in $\mathcal{M}_+$ with $\lim_n f_n(x) = f(x)$, $\forall x \in X$ for some $f \in \mathcal{M}_+$.Suppose $\sup_n \int_X f_n.d\mu < \infty$, and prove that $\int_X f.d\mu < \infty$

(Apply Fatou Lemma **3.6**)

33. Let (f_n) be a decreasing sequence in $\mathcal{M}_+$ such that

$\int_X f_{n_0}.d\mu < \infty$, for some $n_0 \geq 1$

Prove that $\lim_n \int_X f_n\, d\mu = \int_X \lim_n f_n\, d\mu$

(Hint: apply Theorem **3.5** to the increasing positive sequence $(f_{n_0} - f_n)\; n \geq n_0$)

34. Let the interval $]0,1[$ of real numbers be endowed with Lebesgue measure. Apply Fatou Lemma to the following sequence:

$f_n(x) = n, 0 \leq x \leq \frac{1}{n}$ and $f_n(x) = 0, 1 > x > \frac{1}{n}$.

5. Integration of Complex Functions

Definition.5.1.

Let $\mathcal{L}_1(\mu)$ be the subset of $\mathcal{M}(X,\mathbb{C})$ defined by:

$$\mathcal{L}_1(\mu) = \left\{ f \in \mathcal{M}(X,\mathbb{C}) : \int_X |f|\,.d\mu < \infty \right\}$$

where $\mathcal{M}(X,\mathbb{C}) = \{f : X \longrightarrow \mathbb{C} \;\; f \text{ measurable}\}$ (see Definitions **1.1** and **1.2**) if $f = u + iv \in \mathcal{L}_1(\mu)$ we define the integral of f by:

$$\int_X f.d\mu = \int_X u.d\mu + i\int_X v.d\mu = \int_X u^+.d\mu - \int_X u^-.d\mu + i\int_X v^+.d\mu - i\int_X v^-.d\mu$$

this integral is well defined since u^+, u^-, v^+, v^- are less then $|f|$.

If f is real valued, we have $v = 0$ and $\int_X f.d\mu = \int_X u^+.d\mu - \int_X u^-.d\mu$

Definition.5.2.

If $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ we define the integral of f by: $\int_X f.d\mu = \int_X f^+.d\mu - \int_X f^-.d\mu$

provided that $\int_X f^+.d\mu < \infty$ or $\int_X f^-.d\mu < \infty$

Proposition.5.3.

$\mathcal{L}_1(\mu)$ is a vector space on the field $\mathbb{C}$ and we have

$$\int_X (\alpha f + \beta g)\,.d\mu = \alpha\int_X f.d\mu + \beta\int_X g.d\mu$$

Proof. Use the following facts:
$|\alpha f+\beta g| \leq |\alpha| . |f| + |\beta| . |g|$ and
$f = u + iv = u^+ - u^- + iv^+ - iv^-$, $g = z + iw = z^+ - z^- + iw^+ - iw^-$
then apply Definition **5.1.**$\blacksquare$

Lemma.5.4.

Let f, g be in $\mathcal{L}_1(\mu)$ such that $f = g\ \mu - a.e.$ then $\int\limits_X f.d\mu = \int\limits_X g.d\mu$

Proof. Let $E = \{x : f(x) = g(x)\}$ then $\mu(E^c) = 0$
on the other hand we have $\int\limits_{E^c} f.d\mu = \int\limits_{E^c} g.d\mu = 0$ by point (5) Proposition **3.3** applied to the integrals of f^+, f^-, g^+, g^-, since $f.I_E = g.I_E$ we deduce that
$\int\limits_E f.d\mu = \int\limits_E g.d\mu$ that is $\int\limits_X f.d\mu = \int\limits_X g.d\mu.$$\blacksquare$

By the same way one can prove:

Proposition.5.5.

(1) If f, g are real valued in $\mathcal{L}_1(\mu)$ and $f \leq g.\mu - a.e.$ then $\int\limits_X f.d\mu \leq \int\limits_X g.d\mu$

(2) $\left|\int\limits_X f.d\mu\right| \leq \int\limits_X |f|.d\mu$ for all f in $\mathcal{L}_1(\mu)$.

(3) If $\in \mathcal{M}_+$ and $\int\limits_E f.d\mu = 0$ then $f = 0\ \mu - a.e.$ on E

(4) If $f \in \mathcal{L}_1(\mu)$ and $\int\limits_E f.d\mu = 0$ for all $E \in \mathcal{F}$ then $f = 0\ \mu - a.e.$

(5) If $f \in \mathcal{M}\left(X, \overline{\mathbb{R}}\right)$ and $\int\limits_X |f|.d\mu < \infty$ then $\mu\{|f| = +\infty\} = 0$,
i.e f is finite $\mu - a.e.$

Corollary.

Let f, g be in $\mathcal{L}_1(\mu)$:

(a) $\int\limits_E f.d\mu = \int\limits_E g.d\mu\ \forall E \in \mathcal{F} \Longrightarrow f = g.\mu - a.e.$

(b) If f, g are real valued then $\int\limits_E f.d\mu \leq \int\limits_E g.d\mu$, $\forall E \in \mathcal{F} \Longrightarrow f \leq g.\mu - a.e.$

6. The Banach Space $L_1(\mu)$

Definition 6.1
The binary relation $f = g \ \ \mu - a.e$ is an equivalence relation on $\mathcal{L}_1(\mu)$
Let $L_1(\mu)$ be the quotient of $\mathcal{L}_1(\mu)$ by this equivalence relation, that is $L_1(\mu)$ is the set of equivalence classes in $\mathcal{L}_1(\mu)$.
It is well known that $L_1(\mu)$ is a vector space on $\mathbb{R}$ with the operations defined by: class(x) +class(y) =class$(x+y)$ and α.class(x) =class$(\alpha.x)$.
In the sequel we consider elements of $L_1(\mu)$ as functions although they are classes of functions.
If $f \in L_1(\mu)$, formula $\|f\| = \int_X |f|\, d\mu$ defines a norm on $L_1(\mu)$

Theorem.6.2
Endowed with the norm $\|f\| = \int_X |f|\, d\mu$ the space $L_1(\mu)$ is a Banach space.

Proof. Let (f_n) be a Cauchy sequence in $L_1(\mu)$ then we have:

$$\forall j \geq 1, \exists N_j \geq 1 \text{ such that } n, m \geq N_j \implies \|f_n - f_m\| < \frac{1}{2^j}$$

let us define the strictly increasing subsequence $n_1 < n_2 < n_3 < ....$ by the following recipe:

$$n_1 = N_1, n_2 = \max(n_1 + 1, N_2), ..., n_j = \max(n_{j-1} + 1, N_j), ...$$

then we have: $\left\|f_{n_{j+1}} - f_{n_j}\right\| < \frac{1}{2^j}, ..\forall j = 1, 2, ...$

now consider the functions: $g_k = \sum_{j=1}^{k} \left|f_{n_{j+1}} - f_{n_j}\right|$ and $g = \sum_{j=1}^{\infty} \left|f_{n_{j+1}} - f_{n_j}\right|$

$$\|g_k\| \leq \sum_{j=1}^{k} \left\|f_{n_{j+1}} - f_{n_j}\right\| \leq \sum_{j=1}^{k} \frac{1}{2^j} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < 1 \text{ and also } \|g\| < 1$$

so g is integrable $\Longrightarrow g$ is finite $\mu - a.e$
let us define the function $f : X \longrightarrow \mathbb{R}$ by $f(x) = f_{n_1}(x) + \sum_{j=1}^{\infty} \left(f_{n_{j+1}}(x) - f_{n_j}(x)\right)$
then

$$\text{we have obviously } f(x) = \lim_{i \longrightarrow \infty} .f_{n_j}(x)$$

now let us observe that the sequence $\left(f_{n_j}\right)$ is cauchy since it is a subsequence of (f_n) which is cauchy so
$\forall \epsilon > 0, ..\exists N_\epsilon \geq 1 : n_j, m \geq N \Longrightarrow \left\|f_{n_j} - f_m\right\| = \int_X \left|f_{n_j} - f_m\right| .d\mu < \epsilon$
by Fatou lemma **3.6** applied for n_j we get $\int_X .\liminf_{n_j} \left|f_{n_j} - f_m\right| .d\mu = \int_X |f - f_m| .d\mu \leq \liminf_{n_j} . \int_X . \left|f_{n_j} - f_m\right| .d\mu \leq \limsup_{n_j} \int_X . \left|f_{n_j} - f_m\right| .d\mu < \epsilon$. So $f \in L_1(\mu)$ and $\lim_m . \int_X |f - f_m| .d\mu = 0.$ ■

Now we give one of the most famous convergence theorem of Lebesgue integration theory

Theorem.6.3 (Lebesgue's dominated convergence theorem)
Let (f_n) be a sequence in $L_1(\mu)$ such that:

(a) f_n converges $\mu - a.e$ to a function f

(b) there is g in $L_1(\mu)$ such that $\forall n \geq 1 \quad |f_n| \leq |g| \quad \mu - a.e$

Then the function f is in $L_1(\mu)$ and $\lim\limits_{n} \int_X |f_n - f| \, d\mu = 0$

in particular $\lim\limits_{n} \int_X f_n \, d\mu = \int_X f \, d\mu$

Proof. Put $E = \{x : f_n(x) \text{ converges to } f(x)\} \cup \left\{ \underset{n}{\cup} \{|f_n| \leq |g|\} \right\}$
then $\mu(E^c) = 0$
We can assume that f_n converges everywhere to a function f
and that $|f_n| \leq |g|$ everywhere $\forall n \geq 1$
(if necessary replace f_n by $F_n = f_n I_E$ and g by $G = gI_E$)
first since $|f_n| \leq |g|$ everywhere $\forall n \geq 1$ and f_n converges everywhere to f we deduce that

$|f| \leq |g|$ and $|f_n - f| \leq 2g$ so $2g - |f_n - f| \geq 0$

applying Fatou lemma **3.6** to the function $2g - |f_n - f|$ we get:

$\int_X .\liminf\limits_{n} [2g - |f_n - f|] \, .d\mu = \int_X \left[2g. - \limsup\limits_{n} |f_n - f|\right] .d\mu = \int_X 2g.d\mu \leq$ $\liminf\limits_{n}. \int_X [2g - |f_n - f|] \, .d\mu = \int_X 2g.d\mu - .\limsup\limits_{n}. \int_X |f_n - f| \, .d\mu$ and so $\int_X 2g.d\mu \leq$ $\int_X 2g.d\mu - .\limsup\limits_{n}. \int_X |f_n - f| \, .d\mu$, this gives $0 \leq -.\limsup\limits_{n}. \int_X |f_n - f| \, .d\mu$ that is $\limsup\limits_{n}. \int_X |f_n - f| \, .d\mu = 0.$■

Theorem.6.4 (Bounded convergence theorem)
Suppose $\mu(X) < \infty$. Let (f_n) be a sequence in $L_1(\mu)$ such that
$|f_n| \leq M \quad \mu - a.e$ for some constant $M > 0$ then the conclusions of Theorem **6.3** are valid.

