Corrigé type de l'examen Final

EX1

1-) $10e^{i5\frac{\pi}{6}} + 4e^{-i\frac{\pi}{6}} = 10\left(\cos 5\frac{\pi}{6} + i\sin 5\frac{\pi}{6}\right) +$

$4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 10\left(\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right) +$

$4\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 10\left(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) + 4\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= 10\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -5\sqrt{3} + i5 + 2\sqrt{3} - 2i = -3\sqrt{3} + 3i$

Donc : $\boxed{10e^{i5\frac{\pi}{6}} + 4e^{-i\frac{\pi}{6}} = -3\sqrt{3} + 3i}$ 2 points

2-) posons $z_1 = 1+i$ et $z_2 = 1 - \sqrt{3}i$

$|z_1| = \sqrt{2}$ / $|z_2| = 2$

$\cos\theta_1 = \frac{1}{\sqrt{2}}$, $\sin\theta_1 = \frac{1}{\sqrt{2}}$ $\Rightarrow \theta_1 = \frac{\pi}{4}$ / $\cos\theta_2 = \frac{1}{2}$, $\sin\theta_2 = \frac{-\sqrt{3}}{2}$ $\Rightarrow \theta_2 = -\frac{\pi}{3}$

donc $z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}}$ et $z_2 = 2e^{-i\frac{\pi}{4}}$

donc : $\frac{(1+i)^{20}}{(1-\sqrt{3}i)^5} = \frac{\left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{20}}{\left(2e^{-i\frac{\pi}{3}}\right)^5} = \frac{\sqrt{2}^{20}\, e^{i5\pi}}{2^5\, e^{-i5\frac{\pi}{3}}}$

$= 32\, e^{i5\pi}\, e^{i\left(2\pi - \frac{\pi}{3}\right)} = 32\, e^{i5\pi}\, e^{-i\frac{\pi}{3}} = 32\, e^{i2\frac{\pi}{3}}$

2 points

$u \in C^2(\Omega) \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (0,25)

$v \in C^2(\Omega) \Rightarrow \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ (0,25)

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \overset{?}{=} \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \quad (I)$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2} = 0$$ (0,5)

car on a v est harmonique:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

et $u \in C^2(\Omega) \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

donc $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$ (1)

étudions maintenant la 2ème equation de Cauchy:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = -\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) \quad II$$

$$(II) \Leftrightarrow \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}$$

(1,5 point) (2) $= -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right)$

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$$f(x,y) = e^{-x}(x \sin y - y \cos y)$$

$$\frac{\partial f}{\partial x} = -e^{-x}(x \sin y - y \cos y) + e^{-x}(\sin y)$$

$$= -e^{-x}\left(\sin y(-1+x) - y \cos y\right)$$

$$\frac{\partial^2 f}{\partial x^2} = + e^{-x}\left(\sin y(1-x) - y \cos y\right) + e^{-x}(-\sin y)$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = -e^{-x}\left((2-x)\sin y - y \cos y\right)}$$

$$\frac{\partial f}{\partial y} = e^{-x}\left(x \cos y - (\cos y \mp y \sin y)\right)$$

$$e^{-x}(x \cos y - \cos y + y \sin y)$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-x}(-x \sin y + \sin y + \sin y + y \cos y)$$

$$= e^{-x}\left((2-x)\sin y + y \cos y\right)$$

0,75 point

$f(x,y) = e^{-x}(x\sin y - y\cos y)$

$\frac{\partial f}{\partial x} = -e^{-x}(x\sin y - y\cos y) + e^{-x}\sin y$

$= -e^{-x}x\sin y + e^{-x}y\cos y + e^{-x}\sin y$

$= e^{-x}\left((1-x)\sin y + y\cos y\right)$

$\frac{\partial^2 f}{\partial x^2} = -e^{-x}\left((1-x)\sin y + y\cos y\right) + e^{-x}(-\sin y)$

$= -e^{-x}\left((2-x)\sin y + y\cos y\right)$ 01 point

donc $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y} = 0$ 0,5

d'où f est harmoni-que.

