Solution Examen Final

ANA - Num. Mat

MASTER I - M.A

Ex1 (10P)

1)- $1 = \|I_n\| = \|A \cdot A^{-1}\| \leq \|A\| \, \|A^{-1}\| = Cond(A)$ (1P)

- $Cond(\alpha A) = \|\alpha A\| \cdot \|(\alpha A)^{-1}\| = |\alpha| \, \|A\| \cdot \frac{1}{|\alpha|} \|A^{-1}\| = \|A\| \|A^{-1}\| = Cond(A)$ (1P)

- $Cond(AB) = \|(A.B)\| \, \|(A.B)^{-1}\| \leq \|A\| \cdot \|A^{-1}\| \, \|B\| \cdot \|B^{-1}\| = Cond(A) \cdot Cond(B)$ (1P)

2)- $\|A\|_2 = \sqrt{\sigma_n}$, $\|A^{-1}\|_2 = \frac{1}{\sqrt{\sigma_1}} \Rightarrow \|A\|_2 \cdot \|A^{-1}\|_2 = \sqrt{\frac{\sigma_n}{\sigma_1}} = Cond_2(A)$ (3P)

- $A = A^t \Rightarrow \|A\|_2 = \lambda_n$, $\|A^{-1}\|_2 = \frac{1}{\lambda_1} \Rightarrow Cond_2(A) = \frac{\lambda_n}{\lambda_1}$ (2P)

- $A = \alpha Q \Rightarrow Cond_2(A) = Cond_2(Q)$ et $\|Q\|_2 = \|Q^{-1}\|_2 = 1$ (2P)

~~et Cond~~ donc $Cond_2(A) = 1$

Ex2 (10P)

1. $J = \begin{bmatrix} 0 & -\alpha & \beta \\ 0 & 0 & 0 \\ \beta & -\alpha & 0 \end{bmatrix}$ $\det(J - \lambda I) = \lambda(\lambda^2 - \beta^2) \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = +\beta \\ \lambda_3 = -\beta \end{matrix}$ et $\rho(J) = |\beta|$.

$\rho(J) < 1 \iff |\beta| < 1$ et $\alpha \in \mathbb{R}$. (2P)

2)- $\alpha = \beta = 2 \Rightarrow \rho(J) > 1$ Jacobi diverge. (1P)

- $x^0 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $x^1 = Jx^0 + D^{-1}b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $x^2 = Jx^1 + D^{-1}b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. (2P)

Par recurrence $x^k = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\forall k \geq 1$ car x^1 est solution de $Ax = b \iff x = Jx + D^{-1}b$.

3) $\begin{cases} x_1^{k+1} = -\alpha x_2^k + \beta x_3^k + 1 \\ x_2^{k+1} = 1 \\ x_3^{k+1} = \beta x_1^{k+1} - \alpha x_2^{k+1} + 1 = -\alpha\beta x_2^k + \beta^2 x_3^k + \beta - \alpha + 1 \end{cases}$ $\longrightarrow$ (2P)

$\Rightarrow G = \begin{bmatrix} 0 & -\alpha & \beta \\ 0 & 0 & 0 \\ 0 & -\alpha\beta & \beta^2 \end{bmatrix} = (D - E)^{-1} F.$ $\longrightarrow$ (1P)

4) $\det(G - \lambda I) = \lambda^2(\lambda - \beta) = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = \beta \end{matrix}$ et $\rho(G) = |\beta|$. (2P)

La méthode de Gauss-Seidel converge $\iff \rho(G) < 1 \iff |\beta| < 1$ et $\alpha \in \mathbb{R}$

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(8P)

Ex1 1) $\nabla f(x,y) = \begin{pmatrix} 6x^3+6y \\ 6x-6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x^2+y=0 \\ x=y \end{pmatrix} \Leftrightarrow \begin{pmatrix} x^2+x=0 \\ x=y \end{pmatrix} \Leftrightarrow \begin{pmatrix} x(x+1)=0 \\ x=y \end{pmatrix}$

Deux points critiques $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (2P)

2) $H_f(x,y) = \begin{pmatrix} 12x & 6 \\ 6 & -6 \end{pmatrix}$, $H_f(0,0) = \begin{pmatrix} 0 & 6 \\ 6 & -6 \end{pmatrix}$, $\det H_f(0,0) = -36 < 0$, λ_1, λ_2

de signes opposés $\Leftrightarrow$ f ni convexe ni concave (2P)
donc (0, 0) point stationnaire (point - selle).