Application.6.5 (continuity of integrals depending on a parameter)

Let T be an interval of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:

(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$

(b) there is g in $L_1(\mu)$such that$|f(x,t)| \leq |g(x)| \quad \mu - a.e$ for all $t \in T$

then we have $\lim\limits_{t \to t_0} \int\limits_X f(x,t) \, d\mu = \int\limits_X f(x,t_0) \, d\mu$

Application.6.6 (Derivative of integrals depending on a parameter)

Let T be an open set of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{R}$ a function such that:

(a) for each $t \in T$ the function $x \longrightarrow f(x,t)$ is in $L_1(\mu)$

(b) the function $t \longrightarrow f(x,t)$ derivable on T for each $x \in X$

(c) there is $g \in L_1(\mu)$ $\left| \frac{d}{dt} f(x,t) \right| \leq |g(x)| \quad \mu - a.e$ for all $t \in T$

Then the function $t \longrightarrow \int\limits_X f(x,t)\, d\mu$ is differentiable on T

and $\frac{d}{dt}\int\limits_X f(x,t)\, d\mu = \int\limits_X \frac{d}{dt} f(x,t)\, d\mu$

Application.6.7 (Change of variable formula)
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $(Y, \mathcal{G})$ be a measurable space:
If $\varphi : X \longrightarrow Y$ is a measurable mapping from $(X, \mathcal{F})$ into $(Y, \mathcal{G})$ then:
(1) the set function $\nu : \mathcal{G} \longrightarrow [0, \infty]$ given by $G \in \mathcal{G}, \nu(G) = \mu\left(\varphi^{-1}(G)\right)$
is a measure on $(Y, \mathcal{G})$
(2) for every function $g : Y \longrightarrow \mathbb{C}$, $\nu-$integrable the function $g \circ \varphi$ is $\mu-$integrable
and

$$(*) \int\limits_Y g.d\nu = \int\limits_X g \circ \varphi.d\mu$$

$$(**) \int\limits_E g.d\nu = \int\limits_{\varphi^{-1}(E)} g \circ \varphi.d\mu \ \forall E \in \mathcal{G}.$$

As a particular case take $(Y, \mathcal{G}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $\varphi : X \longrightarrow \mathbb{R}$, $\mu-$integrable
put $\nu(B) = \hat{\mu}(B) = \mu\left(\varphi^{-1}(B)\right)$ for $B \in \mathcal{B}_{\mathbb{R}}$

then we get from$(**)$: $\int\limits_{\varphi^{-1}(B)} \varphi.d\mu = \int\limits_B t.d\hat{\mu}$

Application.6.8
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f \in \mathcal{M}_+$ then

the set function $\nu : \mathcal{F} \longrightarrow [0, \infty]$ given by: $A \in \mathcal{F}, \nu(A) = \int\limits_A f.d\mu$

is a positive measure on $\mathcal{F}$ and we have:

$$\int\limits_X g.d\nu = \int\limits_X f.g.d\mu, \text{ for every } g \in \mathcal{M}_+.$$

7. The L_p-Spaces

Let $(X, \mathcal{F}, \mu)$ be a measure space. This section concerns a short description of the L_p-spaces with some important convexity inequalities.

Definition 7.1
Let $\mathcal{L}_p(\mu)$ be the subset of $\mathcal{M}(X, \mathbb{C})$ defined by:

$$\mathcal{L}_p(\mu) = \left\{ f \in \mathcal{M}(X, \mathbb{C}) : \int\limits_X |f|^p .d\mu < \infty \right\}$$

for some real number $0 < p < \infty$.

Definition 7.2
Two real positive numbers $0 < p, q < 1$ such that $p + q = pq$ or equivalently $\frac{1}{p} + \frac{1}{q} = 1$ are called conjugate exponents. If $p \longrightarrow 1$ then $q \longrightarrow \infty$ so $1, \infty$ are considered as conjugate exponents.

Theorem 7.3
Let $f, g \in \mathcal{M}_+$ and let $0 < p, q < 1$ be conjugate exponents then we have:

(1) Hölder's inequality: $\int_X f.g.d\mu \leq \left\{ \int_X f^p.d\mu \right\}^{\frac{1}{p}} . \left\{ \int_X g^q.d\mu \right\}^{\frac{1}{q}}$

(2) Minkowski's inequality: $\left\{ \int_X (f+g)^p.d\mu \right\}^{\frac{1}{p}} \leq \left\{ \int_X f^p.d\mu \right\}^{\frac{1}{p}} + \left\{ \int_X g^p.d\mu \right\}^{\frac{1}{p}}$

Remark: Using Minkowski's inequality it is not difficult to prove that $\mathcal{L}_p(\mu)$ is a vector space over $\mathbb{C}$.

Definition 7.4 Let $0 < p < \infty$ be a positive real number
The binary relation $f = g \ \ \mu - a.e$ is an equivalence relation on $\mathcal{L}_p(\mu)$
Let $L_p(\mu)$ be the quotient of $\mathcal{L}_p(\mu)$ by this equivalence relation, that is $L_p(\mu)$ is the set of equivalence classes in $\mathcal{L}_p(\mu)$.
It is well known that $L_p(\mu)$ is a vector space on $\mathbb{R}$ with the operations defined by: class(x) +class(y) =class$(x + y)$ and α.class(x) =class$(\alpha.x)$.
In the sequel we consider elements of $L_p(\mu)$ as functions although they are classes of functions.

Theorem 7.5
If $f \in L_p(\mu)$, formula $\|f\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}$
defines a norm on $L_p(\mu)$ and with respect to this norm $L_p(\mu)$ is a Banach space. (mimic the proof made for L_1 Theorem **6.2**)

Definition 7.6 The Hilbert Space $L_2(\mu)$

For $p = 2$ it is not difficult to see that the norm $\|f\|_2 = \left\{ \int_X |f|^2 \, d\mu \right\}^{\frac{1}{2}}$ is induced by the inner product $\langle f, g \rangle = \int_X f.\bar{g}.d\mu$, which makes $L_2(\mu)$ a Hilbert space.

8. The Space L_∞

Definition 8.1 Let $(X, \mathcal{F}, \mu)$ be a measure space.
Let $f \in \mathcal{M}_+$ we define the essential supremum of f by:

$$ess - \sup f = \left\{ \begin{array}{c} \alpha \geq 0 : \mu[f > \alpha] = 0 \\ \infty \text{ if } \mu[f > \alpha] > 0, \forall \alpha \geq 0 \end{array} \right\}$$

if $f \in \mathcal{M}(X, \mathbb{C})$ we put $N_\infty(f) = ess - \sup |f|$

Remark.
For $f \in \mathcal{M}(X, \mathbb{C})$ we have:

$\alpha \in \{\alpha \geq 0 : \mu[|f| > \alpha] = 0\} \iff |f| \leq \alpha \quad \mu - a.e$

Lemma.8.2
For $f \in \mathcal{M}(X, \mathbb{C})$ we have:

$\mu[|f| > N_\infty(f)] = 0$, that is $|f| \leq N_\infty(f) \quad \mu - a.e$

Definition 8.3
Let $\mathcal{L}_\infty(\mu)$ be the subset of $\mathcal{M}(X, \mathbb{C})$ defined by:

$\mathcal{L}_\infty(\mu) = \{f \in \mathcal{M}(X, \mathbb{C}) : N_\infty(f) < \infty\}$

It is easy to prove that the binary relation $f = g \quad \mu - a.e$ is an equivalence relation on $\mathcal{L}_\infty(\mu)$ and $N_\infty(f) = N_\infty(g)$ if $f = g \quad \mu - a.e$
Let $L_\infty(\mu)$ be the quotient of $\mathcal{L}_\infty(\mu)$ by this equivalence relation, that is $L_\infty(\mu)$ is the set of equivalence classes in $\mathcal{L}_\infty(\mu)$.
Also one can prove that $L_\infty(\mu)$ is a vector space on $\mathbb{R}$ with the operations defined by: class(f) +class(g) =class$(f+g)$ and α.class(f) =class$(\alpha.f)$.
In the sequel we consider elements of $L_\infty(\mu)$ as functions although they are classes of functions and

Definition 8.3

For any f in $L_\infty(\mu)$ define $\|f\|_\infty$ by $N_\infty(h)$ where h is any function satisfying $f = h \ \mu{-}a.e$ then $L_\infty(\mu)$ is a vector space on $\mathbb{C}$ and $\|f\|_\infty$ is a norm on $L_\infty(\mu)$:

Theorem 8.4
$L_\infty(\mu)$ endowed with the norm $\|f\|_\infty$ defined above is a Banach space.

An important property of the sequences (f_n) in the spaces L_p is the following:

Theorem 8.5

Let (f_n) be a cauchy sequence in L_p that is a sequence (f_n)
satisfying $\lim\limits_{m,n}. \|f_n - f_m\|_p = 0$ then:

(1) For $1 \leq p < \infty$, the sequence (f_n) contains a subsequence $\left(f_{n_j}\right)$
converging $\mu - a.e$ to a function $f \in L_p$

(2) For $p = \infty$ the sequence (f_n) itself
converges uniformly $\mu - a.e$ to a function $f \in L_\infty$.

9. Duality of the L_p-Spaces

Recall.
1 Let X, Y be normed spaces. A linear operator T from a normed space X into a normed space Y is said to be bounded if there is a constant $M > 0$ such that:

$$\|T(x)\| \leq M. \|x\|, \forall x \in X$$

This definition means that if B is a bounded subset of X, the set $\{T(x), x \in B\}$ is bounded in Y. For instance if $B = \{x : \|x\| \leq 1\}$ then $\|T(x)\| \leq M, \forall x \in B$.
2 Let T be a bounded operator from X into Y. Define:

$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\right\}$
$m_1 = \sup\{\|T(x)\| : x \in X, \|x\| = 1\}$
$m_2 = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$
$m_3 = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$
Then $m_1 = m_2 = m_3 = \|T\| < \infty$ and we have:
$\|T(x)\| \leq \|T\| \|x\|, \forall x \in X$
3 If X is a normed space the strong dual of X is the Banach space X^* of continuous linear functionals on X. If $x \in X$ and $x^* \in X^*$, we denote $x^*(x)$ by $\langle x^*, x\rangle$.

Definition 9.1
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \leq p, q \leq \infty$ be conjugate exponents. For g fixed in L_q let us define the functional φ_g on L_p by:
$$\varphi_g : L_p \longrightarrow \mathbb{C}, \quad f \in L_p \quad \varphi_g(f) = \int_X f.g.d\mu$$
It is clear that φ_g is well defined and we have:

Theorem 9.2
(a) φ_g is linear continuous on L_p for any $1 \leq p \leq \infty$.
Moreover if $p > 1$ we have $\|\varphi_g\| = \|g\|_q$
where $\|\varphi_g\| = \sup\{\|\varphi_g(f)\| : f \in L_p, \|f\| \leq 1\}$
(b) If μ is $\sigma-$finite (Definition **3.3** Chapter **2**) then we have
$\|\varphi_g\| = \|g\|_\infty$ for $p = 1$.

Theorem 9.3 **(L_p Duality)**
Let $(X, \mathcal{F}, \mu)$ be a measure space with μ $\sigma-$finite
and let $\varphi : L_p \longrightarrow \mathbb{C}$ be a continuous linear functional on L_p
If $1 \leq p < \infty$ there is a unique $g \in L_q$, for q conjugate exponent of p such that
$$\varphi(f) = \int_X f.g.d\mu \ \forall f \in L_p \text{ and } \|\varphi\| = \|g\|_q$$
In other words the strong dual $(L_p)^*$ of L_p is linearly isometric to L_q for q conjugate exponent of p.

Remark
(a) For $p = 1$ **Theorem 9.3** is not true in general if μ is not $\sigma-$finite
as is shown by the following example:
take $X = \{a, b\}$, $\mu(a) = 1$, $\mu(\phi) = 0$, $\mu(b) = \mu(X) = \infty$
then μ is not $\sigma-$finite. In this case we have
$L_1 = \{f : \{a, b\} \longrightarrow \mathbb{C}, \text{such that } f(b) = 0\} = \mathbb{C}$
so $L_1 = (L_1)^* = \mathbb{C}$, but $L_\infty = \left\{\begin{array}{c} f : \{a, b\} \longrightarrow \mathbb{C}, \text{such that} \\ \sup(f(a), f(b)) < \infty \end{array}\right\} = \mathbb{C}^2$.
(b) The Theorem **9.3** is not true in general for the space L_∞ even if μ is finite
in other words we have $L_1 \subset (L_\infty)^*$ and the inclusion is strict in general.
Here is an example:
(c) Let $[0, 1]$ the unit interval endowed with the Lebesgue μ and let $C[0, 1]$ be the space of real continuous functions on $[0, 1]$ equipped with the uniform norm $\|f\| = \sup\{|f(x)|, x \in [0, 1]\}$. Let us observe that if f, g are continuous

and satisfying $f = g \ \ \mu - a.e$ then $f = g$ everywhere, indeed let $F \subset [0,1]$ be measurable with $\mu(F) = 0$ and $f(x) = g(x)\ \forall x \in [0,1] \backslash F$, so the set $A = \{x \in [0,1] : |f(x) - g(x)| > 0\} = F$, but A is open by the continuity of f,g, then since $\mu(F) = 0$ the equality $A = F$ implies $F = \phi$ and so $f = g$ everywhere on $[0,1]$. Consequently the class of f for the equivalence relation $f = g \ \ \mu - a.e$ is reduced to only f. Since any $f \in C[0,1]$ is bounded we have $C[0,1] \subset L_\infty$.