EX3

$$\int_{\gamma} f(z)\,dz = \int_0^2 f(\gamma(t))\,\gamma'(t)\,dt \quad 0{,}5$$

$$= \int_0^2 \left(\gamma(t)^2 + 3\overline{\gamma(t)} - 2\right)(1)\,dt \quad 0{,}5$$

$$= \int_0^2 \left((t-2i)^2 + 3(t+2i) - 2\right) dt \quad 0{,}5$$

$$= \int_0^2 \left(t^2 - 4it - 4 + 3t + 6i - 2\right) dt \quad 0{,}5$$

$$= \int_0^2 \left(t^2 + 3t - 4it - 6 + 6i\right) dt$$

$$= \left. \frac{t^3}{3} + \frac{3t^2}{2} - 2it^2 - 6t + 6it \right]_0^2 \quad 0{,}5$$

$$= \frac{8}{3} + 6 - 8i - 12 + 12i$$

$$= 4i - 6 + \frac{8}{3} = -\frac{10}{3} + 4i$$

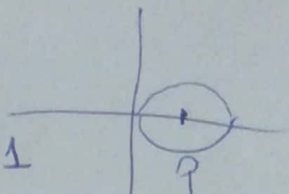
0,5

5

Ex 4

$|z-1| = 1 \Leftrightarrow (x-1)^2 + (y-0)^2 = 1$

cercle de centre $(1,0)$ de rayon 1



$\{\gamma\}$

on a $(z-1)^2(z+2) = 0 \Leftrightarrow \begin{cases} z = 1 \\ \vee \\ z = -2 \end{cases}$

0,25

si $z = 1$ on a : $|z-1| = |1-1| = 0 < 1$

donc $z = 1 \in \{\gamma\}$

0,25

si $z = -2$ on a $|z-1| = |-2-1| = 3 > 1$

$\Rightarrow z = -2 \notin \{\gamma\}$

0,25

donc En utilisant la formule de Cauchy.

$$I = \int_\gamma \frac{\frac{z}{z+2}}{(z-1)^2}\, dz = \int \frac{f(z)}{(z-1)^2}\, dz = \int \frac{f(z)}{(z-1)^{1+1}}\, dz$$

0,5

ou $f(z) = \dfrac{z}{z+2}$, $n = 1$

donc $I = \dfrac{2i\pi}{n!} f'(1) = 2i\pi \left[\dfrac{z+2-z}{(z+2)^2}\right]_{z=1}$ 0,25

0,25 $= 2i\pi \left[\dfrac{2}{(z+2)^2}\right]_{z=1} = 2i\pi \dfrac{2}{9} = \dfrac{4i\pi}{9}$

0,25 / 1,25

2) $\gamma : |z| = 4 \Leftrightarrow (x-0)^2 + (y-0)^2 = 4^2$ (0,25)

cercle de centre $(0,0)$ de rayon 4

on a dans ce cas $z = 1 \in \{\gamma\}$ et $z = -2 \in \{\gamma\}$

car $|1| < 4$ et $|-2| = 2 < 4$ (0,5)

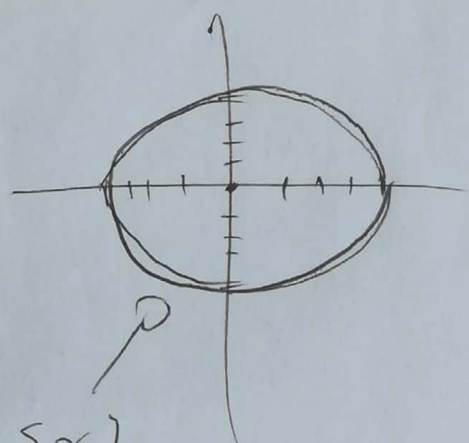
donc $I = \int_\gamma \frac{\frac{z}{(z-1)^2}}{(z+2)}\, dz + \int_\gamma \frac{\frac{z}{z+2}}{(z-1)^2}\, dz$ (0,75)

$= 2i\pi f(-2) + 2i\pi f'(1)$ (0,5)

$= 2i\pi \left(\frac{-2}{9}\right) + 2i\pi \frac{2}{9}$ (0,5)

$= -\frac{4i\pi}{9} + \frac{4i\pi}{9} = 0$

(0,5)



$\{\gamma\}$