$H_f(-1,-1) = \begin{pmatrix} -12 & 6 \\ 6 & -6 \end{pmatrix}$, $\det H_f(-1,-1) = 36 > 0$ et $Tr(H_f(-1,-1)) = -18 < 0 \Rightarrow \lambda_1, \lambda_2 < 0$ (2P)

$H_f(-1,-1)$ définie négative $\Leftrightarrow$ f concave au voisinage $(-1,-1) \Rightarrow (-1,-1)$ maximum local

3) - f n'a pas de minimum local $\Rightarrow$ f n'a pas de minimum global (1P)

- $f(x,0) = 2x^3+2$, donc $f(x,0) \xrightarrow[x \to \mp\infty]{} \mp\infty \Rightarrow$ f n'a pas de maximum global. (1P)

Problème (12P)

(1P) a) $A = 2I$, $b = \begin{pmatrix} 14 \\ 6 \end{pmatrix}$, $\alpha = -7$, $H(f) = 2I$ définie positive $\Rightarrow$ f strictement convexe

(2P) b) $(0,0) \in \Omega$, $\Omega = \underbrace{\theta^{-1}(]-\infty,0] \times]-\infty,0])}_{\text{fermé}}$ / $\theta : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1,x_2) \mapsto \theta(x_1,x_2) = \begin{pmatrix} x_1+x_2-2 \\ x_1+2x_2-3 \end{pmatrix}$ continue

$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 / \theta_1(x_1,x_2) = x_1+x_2-2 \leq 0, \theta_2(x_1,x_2) = x_1+2x_2-3 \leq 0\}$ $\left.\begin{matrix} \theta_1 \text{ affine} \\ \theta_2 \text{ affine} \end{matrix}\right\} \Rightarrow$ convexe

(3P) c) KKT : $\Leftrightarrow \begin{cases} 2x_1 - 14 + \lambda_1 + \lambda_2 = 0 \\ 2x_2 - 6 + \lambda_1 + 2\lambda_2 = 0 \\ \lambda_1(x_1+x_2-2) = \lambda_2(x_1+2x_2-3) = 0 \\ \lambda_1, \lambda_2 \geq 0 \\ x_1+x_2-2 \leq 0, \ x_1+2x_2-3 \leq 0 \end{cases}$

1) cas $\lambda_1 = \lambda_2 = 0 \Rightarrow (\bar{x}_1, \bar{x}_2) = (7,3) \notin \Omega$ Non

2) cas $x_1+x_2-2 = 0$, $\lambda_2 = 0 \Rightarrow (\bar{x}_1, \bar{x}_2) = (3,-1)$, $\lambda_1 = 8$ oui

3) cas $\lambda_1 = 0$, $x_1+2x_2-3 = 0 \Rightarrow (\bar{x}_1, \bar{x}_2) = (5,-1) \notin \Omega$, $\lambda_2 = 4$ Non

4) $x_1+x_2-2 = 0$, $x_1+2x_2-3 = 0 \Rightarrow (\bar{x}_1, \bar{x}_2) = (1,1)$, $\lambda_1 = 20$, $\lambda_2 = -8$ Non.

donc la solution $\bar{x} = (\bar{x}_1, \bar{x}_2) = (3,-1)$ et $f(\bar{x}) = -33$

(1p) d) $L(x,\lambda) = x_1^2 + x_2^2 - 14x_1 - 6x_2 - 7 + \lambda_1(x_1+x_2-2) + \lambda_2(x_1+2x_2-3)$.

(2p) e). $\sup_{\lambda \in \mathbb{R}^2_+} L(x,\lambda) = \begin{cases} +\infty & \text{si } x \notin \Omega \\ f(x) & \text{si } x \in \Omega \end{cases}$ /

$\inf_{x\in\mathbb{R}^2} \sup_{\lambda\in\mathbb{R}^2} L(x,\lambda) = \inf_{x\in\Omega} f(x) = f(\bar{x}) = f(3,-1) = -33$ (*).

(2p) f) $\inf_{x\in\mathbb{R}^2} L(x,\lambda) = L(x^*,\lambda) \Rightarrow$. $\nabla_x L(x^*,\lambda) = 0 \Leftrightarrow \begin{cases} 2x_1^* - 14 + \lambda_1 + \lambda_2 = 0 \\ 2x_2^* - 6 + \lambda_1 + 2\lambda_2 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x_1^* = -\frac{\lambda_1}{2} - \frac{\lambda_2}{2} + 7 \\ x_2^* = -\frac{\lambda_1}{2} - \lambda_2 + 3 \end{cases}$. et $L(x^*,\lambda) = -\frac{\lambda_1^2}{2} - \frac{5}{4}\lambda_2^2 - \frac{3}{2}\lambda_1\lambda_2 + 8\lambda_1 + 1\lambda_2 - 65$

donc $\sup_{\lambda\in\mathbb{R}^2_+} \inf_{x\in\mathbb{R}^2} L(x,\lambda) = \sup_{\lambda\in\mathbb{R}^2_+} L(x^*,\lambda) = \sup_{\lambda\in\mathbb{R}^2_+} D(\lambda)$ / $D(\lambda) = L(x^*,\lambda)$ (le problème dual)

(2p) e) $\sup_{\lambda\in\mathbb{R}^2_+} D(\lambda) = -\inf_{\lambda\in\mathbb{R}^2} -D(\lambda)$.

on résoud le problème $\inf \frac{\lambda_1^2}{2} + \frac{5}{4}\lambda_2^2 + \frac{3}{2}\lambda_1\lambda_2 - 8\lambda_1 - 1\lambda_2 + 65$

$\lambda_1 \geq 0$

$\lambda_2 \geq 0$

par KKT on trouve la solution $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2) = (8,0)$ et $D(\bar{\lambda}) = -33$.

donc le saut de dualité $=$ val(P) $-$ val(D) $= -33 + 33 = 0$.