Now let us consider the linear functional $\varphi : C[0,1] \longrightarrow \mathbb{R}$ given by $\varphi(f) = f(0)$, φ is continuous since $|\varphi(f)| \leq \|f\| = \sup\{|f(x)|, x \in [0,1]\}$ and $\|\varphi\| \leq 1$. By Hahn-Banach Theorem, φ can be extented to a continuous linear functional on all of L_∞; if there were some $g \in L_1$ such that $\varphi(f) = \int_{[0,1]} f.g.d\mu\ \forall f \in L_\infty$, we would have $f(0) = \int_{[0,1]} f.g.d\mu\ \forall f \in C[0,1]$.

Taking $f(x) = \cos(nx)$ we get $f(0) = 1 = \int_{[0,1]} \cos(nx).g.d\mu \quad \forall n \geq 1$, this leads to a contradiction since by the **Riemann-Lebesgue Lemma** ,(see Theorem **10.6** below) we have $\lim\limits_{n \longrightarrow \infty} . \int_a^b f(x) \cos(nx).d\,x = 0$.

10. Riemann Integral and Lebesgue Integral

In this section we consider a **bounded** function $f : [a,b] \longrightarrow \mathbb{R}$,defined on the interval $[a,b]$ with values in $\mathbb{R}$.

10.1 Definition (Darboux sums)
Let $\pi = \{I_1, I_2, ..., I_n\}$ be a finite partition of $[a,b]$ into intervals.
Put $m = \inf\{f(x), x \in [a,b]\}$ and $M = \sup\{f(x), x \in [a,b]\}$
$m_k = \inf\{f(x), x \in I_k\}$ and $M_k = \sup\{f(x), x \in I_k\}$, $1 \leq k \leq n$.
We define the lower and upper Darboux sums of f
with respect to the partition π by:

$$\underline{S}_\pi(f) = \sum_{k=1}^{k=n} m_k.\lambda(I_k) \text{ and } \overline{S}_\pi = \sum_{k=1}^{k=n} M_k.\lambda(I_k)$$

where $\lambda(I)$ is the lengh of the interval I.

10.2 Definition (Lower integral and Upper integral)
The Lower integral of f is defined by:

$$\underline{S}(f) = \sup \underline{S}_\pi(f)$$

The Upper integral of f is defined by:

$$\overline{S}(f) = \inf \overline{S}_\pi$$

where the sup and inf are taken over the finite partitions π of $[a,b]$.
It is clear that $\underline{S}(f) \leq \overline{S}(f)$. We say that f is integrable if $\underline{S}(f) = \overline{S}(f)$.

We define the Riemann integral of f on $[a,b]$ by $\int_a^b f(x)\ dx = \underline{S}(f) = \overline{S}(f)$.

10.3 Theorem
A **bounded** function $f : [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if it is continuous $\mu - a.e$, in this case the Riemann integral is equal to the Lebesgue integral, that is we have:

$$\int_a^b f(x)\ dx = \int_{[a,b]} f\ d\mu, \text{ where } \mu \text{ is the Lebesgue measure on } [a,b].$$

10.4 Theorem
Let f_n:$[a,b] \to \mathbb{R}$ be Riemann integrable functions and assume that f_n converges uniformly to f on $[a,b]$. Then f is Riemann integrable

and $\lim_n \int_a^b f_n\ dx = \int_a^b f\ dx$

If we replace uniform convergence by pointwise convergence, then the above Theorem shows that the limit function f does not have to be Riemann integrable. Therefore the above theorem is not true if we replace uniform convergence by pointwise convergence. There is however a version of the above theorem for pointwise convergence if we add the hypothesis that the limit function is Riemann integrable. This theorem is called **Arzela's Theorem** for the Riemann integral, which is a special case of the Bounded Convergence Theorem of Lebesgue for the Lebesgue integral.

10.5 Theorem (Arzela's Theorem). Let f, f_n:$[a,b] \to \mathbb{R}$ be Riemann integrable functions and assume that f_n converges pointwise to f on $[a,b]$. If there exists M such that $|f_n(x)| \leq M$ for all $n \geq 1$.Then $\lim_n \int_a^b f_n\ dx = \int_a^b f\ dx$.

10.6 Theorem (Riemann-Lebesgue Lemma)
If f is an intégrable function on the interval $[a,b]$, then :

$$\lim_{n \longrightarrow \infty} . \int_a^b f(x) \cos(nx) .dx = 0 \text{ and } \lim_{n \longrightarrow \infty} . \int_a^b f(x) \sin(nx) .dx = 0$$

The proof is easy if f is bounded or if f is C^1 using intégration by parts.

Chapter 5

INTEGRATION IN PRODUCT SPACES
Product Measure and Fubini Theorem

In this chapter we give without proofs the most important results on product spaces useful in applications.Proofs are classical and in general simple.

1. Preliminaries and Notations

1.1 In all what follows, $(X, \mathcal{F}, \mu)$, $(Y, \mathcal{G}, \nu)$ will be fixed measure spaces.
1.2 Let us recall that the product $\sigma-$field $\mathcal{F} \otimes \mathcal{G}$ on $X \times Y$ is generated by the family $\{A \times B, \text{ with } A \in \mathcal{F}, B \in \mathcal{G}\}$, (Definition **3.4** Chapter **1**)
1.3 The set $\mathbb{R}$ will be endowed with its Borel $\sigma-$field $\mathcal{B}_{\mathbb{R}}$. The set $\mathbb{R}^2$ endowed with the σ-field $\mathcal{B}_{\mathbb{R}^2}$=$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$(Theorem**2.9**Chap.**3**)

2. Product Measure

2.1 Definition
For any subset $E \subset X \times Y$ and any $(x, y) \in X \times Y$, define:
the section of E at x, $E_x = \{y \in Y, \quad (x, y) \in X \times Y\}$
the section of E at y, $E_y = \{x \in X, \quad (x, y) \in X \times Y\}$

2.2 Proposition
For every $E \in \mathcal{F} \otimes \mathcal{G}$, $E_x \in \mathcal{G}$ and $E_y \in \mathcal{F}$.

2.3 Theorem
Suppose that the measure μ and ν are $\sigma-$finite
then for every $E \in \mathcal{F} \otimes \mathcal{G}$, we have:
the function $x \longrightarrow \nu(E_x)$ is $\mathcal{F}$ measurable
the function $y \longrightarrow \mu(E_y)$ is $\mathcal{G}$ measurable
Moreover we have $\int\limits_X \nu(E_x)\, d\mu = \int\limits_Y \mu(E_y)\, d\nu$

Corollary.(Product measure)
Under the conditions of Theorem **1.6** the set function $\mu \otimes \nu$ defined on $\mathcal{F} \otimes \mathcal{G}$ by:

$$\mu \otimes \nu(E) = \int\limits_X \nu(E_x)\, d\mu = \int\limits_Y \mu(E_y)\, d\nu,\ E \in \mathcal{F} \otimes \mathcal{G}$$

is a $\sigma-$finite measure on $\mathcal{F} \otimes \mathcal{G}$. Moreover $\mu \otimes \nu$ is the unique $\sigma-$finite measure on $\mathcal{F} \otimes \mathcal{G}$ satisfying $\mu \otimes \nu(A \times B) = \mu(A) . \nu(B)$ for every $A \in \mathcal{F}, B \in \mathcal{G}$.

3 Integration in Product Spaces

3.1 Definition Let $f : X \times Y \longrightarrow \mathbb{R}$ be any function and $(x, y) \in X \times Y$, define:
$f_x : Y \longrightarrow \mathbb{R}$ by $f_x(y) = f(x, y)$ (section of f at x)
$f_y : X \longrightarrow \mathbb{R}$ by $f_y(x) = f(x, y)$ (section of f at y)

3.2 Proposition

Let $f : X \times Y \longrightarrow \mathbb{R}$ be $\mathcal{F} \otimes \mathcal{G}$-measurable then

f_x is $\mathcal{G}$-measurable and f_y is $\mathcal{F}$-measurable

3.3 Theorem (Fubini)

Suppose that the measure μ and ν are $\sigma-$finite and $f : X \times Y \longrightarrow \mathbb{R}$ is $\mathcal{F} \otimes \mathcal{G}$-measurable positive then:

the function $x \longrightarrow \int\limits_Y f(x, y)\ d\nu$ is $\mathcal{F}$-measurable

the function $y \longrightarrow \int\limits_X f(x, y)\ d\mu$ is $\mathcal{G}$-measurable

and we have:

$$\int\limits_{X \times Y} f(x, y)\ d\mu \otimes \nu = \int\limits_X d\mu \int\limits_Y f(x, y)\ d\nu = \int\limits_Y d\nu \int\limits_X f(x, y)\ d\mu$$

3.4 Theorem (Fubini)

For every $f \in L_1(\mu \otimes \nu)$ we have:

(a) $\int\limits_Y f(x, y)\ d\nu \in L_1(\mu)$ and $\int\limits_X f(x, y)\ d\mu \in L_1(\nu)$

(b) $\int\limits_{X \times Y} f(x, y)\ d\mu \otimes \nu = \int\limits_X d\mu \int\limits_Y f(x, y)\ d\nu = \int\limits_Y d\nu \int\limits_X f(x, y)\ d\mu$

3.5 Application. (Convolution of functions)

Let μ be the Lebesgue measure on $\mathbb{R}$,$\mathcal{B}_{\mathbb{R}}$ and $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions in $L_1(\mu)$, then:

$\int\limits_{\mathbb{R}} |f(x-y)| . |g(y)| .d\mu(y) < \infty$ for each x

Let us define the convolution of f and g by the function $h : \mathbb{R} \longrightarrow \mathbb{R}$:

$$h(x) = \int\limits_{\mathbb{R}} f(x-y) .g(y) .d\mu(y)$$

we denote h by $h = f * g$

Since $\left| \int\limits_{\mathbb{R}} f(x-y) .g(y) .d\mu(y) \right| \leq \int\limits_{\mathbb{R}} |f(x-y)| . |g(y)| .d\mu(y) < \infty$ we deduce that $h \in L_1(\mu)$

3.6 Lemma

Under the definition above we have $\|f * g\| \leq \|f\| . \|g\|$.■

4 Convolution of Measures

4.1 Definition
Let us consider on the set $\mathbb{R}^2$ endowed with the σ-field $\mathcal{B}_{\mathbb{R}^2}$=$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $T(x, y) = x + y$ which is measurable because continuous. Let $\mu_1 \otimes \mu_2$ be the product of two finite measures μ_1, μ_2 defined on $\mathbb{R}$, $\mathcal{B}_{\mathbb{R}}$. The convolution $\mu_1 * \mu_2$ of the measures μ_1, μ_2 is the measure on $\mathcal{B}_{\mathbb{R}}$ given by: $B \in \mathcal{B}_{\mathbb{R}}$, $(\mu_1 * \mu_2)(B) = (\mu_1 \otimes \mu_2)\left(T^{-1}(B)\right)$. Then we have:

4.2 Proposition Let $B \in \mathcal{B}_{\mathbb{R}}$ and define:

$\left[T^{-1}(B)\right]_x = \{y \in \mathbb{R}, x + y \in B\} = B - x$

$\left[T^{-1}(B)\right]_y = \{x \in \mathbb{R}, x + y \in B\} = B - y$

then we get: $(\mu_1 * \mu_2)(B) = \int_{\mathbb{R}} .\mu_2(B - x).\mu_1(dx) = \int_{\mathbb{R}} .\mu_1(B - y).\mu_2(dy)$

by applying Fubini Theorem and the relation $(\mu_1 * \mu_2)(B) = (\mu_1 \otimes \mu_2)\left(T^{-1}(B)\right) = \int_{X \times Y} .I_{T^{-1}(B)}(x, y).(\mu_1 \otimes \mu_2)(dx, dy)$.
Moreover if we take a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ integrable with respect to $\mu_1 * \mu_2$ we obtain the following nice relation:

$\int_{\mathbb{R}} f(t).(\mu_1 * \mu_2)(dt) = \int_{\mathbb{R}} \mu_2(dy) \int_{\mathbb{R}} f(x + y).\mu_1(dx) = \int_{\mathbb{R}} \mu_1(dx) \int_{\mathbb{R}} f(x + y).\mu_2(dy)$

4.3 Proposition With the definitions above we have:
(1) $\mu_1 * \mu_2 = \mu_2 * \mu_1$
(2) $(\mu_1 * \mu_2)(\mathbb{R}) = (\mu_1 \otimes \mu_2)\left(T^{-1}(\mathbb{R})\right) = (\mu_1 \otimes \mu_2)\left(\mathbb{R}^2\right) = \mu_1(\mathbb{R}).\mu_2(\mathbb{R})$
(3) $\mu_1 * \delta_0 = \delta_0 * \mu_1 = \mu_1$, δ_0 is the Dirac measure at 0.$\blacksquare$

Chapter 6

VECTOR INTEGRATION AND BOCHNER INTEGRAL

1. Vector Measures and Vector Measurable Functions

In what follows S will be an abstract set, $\mathcal{A} = \mathcal{A}(S)$ a $\sigma-$algebra of subsets of S, X a Banach space and X^* be the topological dual of X.

1.1.Definition . (i) A set function, $\mu : \mathcal{A} \to X$ is called $\sigma-$**additive** if for every pairwise disjoint sequence of sets $\{E_n\}$ in $\mathcal{A}$, the series $\sum\limits_n \mu(E_n)$ is unconditionally convergent in X and we have

$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n).$$

If, in addition, $\mu(\varnothing) = 0$, then μ is called a **vector measure**.

(ii) A set function $\mu : \mathcal{A} \to X$ is called **weakly** $\sigma-$**additive**. if for every pairwise disjoint sequence of sets $\{E_n\}$ in $\mathcal{A}$ we have

$$x^*\mu\left(\bigcup_n E_n\right) = \sum_n x^*\mu(E_n)$$

for each $x^* \in X^*$, in other words the real set function $x^*\mu$ is a real measure.

1.2.Remark . It is clear that a vector measure is weakly $\sigma-$additive. The converse is also true (see **Theorem 1.5** below).

1.3.Definition . The **semi-variation** of the vector measure μ is defined by the set function:

$$\|\mu\|(E) = \sup\left\|\sum_{i=1}^{n} \varepsilon_i \ \mu(E_i)\right\|, \quad E \in \mathcal{A},$$

the supremum being taken over all finite partitions $\{E_i\}$ of E in $\mathcal{A}$, and all finite systems of scalars$\{\varepsilon_i\}$ with $|\varepsilon_i| \leq 1$.

The semi-variation so defined is needed for some estimations in the integration process which will be used.

1.4.Definition . If $\mu : \mathcal{A} \to X$ is a vector measure, the **variation** of μ is defined by the positive set function $|\mu|(\bullet)$ given by

$$|\mu|(E) = \sup\sum_i \|\mu(E_i)\|, \qquad E \in \mathcal{A}.$$

the supremum being taken over all finite partitions $\{E_i\}$ of E in $\mathcal{A}$.The notation $v(\mu, E)$ is also used for $|\mu|(E)$ by some authors**.**

We say that μ is of bounded variation if $|\mu|(S) < \infty$. Moreover, if μ is scalar valued, then we have**:**

$$|\mu|(E) \leq 4\sup\{|\mu(F)|, F \in \mathcal{A},\ F \subseteq E\}. \tag{\#1.1}$$

1.5.Theorem . *Let $\mu : \mathcal{A} \to X$ be a weakly $\sigma-$additive set function. Then:*
(a) μ is a vector measure.
(b) Moreover we have:

$$\|\mu\|(E) \leq 4\sup\{\|\mu(F)\|, F \in \mathcal{A},\ F \subseteq E\}, E \in \mathcal{A}.$$

1.6. Definition. (a) A $\mu-$**null set** is a subset of a set $E \in \mathcal{A}$ such that, $\|\mu\|(E) = 0$. A property on S is said to be valid $\mu-$**almost everywhere** if it is valid on the complement of a $\mu-$null set. From now on, we will assume that $\mathcal{A}$ contains the $\mu-$null sets of S, otherwise, the Lebesgue completion process can be applied to $\mathcal{A}, \|\mu\|$.

(b) We say that a function $f : S \to \mathbb{R}$ is **measurable** if for every Borel set B of $\mathbb{R}$, $f^{-1}(B) \in \mathcal{A}$.

(c) A **simple measurable function** of S into $\mathbb{R}$ is a finite linear combination of characteristic functions of pairwise disjoint sets in $\mathcal{A}$.

As is well known, we have:

1.7. Proposition . *A function $f : S \to \mathbb{R}$ is measurable if and only if it is the limit $\mu-$almost everywhere of a sequence of simple measurable functions. Also, the limit $\mu-$almost everywhere of a sequence of measurable functions, is measurable.*

1.8.Definition . Let $f : S \to \mathbb{R}$ be a real measurable function. We define the $\mu-$**essential supremum** of f on a set $E \in \mathcal{A}$ by:

$$\mu - ess\ \sup_{s\in E} |f(s)| = \inf\{A : \|\mu\|(E \cap \{s : |f(s)| > A\} = 0)\}$$

if

$$\mu - ess \sup_{s\in E} |f(s)| < \infty,$$

we say that f is $\mu-$**essentially bounded** on the set E and $\mu-$essentially bounded if $E = S$.

In **Section 3** below, we need to integrate vector valued functions, so we have to make precise the concept of measurability for such functions.

1.9.**Definition** . (i) .An **elementary measurable function** $f : S \to X$ is a function of the form

$$f(\bullet) = \sum_i \chi_{A_i}(\bullet)\, x_i, \text{ or in short } f = \sum_i \chi_{A_i} \otimes x_i,$$

where $\{A_i\}$ is a countable partition of S in $\mathcal{A}$ and $\{x_i\}$ a sequence of vectors in X. We denote by $\mathcal{E}(S, X)$ the set of all elementary measurable functions $f : S \to X$.

(ii) .A function $f : S \to X$ is said to be **strongly measurable** if there is a sequence of elementary measurable functions f_n converging $\mu-$almost everywhere to f. Let $F_{sm}(S, X)$ be the set of all strongly measurable functions $f : S \to X$.

(iii) .A function $f : S \to X$ is said to be **weakly measurable** if for each $x^* \in X^*$, the real function $x^* \circ f : S \to \mathbb{R}$ is measurable.

1.10. Proposition *(1) The sets $E(S, X)$ and $F_{sm}(S, X)$, with addition and scalar multiplication pointwise defined, are vector spaces.*

(2) Let u be a function from X into a Banach space Y and

$$f = \sum_i \chi_{A_i} \otimes x_i,$$

an elementary measurable function from S into X, then we have:

$$u \circ f = \sum_i \chi_{A_i} \otimes u(x_i).$$

so that $u \circ f$ is elementary with values in Y. This gives the proof of the following:

1.11.**Proposition** . *If u is a continuous function from X into a Banach space Y and if $f : S \to X$ is strongly measurable, then $u \circ f$ is strongly measurable. In particular the function $s \to \|f(s)\|$ is measurable in the usual sense.*

Proof: Let $\{s_n\}$ be a sequence of elementary measurable functions converging $\mu-$almost everywhere to f. Then $u(s_n)$ is elementary by the preceding Proposition and $u(s_n)$ converges $\mu-$almost everywhere to $u \circ f$ by the continuity of u. To see that $s \to \|f(s)\|$ is measurable take $u(\cdot) = \|\cdot\|$. $\square$

As for the relation between the two types of measurability, weak-strong, this is given by the following theorem of **Pettis**:

1.12.**Theorem** *A function $f : S \to X$ is strongly measurable if and only if the following conditions are satisfied:*

(a) f is weakly measurable

(b) There is a set $S_0 \in \mathcal{A}$ such that $\mu(S \setminus S_0) = 0$ and the image $f(S_0)$ of S_0 by f is separable. We say that f is $\mu-$almost separably valued. In particular, if X is a separable Banach space, the weak and strong measurability are equivalent.

Proof: $(\Rightarrow)$ Let f be strongly measurable.

To see (a), take $u = x^*$, for $x^* \in X^*$, in Proposition **1.11**.

To prove (b), let f_n be elementary converging $\mu-$almost everywhere to f. So let $S_0 \in \mathcal{A}$ be such that $\mu(S \setminus S_0) = 0$ and $f_n(s) \longrightarrow f(s)$, $\forall s \in S_0$. We prove that $f(S_0)$ is separable. Put $A_n = f_n(S_0)$, then A_n is countable since f_n

is elementary, and so $A = \bigcup_n A_n$ is also countable. We deduce that the closure A^- of A is separable. Since $f_n(s) \longrightarrow f(s)$, $\forall s \in S_0$, we have $f(S_0) \subseteq A^-$, and then $f(S_0)$ is separable.

($\Leftarrow$) Assume (a) and (b). We can suppose X separable, otherwise we replace it by the closed subspace generated by $f(S_0)$ which is separable by (b). Let $\{z_n, n = 1, 2, ...\}$ be a countable dense set in X. Then the family of balls $\{B\left(z_j, \frac{1}{n}\right),$ $j = 1, 2, ...\}$. covers X for each n. Moreover we have:

$$f^{-1}\left(B\left(z_j, \frac{1}{n}\right)\right) = \left\{s \in S : \quad \|f(s) - z_j\| < \frac{1}{n}\right\} \in \mathcal{A},$$

by the measurability of the function $s \to \|f(s) - z_j\|$. Now form the disjoint sets

$$C_{j,n} = f^{-1}\left(B\left(z_i, \frac{1}{n}\right)\right) \setminus \bigcup_{i<j} f^{-1}\left(B\left(z_i, \frac{1}{n}\right)\right)$$

making a partition of S and define for each n the function $f_n : S \to X$ by $f_n(s) = z_{j_n}$, where j_n is the unique j such that $s \in C_{j,n}$. Then (f_n) is a sequence of elementary measurable functions satisfying $\|f_n(s) - f(s)\| < \frac{1}{n}$. So f_n converges (uniformly on S) to f, consequently f is strongly measurable. $\square$

By the way we proved the following lemma:
Lemma: *Let X be a separable Banach space. Then any function $f : S \to X$ is the uniform limit of a sequence of elementary functions.*

2.Integration of scalar-valued functions

Now let X be a Banach space, and let $\mu : \mathcal{A} = \mathcal{A}(S) \to X$ be a vector measure. If $f : S \to \mathbb{R}$ is a real measurable function, we will define the integral of f with respect to vector measure μ and give some of its properties needed in integral representation. First we consider simple functions.

2.1.Definition Let $f(\bullet) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(\bullet)$ be a simple measurable function. The **integral** *of* f with respect to μ over the set $E \in \mathcal{A}$ is defined by:

$$\int_E f \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(E \cap A_i).$$

Just as in the customary real case, this integral does not depend on the representation of f.

It is clear that the integral so defined is linear as a function of f, and

$\sigma-$additive as a set function of E. Moreover if $M = \sup_{s\in E} |f(s)|$, then:

$$\begin{aligned} \left\| \int_E f\, d\mu \right\| &= \left\| M \sum_{i=1}^{n} \left(\frac{\alpha_i}{M}\right) \mu(E\cap A_i) \right\| \\ &\leq M \left\| \sum_{i=1}^{n} \frac{\alpha_i}{M} \mu(E\cap A_i) \right\| \leq M\ \|\mu\|(E), \end{aligned}$$

so we deduce that:

$$\left\| \int_E f\, d\mu \right\| \leq \left(\sup_{s\in E} |f(s)| \right) \|\mu\|(E). \qquad (\#1.2)$$

2.2.Definition A measurable function $f : S \to \mathbb{R}$ is said to be $\mu-$**integrable**, if there is a sequence f_n of simple functions such that:

(a) f_n converges to f $\mu-$almost everywhere

(b) The sequence $\left\{ \int_E f_n\, d\mu \right\}$ converges in the norm of X for each $E \in \mathcal{A}$.

The limit of the sequence $\int_E f_n\, d\mu$ in (b) is called the **integral of f with respect to μ over E** and is denoted by $\int_E f\, d\mu$.

The integral so defined does not depend on the sequence f_n chosen. This fact is not trivial at all (as it involves applications of **Vitali-Hahn-Saks Theorem** and **Egorov Theorem**; . On the other hand, it is straightforward that the integral $\int_E f\, d\mu$ is linear in f.

We record some properties of this integral in the following theorem:

2.3.Theorem *(a). If f is $\mu-$essentially bounded on the set E, then f is $\mu-$integrable over E and :*

$$\left\| \int_E f\, d\mu \right\| \leq \left(\mu - ess \sup_{s\in E} |f(s)| \right) . \|\mu\|(E).$$

(b). Let T be a linear bounded operator from X into the Banach space Y. Then $T\mu$ is a $Y-$valued vector measure on $\mathcal{A}$, and for any $\mu-$integrable f and any $E \in \mathcal{A}$ we have

$$T\left(\int_E f\, d\mu \right) = \int_E f\, dT\mu.$$

Proof: It is easy to see that the conclusions are valid for f simple (see the inequality **(#1.2)** in Definition **2.1**). The fact that $T\mu$ is a vector measure comes from the boundedness of T.

Now let f be measurable with $\mu-$essential bound B on E. Let $\varepsilon > 0$ and let $F_1, F_2, ..., F_n$ be a covering of $f(E)$ by disjoint Borel sets of scalars. Define $E_j = f^{-1}(F_j)$. Let $\alpha_j \in F_j$ and define $f_\varepsilon(s) = \alpha_j$ for $s \in E_j$. Then f_ε is simple and we can arrange matters so that

$$\mu - ess \sup_{s \in E} |f_\varepsilon(s) - f(s)| < \varepsilon.$$

Let $\varepsilon_n \to 0$, we have $\lim_{m,n} \ \mu - ess \sup_{s\in E} |f_{\varepsilon_n}(s) - f_{\varepsilon_m}(s)| = 0$. By the inequality **(#1.2)** in Definition **2.1,** we deduce that

$$\lim_{m,n} \left| \int_E f_{\varepsilon_n} d\mu - \int_E f_{\varepsilon_m} d\mu \right| = 0.$$

This yields the convergence of $\int_E f_{\varepsilon_n} d\mu$ for each $E \in \mathcal{A}$ and we have

$$\int_E f \, d\mu = \lim_n \int_E f_{\varepsilon_n} d\mu.$$

On the other hand, since

$$|f_{\varepsilon_n}| \leq |f_{\varepsilon_n}(s) - f(s)| + |f(s)|,$$

we deduce that

$$\mu - ess \sup_{s\in E} |f_{\varepsilon_n}(s)| \leq B + \varepsilon_n,$$

so that the validity of (a) follows from its validity for simple functions (see the inequality (#1.2) in definition **2.1**). Part (b) comes from its trivial validity for simple functions and the boundedness of T. ■

3.Bochner Integral of Vector-valued Functions

Let $(S, \mathcal{A}, \mu)$ be a measure space, with μ a finite positive measure. We will assume that $(S, \mathcal{A}, \mu)$ is complete. As considered in **Sections 1, 2**, X will be a Banach space with topological dual X^*. In this section we define the Bochner integral with respect to the scalar measure μ, for functions $f : S \to X$. (for all details on Bochner integral, see [5]).

3.1.Definition .We say that the elementary measurable function $f(\bullet) = \sum_i x_i.\chi_{A_i}(\bullet)$ is $\mu-$**integrable** if

$$\sum_i \|x_i\| .\mu(A_i) < \infty.$$

In this case we define the integral of f with respect to μ by

$$\textstyle\int_S f\, d\mu = \sum_i x_i.\mu(A_i).$$

Likewise the integral of f over the set $E \in \mathcal{A}$ is

$$\int_E f\, d\mu = \sum_i x_i.\mu\left(A_i \cap E\right).$$

3.2.Proposition *The integral of an elementary measurable function has the following properties:*

(a) If f, g are elementary $\mu-$integrable and if α, β are scalars then $\alpha.f + \beta.g$ is elementary $\mu-$integrable and

$$\int_E (\alpha.f + \beta.g)\, d\mu = \alpha \int_E f\, d\mu + \beta \int_E g\, d\mu.$$

(b) If f is elementary $\mu-$integrable,then

$$\left\| \int_E f\, d\mu \right\| \leq \int_E \|f\|\, d\,\mu,$$

where $\|f(\bullet)\| = \sum_i \|x_i\|\,.\chi_{A_i}(\bullet)$.

(c) If Y is a Banach space and if $T : X \to Y$ is a linear bounded operator, then for each elementary $\mu-$integrable function f, the function Tf is elementary $\mu-$integrable and we have

$$T\left(\int_E f\, d\mu\right) = \int_E Tf\, d\mu.$$

Proof: To see (a), write $f = \sum_n s_n.\chi_{A_n}$, and $g = \sum_k t_k.\chi_{B_k}$. Then

$$f + g = \sum_{n,k} (s_n + t_k)\, \chi_{A_n \cap B_k} \quad \text{and } \alpha f = \sum_n \alpha s_n.\chi_{A_n}.$$

This implies that

$$\begin{aligned} \|f + g\| &= \sum_{n,k} \|s_n + t_k\|\, \chi_{A_n \cap B_k} \leq \sum_{n,k} (\|s_n\| + \|t_k\|)\, \chi_{A_n \cap B_k} \\ &= \sum_n \|s_n\|\,.\chi_{A_n} + \sum_k \|t_k\|\,.\chi_{B_k}. \end{aligned}$$

So we deduce that

$$\sum_{n,k} \|s_n + t_k\| \, \mu\,(A_n \cap B_k) \leq \sum_n \|s_n\| \, \mu(A_n) + \sum_k \|t_k\| \, \mu\,(B_k) < \infty.$$

Consequently $f + g$ is elementary $\mu-$integrable and

$$\int\limits_E (f+g)\, d\mu = \int\limits_E f\, d\mu + \int\limits_E g.d\mu.$$

Likewise αf is elementary $\mu-$integrable and

$$\int\limits_E \alpha f\, d\mu = \alpha \int\limits_E f\, d\mu.$$

Part (b) is trivial. To prove (c), we have $Tf = \sum\limits_n Ts_n.\chi_{A_n}$ and

$$\int Tf\, d\mu = \sum_n Ts_n.\mu(A_n) = T\left(\sum_n s_n.\mu(A_n)\right) = T\left(\int\limits_E f\, d\mu\right),$$

by the boundedness of T.■

Remark: If we take in (c) $Y = \mathbb{R}$ and $T = x^*$ for some $x^* \in X^*$ we get

$$x^*\left(\int\limits_E f\, d\mu\right) = \int\limits_E x^* f\, d\mu.$$

3.3.Definition . A function $f : S \to X$ is said to be $\mu-$**integrable** if there is a sequence f_n of elementary $\mu-$integrable functions such that:

(1).f_n converges $\mu-$almost everywhere to f

(2). $\lim_n \int\limits_E \|f_n - f\|\, d\,\mu = 0$

In this case the Bochner integral of f is defined by

$$\int\limits_E f\, d\mu = \lim_n \int\limits_E f_n\, d\mu.$$

The following observations legitimate this definition:

First f is strongly measurable by (1) (Definition **1.9** (ii)). Next the function $\|f_n - f\|$ is positive measurable by Proposition **1.11**, so the integrals $\int\limits_E \|f_n - f\|\, d\,\mu$

make sense. Finally the sequence $\int_E f_n \, d\mu$ is Cauchy in the Banach space X. For we have

$$\left\| \int_E f_n \, d\mu - \int_E f_m \, d\mu \right\| = \left\| \int_E (f_n - f_m) \, d\mu \right\| \leq \int_E \|(f_n - f_m)\| \, d\mu$$

by Proposition **3.2** (b), since $f_n - f_m$ is elementary; so we get

$$\left\| \int_E f_n \, d\mu - \int_E f_m \, d\mu \right\| \leq \int_E \|f_n - f\| \, d\mu + \int_E \|f_m - f\| \, d\mu \to 0, n, m \to \infty,$$

by (2).

Now, if f_n , g_n are two sequences of elementary $\mu-$integrable functions satisfying $(1) - (2)$we have

$$\begin{aligned} \left\| \int_E f_n \, d\mu - \int_E g_n \, d\mu \right\| &\leq \int_E \|(f_n - g_n)\| \, d\mu \\ &\leq \int_E \|f_n - f\| \, d\mu + \int_E \|g_n - f\| \, d\mu \to 0, n \to \infty. \end{aligned}$$

Consequently the Bochner integral of f is well defined. The following theorem gives one of the outstanding facts about the Bochner integral.
3.4.Theorem For every $\mu-$integrable function $f : S \to X$ and every $x^* \in X^*$ we have

$$x^* \left(\int_E f \, d\mu \right) = \int_E x^* f \, d\mu.$$

Proof: Let $\{f_n\}$ be a sequence of elementary functions defining $\int_E f \, d\mu$. Since $f_n \to f$, $\mu.a.e$ and since x^* is continuous, we have $x^* f_n \to x^* f$, $\mu.a.e$. On the other hand $x^* \left(\int_E f_n d\mu \right) = \int_E x^* f_n d\mu$ by Proposition **3.2** (c). We deduce

$$\begin{aligned} \left| x^* \left(\int_E f_n d\mu \right) - \int_E x^* f \, d\mu \right| &= \left| \int_E x^* f_n d\mu - \int_E x^* f \, d\mu \right| \\ &\leq \int_E |x^* f_n - x^* f| \, d\mu \leq \|x^*\| \int_E \|f_n - f\| \, d\mu \to 0. \end{aligned}$$

But $\int\limits_E f_n d\mu \rightarrow \int\limits_E f\, d\mu$, so

$$x^*\left(\int\limits_E f_n d\mu\right) \rightarrow x^*\left(\int\limits_E f\, d\mu\right).$$

Consequently $x^*\left(\int\limits_E f\, d\mu\right) = \int\limits_E x^* f\, d\mu$. $\square$

3.5.Theorem.(Bochner) *A function $f : S \rightarrow X$ is $\mu-$integrable if and only if f is strongly measurable and $\int_S \|f\|\, d\,\mu < \infty$.*

In other words, for a strongly measurable function f, the $\mu-$integrability is equivalent to the integrability of $\|f\|$.

Proof: Suppose f $\mu-$integrable.There exists a sequence $(f_n)$of $\mu-$integrable elementary functions such that $f_n \rightarrow f$, $\mu.a.e$ and $\int\limits_E \|f_n - f\|\, d\mu \rightarrow 0$; in particular f is strongly measurable and

$$\int_S \|f\|\, d\,\mu \leq \int_S \|f_n - f\|\, d\mu + \int_S \|f_n\|\, d\,\mu < \infty.$$

Conversely let f be strongly measurable and satisfies

$$\int_S \|f\|\, d\,\mu < \infty.$$

By **Definition 3.3**, we have to show the existence of a sequence g_n of elementary $\mu-$integrable functions such that g_n converges $\mu-$almost everywhere to f and $\lim_n \int\limits_E \|g_n - f\|\, d\,\mu = 0$. Fix $\alpha > 0$ and let us consider the sets

$$K_n = \{s \in S, \quad \|f_n(s)\| \leq \|f(s)\| . (1 + \alpha)\}.$$

Now define the function g_n by $g_n = f_n \chi_{K_n}$. Since $\|f_n\|$ and $\|f\|$ are measurable, it results that $K_n \in \mathcal{A}$ and so g_n is elementary measurable. On the otherhand we have

$$\|g_n\| = \|f_n\|\, \chi_{K_n} \leq \|f\| . (1 + \alpha) \quad \text{on all of } \; S.$$

From the condition $\int_S \|f\|\, d\,\mu < \infty$ we deduce that g_n is $\mu-$integrable. Since $f_n \rightarrow f$, $\mu.a.e$, we have: for almost every $s \in S$, there exists $N(s) \geq 1$ such that

$$\|f_n(s)\| - \|f(s)\| \leq \alpha\, \|f(s)\|\, \forall n \geq N(s);$$

that is $s \in K_n$. This means that $g_n(s) = f_n(s)\, \chi_{K_n}(s) = f_n(s)$, $\forall n \geq N(s)$. This proves that g_n converges $\mu-$almost everywhere to f and then $\|g_n - f\| \rightarrow 0$ $\mu.a.e.$ Since we have

$$\|g_n - f\| \leq \|g_n\| + \|f\| \leq \|f\| . (1 + \alpha) + \|f\| = \|f\| . (2 + \alpha),$$

we deduce from dominated convergence theorem in the classical space $L_1(\mu)$ that

$$\lim_n \int_E \|g_n - f\| \, d\mu = 0. \ \square$$

We denote by $L_1(\mu, X)$ the set of all $\mu-$integrable functions.

As usual we identify two functions that are equal $\mu-$almost everywhere, in symbols $f = g \ \ \mu - a.e.$ it is easy to check that:

(a) $L_1(\mu, X)$ is a vector space and the map $f \to \int_E f \, d\mu$ is linear;

(b) $f = g \ \ \mu - a.e \Longleftrightarrow \int_E f \, d\mu = \int_E g \, d\mu$, for all $E \in \mathcal{A}$.

3.6.Theorem . Let (f_n) be a sequence in $L_1(\mu, X)$ such that $f_n \to f \quad \mu - a.e.$ Suppose there exists $g \in L_1(\mu)$ such that for each n we have $\|f_n\| \leq g, \ \mu - a.e.$ Then $f \in L_1(\mu, X)$ and $\int_S f \, d\mu = \lim_n \int_S f_n \, d\mu$.

Proof: For each $x^* \in X^*$ the scalar functions $x^* f_n$ converge to $x^* f$ $\mu-a.e.$, so f is weakly measurable. On the other hand, since f_n is strongly measurable $\underline{f_n}(S)$ is separable. We deduce that $\bigcup_n f_n(S)$ is separable. Since $f(S) \subset \overline{\bigcup_n f_n(S)}$, it follows that $f(S)$ is separable. Now we apply **Pettis Theorem 1.12**.to get that f is strongly measurable. Since $\|f_n\| \leq g, \ \ \mu - a.e.$ we deduce $\|f\| \leq g, \ \ \mu - a.e.$and then $\int_S \|f\| \, d\mu < \infty$, consequently f is Bochner integrable by **Theorem 3.5**, so $f \in L_1(\mu, X)$. On the other hand,

$$\|f_n - f\| \leq 2g \ \text{ and } \ 2g - \|f_n - f\| \geq 0.$$

By **Fatou lemma**

$$\begin{aligned}
0 &\leq \int_S \liminf_n 2g - \|f_n - f\| \ d\mu = \int_S 2g \, d\mu \\
&\leq \liminf_n \int_S (2g - \|f_n - f\|) \ d\mu \\
&= \int_S 2g \, d\mu - \limsup_n \int_S \|f_n - f\| \ d\mu.
\end{aligned}$$

It follows that $\limsup_n \int_S \|f_n - f\| \ d\mu \leq 0$, so $\lim_n \int_S \|f_n - f\| \ d\mu = 0.$■

Chapter 7

Bochner integral in locally convex spaces

Let X be a locally convex Hausdorff space, whose topology is generated by a family $\{p_\alpha\}$ of continuous seminorms. We assume that $\{p_\alpha\}$ is separating, this means that for each nonzero $x \in X$ there is a p_α such that $p_\alpha(x) \neq 0$. Moreover we assume that X is sequentially complete, that is, every Cauchy sequence in X is convergent. For all details on such spaces, the reader is referred to [13], especially the sections $1.25, 1.36, 1.37$ there. The construction of the Bochner integral we give in this context is, as far as we know, new. (for other approachs see $[1, 5, 14]$). On the space $L_1(\mu, X)$ of Bochner integrable functions we define a family of separating seminorms that make this space locally convex. Finally we introduce a special class of bounded operators from $L_1(\mu, X)$ into X whose structure is , in many respects, similar to some well known operators from $L_1(\mu)$ into $\mathbb{R}$.

For the needs of measurability and integration, we fix an abstract measure space $(S, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a $\sigma-$field on the set S and μ a finite positive measure on $\mathcal{F}$.

1. Measurability

1.1. Definition: A function $f : S \longrightarrow X$ is called elementary if its range $f(S)$ is finite.
If we put $f(S) = \{x_1, x_2, ..., x_n\}$ and $A_j = \{s : f(s) = x_j\}$ then the sets A_j form a partition of S and we can write f in the consolidated form
$f(\bullet) = \sum\limits_{j=1}^{n} x_j \, 1_{A_j}(\bullet)$, where 1_{A_j} is the characteristic function of the set A_j.

1.2. Definition: An elementary function $f(\bullet) = \sum\limits_{j=1}^{n} x_j \, 1_{A_j}(\bullet)$ is measurable if we have $A_j \in \mathcal{F}$ for every j. We denote by $\mathcal{E}(X)$ the set of all elementary measurable functions $f : S \longrightarrow X$. Then we have:

1.3 Proposition: $\mathcal{E}(X)$ is a vector space on $\mathbb{R}$.

Proof: Let f, g be in $\mathcal{E}(X)$ and $\lambda \in \mathbb{R}$. Put $f(\bullet) = \sum\limits_{n} x_n \, 1_{A_n}(\bullet)$

$g(\bullet) = \sum\limits_{m} y_m \, 1_{B_m}(\bullet)$, then $(f+g)(\bullet) = \sum\limits_{n,m} (x_n + y_m) \; 1_{A_n \cap B_m}(\bullet)$ and $(\lambda f)(\bullet) = \sum\limits_{n} \lambda x_n \, 1_{A_n}(\bullet)$. ■

1.4. Remark: Let T be any mapping from X into Y.
If $f(\bullet) = \sum\limits_{n} x_n \, 1_{A_n}(\bullet)$ then $(T \circ f)(\bullet) = \sum\limits_{n} T(x_n) \; 1_{A_n}(\bullet)$.

1.5. Definition: A function $f : S \longrightarrow X$ is measurable if there is a sequence (f_n) of elementary measurable functions such that:

$$\underset{n}{Lim}\, p_\alpha (f_n - f) = 0$$

for each p_α.

This means that for each $s \in S$, each $\epsilon > 0$, and each p_α, there is $N = N_{s,\epsilon,p_\alpha} \geq 1$ such that $\forall n \geq N$, $p_\alpha (f_n (s) - f (s)) < \epsilon$.

1.6. Proposition: The set $M (X)$ of all measurable functions $f : S \longrightarrow X$ is a vector space on $\mathbb{R}$.

Proof: Let f, g be in $M (X)$ and let f_n, g_n be sequences of elementary functions such that $p_\alpha (f_n - f) \longrightarrow 0$ and $p_\alpha (g_n - g) \longrightarrow 0$, for each p_α. Then we have $p_\alpha ((f_n + g_n) - (f + g)) \leq p_\alpha (f_n - f) + p_\alpha (g_n - g)$, so the sequence of elementary functions $f_n + g_n$ gives the measurability of $f + g$.

Likewise for $\lambda \in \mathbb{R}$,we have $p_\alpha (\lambda f_n - \lambda f) = |\lambda| p_\alpha (f_n - f) \longrightarrow 0$, which gives $\lambda f \in M (X)$.$\blacksquare$

2. Bochner integration

2.1. Definition: Let $f (\bullet) = \sum\limits_{j=1}^{n} x_j\, 1_{A_j} (\bullet)$ be an elementary measurable function. We define the integral of f by the vector $\int_S f\, d\mu \in X$:

$$\int_S f\, d\mu = \sum_{j=1}^{n} \mu (A_j) .x_j$$

Since μ is finite this integral is well defined.

2.2. Proposition: (a)The integral is linear from $\mathcal{E} (X)$ into X.

(b). For every $f \in \mathcal{E} (X)$ and every p_α we have

$$p_\alpha \left(\int_S f\, d\mu\right) \leq \int_S p_\alpha (f)\ d\mu$$

where $p_\alpha (f)$ is the positive elementary function given by

$p_\alpha (f) (\bullet) = \sum\limits_{j=1}^{n} p_\alpha (x_j)\ 1_{A_j} (\bullet)$ whose integral is $\int_S p_\alpha (f)\ d\mu = \sum\limits_{j=1}^{n} p_\alpha (x_j)\ \mu (A_j)$.

Proof: (a) Put $f (\bullet) = \sum\limits_{j=1}^{n} x_j\, 1_{A_j} (\bullet)$, $g (\bullet) = \sum\limits_{k=1}^{m} y_k\, 1_{B_k} (\bullet)$

then $(f + g) (\bullet) = \sum\limits_{j,k} (x_j + y_k)\ 1_{A_j \cap B_k} (\bullet)$ and $(\lambda f) (\bullet) = \sum\limits_{j} \lambda x_j\, 1_{A_j} (\bullet)$.

This yields $\int_S (f + g)\ d\mu \sum\limits_{1\leq k\leq m} \sum\limits_{1\leq j\leq n} (x_j + y_k)\ \mu (A_j \cap B_k) = \sum\limits_{j=1}^{n} \mu (A_j) .x_j + \sum\limits_{k=1}^{m} \mu (B_k) .y_k = \int_S f\, d\mu + \int_S g\, d\mu$

Likewise we can prove that $\int_S \lambda.f\, d\mu = \lambda. \int_S f\, d\mu$ for $\lambda \in \mathbb{R}$.

(b) We have $p_\alpha\left(\int_S f\, d\mu\right) = p_\alpha\left(\sum_{j=1}^{n} \mu(A_j).x_j\right) \leq$

$\sum_{j=1}^{n} \mu(A_j).p_\alpha(x_j) = \int_S p_\alpha(f)\ d\mu.\blacksquare.$

2.3. Proposition: Let $T : X \longrightarrow Y$ be a linear operator from X into a locally convex space Y.
Let $f \in \mathcal{E}(X)$, then $T \circ f \in \mathcal{E}(Y)$ and we have:

$$T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.$$

Proof: Let $f(\bullet) = \sum_{j=1}^{n} x_j\, 1_{A_j}(\bullet)$, with $\int_S f\, d\mu = \sum_{j=1} \mu(A_j).x_j$, then $(T \circ f)(\bullet) = \sum_{j=1}^{n} T(x_j)\ 1_{A_j}(\bullet)$ and

$\int_S T \circ f\, d\mu = \sum_{j=1}^{n} \mu(A_j).T(x_j) = T\left(\sum_{j=1}^{n} \mu(A_j).x_j\right)$, by the linearity of T, so we deduce that $T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.\blacksquare.$

2.4. Definition: A measurable function $f : S \longrightarrow X$ is Bochner integrable if there is a sequence f_n of elementary measurable functions such that for each p_α, $\underset{n}{Lim}\ p_\alpha(f_n - f) = 0$ uniformly on S. Since the measure μ is assumed finite, this implies that $\underset{n}{Lim}\ \int_S p_\alpha(f_n - f)\ d\mu = 0$, for each p_α.
To define the Bochner integral of f let us observe that if f_n is such a sequence of elementary functions we have:
$\int_S p_\alpha(f_n - f_m)\ d\mu \leq \int_S p_\alpha(f_n - f)\ d\mu + \int_S p_\alpha(f_m - f)\ d\mu.$
So $\underset{n,m}{Lim}\ \int_S p_\alpha(f_n - f_m)\ d\mu = 0$. But $p_\alpha \int_S (f_n - f_m)\ d\mu \leq \int_S p_\alpha(f_n - f_m)\ d\mu$ by **Proposition 2.2**(b), this implies that the sequence of integrals $\int_S f_n\ d\mu$ is Cauchy. As the space X is assumed sequentially complete, $\int_S f_n\ d\mu$ converges. This allows to define the Bochner integral of f by the vector:

$$\int_S f\, d\mu = \underset{n}{Lim}\ \int_S f_n\ d\mu.$$

If g_n is another sequence of elementary functions such that $p_\alpha(g_n - f) \longrightarrow 0$ uniformly on S, it is easy to check, from the continuity of p_α that $\underset{n}{Lim}\ \int_S f_n\ d\mu = \underset{n}{Lim}\ \int_S g_n\ d\mu$, so the Bochner integral $\int_S f\, d\mu$ is well defined.

In the sequel we will denote by $L_1(\mu, X)$ the set of all Bochner integrable functions $f : S \longrightarrow X$, where as usual two integrable functions are considered as identical if they are equal $\mu-$almost everywhere.

2.5. Proposition: $L_1(\mu, X)$ is a vector space on $\mathbb{R}$ and we have:
(a). The integral as defined is linear from $L_1(\mu, X)$ into X.
(b). For every $f \in L_1(\mu, X)$ and every p_α we have
$p_\alpha\left(\int_S f\, d\mu\right) \leq \int_S p_\alpha(f)\ d\mu$

Proof:

(a) Let f, g be in $L_1(\mu, X)$ and let f_n, g_n be in $\mathcal{E}(X)$ such that $p_\alpha(f_n - f) \longrightarrow 0$ and $p_\alpha(g_n - f) \longrightarrow 0$, uniformly on S. Since we have $p_\alpha((f+g) - (f_n + g_n)) \leq p_\alpha(f_n - f) + p_\alpha(g_n - f) \longrightarrow 0$, it follows that $p_\alpha((f+g) - (f_n + g_n)) \longrightarrow 0$ uniformly on S. This yields
$\int_S (f+g)\, d\mu = \underset{n}{Lim} \int_S (f_n + g_n)\, d\mu = \underset{n}{Lim} \int_S f_n\, d\mu + \underset{n}{Lim} \int_S g_n\, d\mu = \int_S f\, d\mu + \int_S g\, d\mu$. Likewise we have $\int_S \lambda.f\, d\mu = \lambda. \int_S f\, d\mu$.

(b) Let f_n be in $\mathcal{E}(X)$ defining $\int_S f\, d\mu$. By proposition **2.2**(b)
$p_\alpha\left(\int_S f_n\, d\mu\right) \leq \int_S p_\alpha(f_n)\, d\mu$ for all n. This implies $p_\alpha\left(\int_S f\, d\mu\right) = p_\alpha\left(\underset{n}{Lim} \int_S f_n\, d\mu\right) = \left(\underset{n}{Lim}\, p_\alpha\left(\int_S f_n\, d\mu\right)\right) \leq \liminf_n \int_S p_\alpha(f_n)\, d\mu \leq \liminf_n \left(\int_S p_\alpha(f_n - f)\, d\mu + \int_S p_\alpha(f)\, d\mu\right) = \int_S p_\alpha(f)\, d\mu.\blacksquare$.

2.6. Proposition: Let $T : X \longrightarrow Y$ be a linear continuous operator from X into a locally convex space Y.
Let $f \in L_1(\mu, X)$, then $T \circ f \in L_1(\mu, Y)$ and we have:

$$T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.$$

Proof: Let f_n be in $\mathcal{E}(X)$ defining $\int_S f\, d\mu$, i.e $\underset{n}{Lim}\, p_\alpha(f_n - f) \longrightarrow 0$ uniformly on S. By the continuity of T, if q is a seminorm on Y there is a seminorm p_α on X such that $q(Tx) \leq p_\alpha(x)$, for every $x \in X$. It follows that $q(Tf_n - Tf) = qT(f_n - f)\, d\mu \leq p_\alpha(f_n - f) \to 0$ uniformly on S. We deduce that $q(Tf_n - Tf) \longrightarrow 0$ uniformly on S for each q. So the sequence Tf_n, which is in $\mathcal{E}(Y)$ by Proposition **2.3,** is defining the integral of Tf by $\int_S Tf\, d\mu = \underset{n}{Lim} \int_S Tf_n d\mu$. By **Proposition 2.3** once more we have
$\int_S Tf_n d\mu = T \int_S f_n d\mu$ for all n.
Since $\underset{n}{Lim} \int_S f_n d\mu = \int_S f\, d\mu$, we get $\underset{n}{Lim} \int_S Tf_n d\mu = T\left(\int_S f\, d\mu\right)$, by the continuity of T. this gives $T\left(\int_S f\, d\mu\right) = \int_S T \circ f\, d\mu.\blacksquare$.

3. Bounded operators on $\mathbf{L}_1(\mu, X)$

First we start by defining on $L_1(\mu, X)$ a family $\{\widetilde{p_\alpha}\}$ of continuous seminorms which will make $L_1(\mu, X)$ a locally convex space.

Let us observe that for each p_α, we have $p_\alpha(f)$ bounded on S if $f \in L_1(\mu, X)$.To see this let f_n be in $\mathcal{E}(X)$ defining $\int_S f\, d\mu$, i.e $Lim p_\alpha(f_n - f) = 0$ uniformly on S, (**Definition 2.4**), so if $\epsilon > 0$, there is $N \geq 1$ such that $|p_\alpha(f) - p_\alpha(f_N)| \leq p_\alpha(f_N - f) < \epsilon$ uniformly on S. We deduce that $p_\alpha(f) < \epsilon + p_\alpha(f_N)$ on S and $p_\alpha(f_N)$ is bounded on S since $f_N \in \mathcal{E}(X)$.

Now define $\widetilde{p_\alpha}$ on $L_1(\mu, X)$ by:

$$f \in L_1(\mu, X) \qquad \widetilde{p_\alpha}(f) = \underset{t \in S}{Sup}\, p_\alpha(f(t)) \tag{3.1}$$

Then $\widetilde{p_\alpha}$ is a seminorm on $L_1(\mu, X)$ and the family $\{\widetilde{p_\alpha}\}$ is separating. To see this, let f be in $L_1(\mu, X)$ with $f \neq 0$, that is $f(t) \neq 0$ for some $t \in S$. Since the family $\{p_\alpha\}$ is assumed separating on X, there is a p_α such that $p_\alpha(f(t)) > 0$, so that $\widetilde{p_\alpha}(f) > 0$.
Since the family of seminorms $\{\widetilde{p_\alpha}\}$ is separating, it makes $L_1(\mu, X)$ a locally convex space such that each $\widetilde{p_\alpha}$ is continuous ([13], section 1.37).

In what follows we define a special class of bounded operators from $L_1(\mu, X)$ into X which are, in many respects, similar to some well known operators from $L_1(\mu)$ into $\mathbb{R}$. First let us observe:

3.2. Lemma: Let $g \in L_\infty(\mu)$, then for every $f \in L_1(\mu, X)$
$g.f \in L_1(\mu, X)$.

Proof: Since $g \in L_\infty(\mu)$, there is a sequence (g_n) of simple measurable functions $g_n : S \longrightarrow \mathbb{R}$ converging uniformly to g on S. Since $f \in L_1(\mu, X)$, there is a sequence f_n of elementary measurable functions such that for each p_α, $\underset{n}{Lim}\ p_\alpha(f_n - f) = 0$ uniformly on S. But $g_n.f_n$ is elementary measurable, and we have:
$p_\alpha(g_n.f_n - g.f) = p_\alpha[(g_n.f_n - g_n.f) + (g_n.f - g.f)]$
$\leq |g_n - g|.p_\alpha(f) + |g_n|.p_\alpha(f_n - f)$
$\leq |g_n - g|.\widetilde{p_\alpha}(f) + |g_n|.p_\alpha(f_n - f) \longrightarrow 0,\ n \longrightarrow \infty,$ uniformly on S.
Consequently we have $f.g \in L_1(\mu, X)$.■.

Now we define a class $\{T_g, g \in L_\infty(\mu)\}$ of operators T_g, by the following recipe:

3.3. Definition: For each fixed $g \in L_\infty(\mu)$, T_g sends $L_1(\mu, X)$ into X by the formula:

$$f \in L_1(\mu, X), \qquad T_g(f) = \int_S fg\, d\mu$$

3.4. Theorem:. Let $L_1(\mu, X)$ be endowed with the seminorms $\{\widetilde{p_\alpha}\}$ given by (3.1), and let X be equipped with the seminorms $\{p_\alpha\}$, then the operators T_g are linear and bounded.

Proof: The linearity is clear from **2.5** (a). To see boundedness, let p_α be a seminorm on X, by **2.5** (b) we have:
$p_\alpha(T_g(f)) = p_\alpha\left(\int_S fg\, d\mu\right) \leq \int_S p_\alpha(fg)\, d\mu$. Since $p_\alpha(fg) = |g|\ p_\alpha(f)$, we deduce that $p_\alpha(T_g(f)) \leq \int_S |g|\ p_\alpha(f)\, d\mu \leq \|g\|_\infty .\widetilde{p_\alpha}(f).\mu(X)$, which proves that T_g is bounded.■.

In what follows, we quote some properties of the operators T_g, whose proof comes from facts about Bochner integral (**2.5-2.6**).We denote by E' the strong dual of the space E :

3.5. Proposition: (a) If $\theta \in X'$, then $\theta \circ T_g \in L_1'(\mu, X)$.
(b) If $\theta \in X'$, then $\theta \circ T_g(f) = \int_S g\,\theta f\, d\mu$, for every $f \in L_1(\mu, X)$.
(c) If θ, σ are in X', and φ, ψ in $L_1(\mu, X)$, then:

$$\theta \circ \varphi = \sigma \circ \psi \Longrightarrow \theta \circ T_g(\varphi) = \sigma \circ T_g(\psi)$$

These properties, especially property (c), lead to the following:
Open problem: Let $T : L_1(\mu, X) \longrightarrow X$ be a linear bounded operator from $L_1(\mu, X)$ into X satisfying condition **3.5**(c), that is:
If θ, σ are in $X^{'}$, and φ, ψ in $L_1(\mu, X)$, then:

$$\theta \circ \varphi = \sigma \circ \psi \Longrightarrow \theta \circ T(\varphi) = \sigma \circ T(\psi)$$

Does there exist a $g \in L_\infty(\mu)$ such that:

$$T(f) = \int_S fg\, d\mu, \text{ for all } f \in L_1(\mu, X).$$

Appendix

Operators in Banach Spaces

1. Linear Bounded Operators

1.1 Let X, Y be normed space. A linear operator T from a normed space X into a normed space Y is said to be bounded if there is a constant $M > 0$ such that:

$$\|T(x)\| \leq M. \|x\|, \forall x \in X$$

This definition means that if B is a bounded subset of X, the set $\{T(x), x \in B\}$ is bounded in Y. For instance if $B = \{x : \|x\| \leq 1\}$ then $\|T(x)\| \leq M, \forall x \in B$.

1.2.Example Let the space $C[0,1]$ of continuous functions $f : [0,1] \longrightarrow \mathbb{R}$, be equipped with the uniform norm $\|\cdot\|_\infty$. Define $T : C[0,1] \longrightarrow \mathbb{R}$ by
$T(f) = \int_0^1 f(x)\,dx$ (Riemann integral), then it is clear that $|T(f)| \leq \|f\|_\infty$ and T is bounded with the choice $M = 1$.

1.3 Proposition Let T be a bounded operator from X into Y. Define:

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\right\}$$
$$m_1 = \sup\{\|T(x)\| : x \in X, \|x\| = 1\}$$
$$m_2 = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$$
$$m_3 = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$$

Then $m_1 = m_2 = m_3 = \|T\| < \infty$ and we have:

$$\|T(x)\| \leq \|T\| \|x\|, \forall x \in X$$

Proof: First T bounded operator $\implies m_i < \infty, i = 1, 2, 3$, and $\|T\| < \infty$; next we have from the definition, $m_1 \leq \|T\|$. On the other hand if $x \neq 0$, then $\left\|\frac{x}{\|x\|}\right\| = 1$, and $\left\|T\left(\frac{x}{\|x\|}\right)\right\| = \frac{\|T(x)\|}{\|x\|} \leq m_1$, this yields $\|T\| \leq m_1$, so $\|T\| = m_1$.
Since $\{x : \|x\| = 1\} \subset \{x : \|x\| \leq 1\}$, we get $m_1 \leq m_3$; on the other hand for $\|x\| \leq 1, x \neq 0$, we have $\left\|T\left(\frac{x}{\|x\|}\right)\right\| = \frac{\|T(x)\|}{\|x\|} \leq m_1$, whence $\|T(x)\| \leq \|x\| m_1 \leq m_1$. Taking supremum over $\{x : \|x\| \leq 1\}$ we get $m_3 \leq m_1$,so $m_1 = m_3$. By the same trick we obtain $m_1 = m_2$. Finally it is clear that $\|T(x)\| \leq \|T\| \|x\|, \forall x \in X$.■

1.4 Theorem The following properties are equivalent for a linear operator T from X into Y:

(a) T is bounded.

(b) T is uniformly continuous.

(c) T is continuous at a point $x_0 \in X$.

Proof:$(a) \implies (b)$

We have $\|T(x) - T(x')\| = \|T(x - x')\| \leq \|T\| \|x - x'\|, \forall x, x' \in X$, so if $\epsilon > 0$ then we have $\|x - x'\| < \epsilon \|T\|^{-1} \implies \|T(x) - T(x')\| < \epsilon$.

$(b) \implies (c)$ is trivial.

$(c) \implies (a)$

By (c) there is $\sigma > 0$ such that $\|x - x_0\| < \sigma \implies \|T(x) - T(x_0)\| < 1$. Now if $\|x\| < 1$, we get $\|\sigma x + x_0 - x_0\| < \sigma$, and since $x = \dfrac{1}{\sigma}(\sigma x + x_0 - x_0)$, we deduce $\|T(x)\| = \dfrac{1}{\sigma} \|T(\sigma x + x_0) - T(x_0)\| < \dfrac{1}{\sigma}$. From Proposition **1.3**, $\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| < 1\}$, and then $\|T\| \leq \dfrac{1}{\sigma}$, this yields T bounded.■

We denote by $B(X,Y)$ the set of linear bounded operators from a normed space X into a normed space Y, on the same field $\mathbb{K}$ of scalars. Let $S, T \in B(X,Y)$ and $\alpha \in \mathbb{K}$, we define for $x \in X$:

$(S + T)(x) = S(x) + T(x)$

$(\alpha T)(x) = \alpha T(x)$

Then we have:

1.5 Proposition $B(X,Y)$ is a vector space with these defined operations. Moreover the function $T \longrightarrow \|T\|$ is a norm on $B(X,Y)$.

Proof: It is immediate that $B(X,Y)$ is a vector space, the null vector being the operator T with $T(x) = 0, \forall x \in X$. To see that $T \longrightarrow \|T\|$ is a norm, let $S, T \in B(X,Y)$, then we have $\|S + T\| = \sup\{\|S(x) + T(x)\|, \|x\| = 1\} \leq \sup\{\|S(x)\| + \|T(x)\|, \|x\| = 1\} \leq \sup\{\|S(x)\|, \|x\| = 1\} + \sup\{\|T(x)\|, \|x\| = 1\} = \|S\| + \|T\|$. Likewise $\|\alpha T\| = |\alpha| \|T\|, \alpha \in \mathbb{K}$. Finally, since $\|T(x)\| \leq \|T\| \|x\|$, we have $\|T\| = 0 \implies T(x) = 0, \forall x \in X$.■

1.6 Theorem If Y is a Banach space, $B(X,Y)$ is a Banach space.

Proof: Let (T_n) be Cauchy in $B(X,Y)$. For each $x \in X$ we have $\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| \|x\|, \forall n, m \geq 1$, so $(T_n(x))$ is Cauchy in Y and since Y is Banach $\lim\limits_n T_n(x)$ exists in Y; we denote this limit by $T(x)$.

The mapping so defined from X into Y is linear.

Indeed, for each $n \geq 1$ we have $\|T(x + y) - (T(x) + T(y))\| \leq$
$\|T(x + y) - T_n(x + y)\| + \|T_n(x + y) - (T(x) + T(y))\| \leq$
$\|T(x + y) - T_n(x + y)\| + \|T_n(x) - T(x)\| + \|T_n(y) - T(y)\| \longrightarrow 0, n \longrightarrow \infty$.

So we get $T(x + y) = T(x) + T(y)$. Similarly $T(\alpha x) = \alpha T(x)$, $\alpha \in \mathbb{K}$, $x \in X$.

It remains to prove that $T \in B(X,Y)$ and that $\|T_n - T\| \longrightarrow 0$. If $\epsilon > 0$ there is $N_\epsilon \geq 1$: $n, m \geq N_\epsilon \implies \|T_n - T_m\| < \epsilon$. For $n \geq N_\epsilon$, we have $\|T_n(x) - T(x)\| = \lim\limits_m \|T_n(x) - T_m(x)\| \leq \limsup\limits_m \|T_n - T_m\| \|x\| \leq \epsilon \|x\|$.

Consequently if $n \geq N_\epsilon$, we get $\|T_n(x) - T(x)\| \leq \epsilon \|x\|, \forall x \in X$, and this yields $T_n - T \in B(X,Y)$ and $T \in B(X,Y)$; on the other hand for $n \geq N_\epsilon$
$\|T_n - T\| = \sup\{\|T_n(x) - T(x)\|, \|x\| = 1\} \leq \epsilon$, that is $\|T_n - T\| \longrightarrow 0$.■

2. Duality Hahn-Banach Theorem

2.1 Definition Let X be a normed space on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear functional or linear form on X is a linear operator from X into $\mathbb{K}$. We denote by X^*, instead of $B(X, \mathbb{K})$, the Banach space of continuous linear functionals on X. The space X^* is called the dual space of X. If $x \in X$ and $x^* \in X^*$, we denote $x^*(x)$ by $\langle x^*, x\rangle$.
We are often faced to the following problem: given a subspace $M \subset X$ and a bounded linear functional y^* on M, i.e $y^* \in M^*$, how to extend y^* to a bounded linear functional x^* on X. The extension process solution to this problem is given by the famous **Hahn-Banach Theorem.**

2.2 Theorem (Hahn-Banach)
Let X be a vector space on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $p : X \longrightarrow [0, \infty)$ be a seminorm on X, that is p satisfies:
(1) $p(x + y) \leq p(x) + p(y), \forall x, y \in X$.
(2) $p(\alpha x) = |\alpha| p(x), \forall \alpha \in \mathbb{K}, \forall x \in X$.
Let M be a subspace of X and $g : M \longrightarrow \mathbb{K}$, a linear form on M such that:$|g(y)| \leq p(y), \forall y \in M$
Then g can be extended to a linear functional $f : X \longrightarrow \mathbb{K}$, on X such that:$|f(x)| \leq p(x), \forall x \in X$

For applications, the following corollaries are the most useful:

Corollary 1: Let X be a vector space on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
and let $p : X \longrightarrow [0, \infty)$ be a seminorm on X, then for each $a \in X$, there is a linear functional f on X such that $f(a) = p(a)$ and $|f(x)| \leq p(x), \forall x \in X$.

The next corollary shows that in a normed space, continuous linear functionals exist in profusion:

Corollary 2: Let X be a normed space, then for each $a \in X$, there is $x^* \in X^*$ such that $\langle x^*, a\rangle = \|a\|$, $|\langle x^*, x\rangle| \leq \|x\|, \forall x \in X$ and $\|x^*\| = 1$.

Corollary 3: Let X be a normed space and let M be a subspace of X.
If $y^* : M \longrightarrow \mathbb{K}$, is a continuous linear functional on M, i.e $y^* \in M^*$, there is an extension x^* of y^* to X with $x^* \in X^*$ and $\|x^*\| = \|y^*\|$.

3. Uniform Boundedness Theorem

3.1 Theorem (Uniform Boundedness Theorem)
Let X be a Banach space and let $(E_i, i \in I)$ be a family of normed spaces. For each $i \in I$ let $T_i : X \longrightarrow E_i$ a linear continuous operator such that: $\sup\{\|T_i(x)\|, i \in I\} < \infty$, for each $x \in X$. Then we have: $\sup\{\|T_i\|, i \in I\} < \infty$.

Proof: For each $i \in I$ define the continuous function $f_i : X \longrightarrow \mathbb{R}$ by $f_i(x) = \|T_i(x)\|$. The condition reads: for each $x \in X$ there is $M_x > 0$ such that $f_i(x) = \|T_i(x)\| \leq M_x, \forall i \in I$. Since X is a Baire space there is a nonempty open U of X and a constant $M > 0$ such that $\sup\{\|T_i(x)\|, i \in I\} \leq M, \forall x \in U$ (Theorem **5.7.7**). Let $B(a, r)$ be an open ball contained in U with $a \in U$ and $r > 0$. So we have $\|x - a\| < r \implies \sup\{\|T_i(x)\|, i \in I\} \leq M$. We show that $\|T_i\| \leq \dfrac{2M}{r}, \forall i \in I$. To this end it is enough to have $\|T_i(y)\| \leq \dfrac{2M}{r}, \forall i \in I$ for $\|y\| < 1$. For such y put $x = a + ry$, we get $\|x - a\| = r\|y\| < r$ and then $\|T_i(x)\| \leq M, \forall i \in I$. This gives $\|T_i(a) + rT_i(y)\| \leq M, \forall i$. But $a \in U$, so $\|T_i(a)\| \leq M$.
Now we make the following estimation:
$\|rT_i(y)\| \leq \|T_i(a) + rT_i(y)\| + \|T_i(a)\| \leq M + M = 2M$ whence $\|T_i(y)\| \leq \dfrac{2M}{r}, \forall i \in I$. Finally we get $\|T_i\| = \sup\{\|T_i(y)\|, \|y\| < 1\} \leq \dfrac{2M}{r}, \forall i \in I.$■

Corollary: (Banach-Steinhauss)
Let X be a Banach space and let E be a normed space. Let (T_n) be a sequence of linear bounded operators from X into E. Suppose that $T(x) = \lim\limits_n T_n(x)$ exists in E for each $x \in X$. Then $T(x)$ defines a linear bounded operator T from X into E with $\|T\| \leq \liminf\limits_n \|T_n\|$.

Proof: It is clear that T is linear. Let $\epsilon > 0$ and $x \in X$, there is $N = N_{\epsilon,x} \geq 1$ such that $\forall n \geq N : \|T_n(x)\| \leq \|T(x)\| + \epsilon$. Since $\sup\limits_{n \leq N} \|T_n(x)\| < \infty$, we deduce that $\sup\limits_n \|T_n(x)\| < \infty$, for each $x \in X$. By the uniform boundedness theorem, there is $M > 0$ such that $\sup\limits_n \|T_n\| \leq M$, this yields $\|T_n(x)\| \leq \|T_n\| \|x\| \leq M\|x\|, \forall n \geq 1$ and $\|T(x)\| = \lim\limits_n \|T_n(x)\| \leq M\|x\|$, this proves that T is bounded. On the other hand $\|T_n(x)\| \leq \|T_n\| \|x\|, \forall n \geq 1 \implies \|T(x)\| = \lim\limits_n \|T_n(x)\| \leq \liminf\limits_n \|T_n\| \|x\|$ and then $\|T\| \leq \liminf\limits_n \|T_n\|$.■

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